ON RECURRENT SPACES

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In this paper some properties of recurrent and Ricci-recurrent spaces of the first and second order have been obtained. Recurrent and Ricci-recurrent spaces of the nth order have been defined and studied. A geometrical interpretation of the vector of recurrence and the necessary and sufficient condition for the existence of parallel vector field over a coordinate neighbourhood of a point in a recurrent space have been obtained. It has been proved that a Ricci-recurrent space is projectively recurrent if it is conformally recurrent and is conformally recurrent if it is projectively recurrent. The condition that a space which is both projectively recurrent and conformally recurrent may be Ricci-recurrent has been found. It has also been proved that every recurrent space is conformally/projectively recurrent but a conformally/projectively recurrent space is a recurrent space only if it is Ricci-recurrent. Necessary and sufficient conditions that (i) a Riemannian space be 2-recurrent, (ii) a Riemannian space be n-Ricci-recurrent and (iii) a n-Ricci-recurrent space be n-recurrent have also been obtained.

1. Introduction

A space for which the covariant derivative of the curvature tensor is zero at all points is called a symmetric space. Thus

$$R_{hijk,l} = 0 \qquad \text{(1.1)}$$

defines a symmetric space.

A space of constant curvature is characterized by

$$R_{hijk} = 2Kg_{h[j}g_{k]i}. \qquad \text{(1.2)}$$

Transvecting (1.2) by g^{hk},

$$R_{ij} = K(1-n)g_{ij}. \qquad \text{(1.3)}$$

From (1.1) and (1.2) it immediately follows that all spaces of constant curvature are symmetric but the converse is not true, in general.

A non-flat n-dimensional Riemannian space for which the curvature tensor satisfies the relation of the form

$$R_{hijk,l} = \lambda_l R_{hijk}, \qquad \text{(1.4)}$$

where λ_l is a non-zero vector and a comma followed by an index l denotes covariant differentiation with respect to x^l, is called a recurrent or kappa-space and is denoted by K_n. Using Bianchi identity

$$3R_{hi[jk,l]} \equiv R_{hijk,l} + R_{hikl,j} + R_{hilj,k} = 0,$$

we get from (1.4)

$$R_{hi[jk}\lambda_{l]} = 0 \qquad \ldots \qquad (1.5)$$

which is always true for a recurrent space.

An n-dimensional space which is either a K_n or which satisfies both (1.1) and (1.5) for some non-zero vector λ_l has been denoted by K_n^* by Walker (1950), who has shown that in some cases K_n and K_n^* can be treated alike.

A space of n dimensions for which the conformal curvature tensor satisfies the relation of the form

$$C_{ijkl,\, m} = \lambda_m C_{ijkl}, \qquad (1.6)$$

for some non-zero vector λ_m at all points will be called conformally recurrent.

Similarly, a space of n dimensions for which the projective tensor W_{ijkl} satisfies

$$W_{ijkl,\, m} = \lambda_m W_{ijkl}, \qquad \ldots \qquad (1.7)$$

for some non-zero vector λ_m will be called projectively recurrent.

A Riemannian space of more than two dimensions whose Ricci tensor R_{ij} ($\neq 0$) satisfies

$$R_{ij,\, l} = \lambda_l R_{ij}, \qquad \ldots \qquad (1.8)$$

for some non-zero vector λ_l is called a Ricci-recurrent space and has been denoted by R_n (Patterson 1952).

Obviously every K_n is an R_n but the converse is not, in general, true. Chaki (1956) has shown that every R_n ($n > 3$) is a K_n if it is conformal to a flat space and that every R_3 is a K_3. Walker (1950) has shown that the vector of recurrence λ_l is a gradient for K_n-space, whereas for a K_n^*-space, it is not necessarily a gradient. However, it can be chosen to be a gradient subject to certain conditions. Chaki (1956) has shown that a necessary and sufficient condition that λ_l be a gradient is that at every point of the R_n,

$$2R_{p(j}R^{p}_{i)kl} \equiv R_{pj}R^{p}_{ikl} + R_{pi}R^{p}_{jkl} = 0. \qquad \ldots \qquad (1.9)$$

According to Lichnerowicz (1952), an n-dimensional Riemannian space is said to be second order recurrent or briefly 2-recurrent if

$$R_{ijkl,\, mp} = a_{mp} R_{ijkl}, \qquad \ldots \qquad (1.10)$$

for some non-zero tensor a_{mp}. Using Bianchi identity

$$R_{hi[jk,\, l]m} = 0,$$

we get from (1.10)

$$R_{ij[kl}a_{m]p} = 0. \qquad \ldots \qquad (1.11)$$

A non-flat second order recurrent space with $a_{mp} \neq 0$ is said to be an essentially 2-recurrent or a K_{II}-space.

n-Spaces satisfying the equations

$$R_{ijkl,\, mp} = 0$$

and

$$R_{ij[kl}a_{m]p} = 0,$$

for a non-zero tensor a_{mp} have many common properties with K_{II}-spaces and, together with these, they form a class denoted by K_{II}^*. The class K_{II}^* is similar to the class K_n^* considered by Walker (1950).

Lichnerowicz (1952) has proved that any compact 2-recurrent space with non-zero scalar curvature ($R \neq 0$) is either symmetric in the sense of Cartan or recurrent. Roter (1964*a*) has shown that this result holds for K_{II}^*-spaces without the assumption of compactness or $R \neq 0$.

A space whose Ricci tensor R_{ij} ($\neq 0$) satisfies the relation

$$R_{ij,\,lm} = a_{lm}R_{ij} \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (1.12)$$

for some non-zero tensor a_{lm}, is called a 2-Ricci-recurrent space. Evidently every K_{II}-space with $R_{ij} \neq 0$ is 2-Ricci-recurrent. Roter (1964*a*) has obtained the necessary and sufficient conditions for 2-Ricci-recurrent spaces to be recurrent spaces.

2-Recurrent spaces with definite metrics have been studied by Lichnerowicz (1952) and by Roter (1964*b*). Roter has proved that when one of the scalars $a^{rs}a_{rs}$ or a^r_r is non-zero, then the curvature tensor of 2-recurrent space with non-zero scalar curvature R can be expressed by R and the Ricci tensor R_{ij} only. He has also given a necessary and sufficient condition for such 2-recurrent spaces to be recurrent.

An n-dimensional space for which the curvature tensor satisfies the relation

$$R_{hijk,\,l_1l_2\ldots l_n} = a_{l_1l_2\ldots l_n}R_{hijk} \qquad \ldots \quad \ldots \quad \ldots \quad (1.13)$$

for some non-zero tensor of the nth order, $a_{l_1l_2\ldots l_n}$ will be called n-recurrent.

2. Geometrical Interpretation of the Vector of Recurrence

In a recurrent space, we have

$$R_{hijk,\,m} = \lambda_m R_{hijk}.$$

Using Bianchi identity, we get

$$\lambda_m R_{hijk} + \lambda_j R_{hikm} + \lambda_k R_{himj} = 0.$$

Multiplying by $g^{hk}g^{ij}$, we get

or

$$\left.\begin{aligned} R\lambda_m - R^j_m\lambda_j - R^k_m\lambda_k &= 0 \\ R^j_m\lambda_j - \tfrac{1}{2}R\lambda_m &= 0 \\ (R^j_m - \tfrac{1}{2}R\delta^j_m)\lambda_j &= 0 \end{aligned}\right\} \qquad \ldots \quad \ldots \quad \ldots \quad (2.1)$$

Therefore λ_j is the eigenvector of R^j_m, the corresponding eigenvalue being $\tfrac{1}{2}R$. We get the same result for a Ricci-recurrent space.

3. Parallel Vector Field

It is known (Mishra 1965) that the necessary and sufficient conditions for the existence of a parallel vector field λ over a coordinate neighbourhood of a point in a Riemannian space are

$$\lambda^k R^l_{ijk} = 0, \qquad (3.1)$$

$$\lambda^k R^l_{ijk,\, m_1} = 0, \qquad (3.2)$$

$$\left.\begin{array}{l} \lambda^k R^l_{ijk,\, m_1 m_2} = 0 \\ \cdot \quad \cdot \quad \cdot \\ \cdot \quad \cdot \quad \cdot \\ \lambda^k R^l_{ijk,\, m_1 m_2 \ldots m_r} = 0 \end{array}\right\}. \qquad (3.3)$$

For a recurrent space $R_{hijk,\, m} = \mu_m R_{hijk}$ for some vector μ_m.

Substituting in (3.2) we get

$$\lambda^k \mu_{m_1} R^l_{ijk} = 0$$

which is identically satisfied by virtue of (3.1). Similarly, substituting in (3.3) we find that this is also identically satisfied. Consequently, all the other equations are identically satisfied. Hence we have the theorem that

The necessary and sufficient condition for the existence of a parallel vector field over a coördinate neighbourhood of a point in a recurrent space is

$$\lambda^k R^l_{ijk} = 0.$$

4. Projectively Recurrent and Conformally Recurrent Spaces

The conformal curvature tensor (Eisenhart 1949) is given by

$$C_{ijkl} = R_{ijkl} + \frac{2}{n-2}\{g_{i[k}R_{l]j} - g_{j[k}R_{l]i}\} + \frac{2R}{(n-1)(n-2)}\{g_{j[k}g_{l]i}\} \qquad (4.1)$$

and the Weyl projective tensor is given by

$$W_{ijkl} = R_{ijkl} - \frac{2}{n-1}(g_{i[l}R_{k]j}). \qquad (4.2)$$

Subtracting, we get

$$C_{ijkl} - W_{ijkl} = \frac{2R}{(n-1)(n-2)}(g_{j[k}g_{l]i}) - \frac{2}{(n-1)(n-2)}(g_{i[l}R_{k]j}) - \frac{2}{n-2}(g_{j[k}R_{l]i}). \qquad (4.3)$$

Differentiating covariantly with respect to x^m, we get

$$C_{ijkl,\, m} - W_{ijkl,\, m} = \frac{2R_{,m}}{(n-1)(n-2)}(g_{j[k}g_{l]i}) - \frac{2}{(n-1)(n-2)}(g_{i[l}R_{k]j,\, m}) - \frac{2}{(n-2)}(g_{j[k}R_{l]i,\, m}). \qquad (4.4)$$

Multiplying (4.3) by λ_m and subtracting from (4.4), we get

$$(C_{ijkl,\,m}-\lambda_m C_{ijkl})-(W_{ijkl,\,m}-\lambda_m W_{ijkl}) = \frac{2}{(n-1)(n-2)}(g_{j[k}g_{l]i})(R_{,\,m}-\lambda_m R)$$

$$-\frac{1}{(n-1)(n-2)}\{(R_{jk,\,m}-\lambda_m R_{jk})g_{il}-(R_{jl,\,m}-\lambda_m R_{jl})g_{ik}\}$$

$$-\frac{1}{n-2}\{(R_{il,\,m}-\lambda_m R_{il})g_{jk}-(R_{ik,\,m}-\lambda_m R_{ik})g_{jl}\}. \quad \ldots \quad (4.5)$$

Hence it follows that

A Ricci-recurrent space is projectively recurrent if it is conformally recurrent and is conformally recurrent if it is projectively recurrent.

A space which is both projectively recurrent and conformally recurrent is Ricci-recurrent if

$$\lambda_m = \frac{\partial}{\partial x^m}\log R.$$

Again, differentiating (4.1) covariantly with respect to x^m, we get

$$C_{ijkl,\,m} = R_{ijkl,\,m}+\frac{2}{n-2}\{g_{i[k}R_{l]j,\,m}-g_{j[k}R_{l]i,\,m}\}+\frac{2R_{,\,m}}{(n-1)(n-2)}\{g_{j[k}g_{l]i}\}. \qquad (4.6)$$

For a recurrent space $R_{ijkl,\,m} = \lambda_m R_{ijkl}$. Therefore (4.6) gives

$$C_{ijkl,\,m} = \lambda_m C_{ijkl}.$$

Thus

$$[R_{ijkl,\,m} = \lambda_m R_{ijkl}] \Longrightarrow [C_{ijkl,\,m} = \lambda_m C_{ijkl}].$$

The converse is not, in general, true. For $C_{ijkl,\,m} = \lambda_m C_{ijkl}$ yields

$$R_{ijkl,\,m}-\lambda_m R_{ijkl} = \frac{1}{n-2}[(R_{jk,\,m}-\lambda_m R_{jk})g_{il}-(R_{jl,\,m}-\lambda_m R_{jl})g_{ik}+(R_{il,\,m}-\lambda_m R_{il})g_{jk}$$

$$-(R_{ik,\,m}-\lambda_m R_{ik})g_{jl}]+\frac{1}{(n-1)(n-2)}[g_{jk}g_{il}-g_{jl}g_{ik}][R_{,\,m}-\lambda_m R].$$

The space is therefore recurrent if the R.H.S. is zero. Hence we have the result that

Every recurrent space is conformally recurrent but a conformally recurrent space is not, in general, a recurrent space. A conformally recurrent space is a recurrent space if it is Ricci-recurrent. Similarly, it can be shown that every recurrent space is projectively recurrent, but a projectively recurrent space is not, in general, a recurrent space. A projectively recurrent space is a recurrent space if it is Ricci-recurrent.

5. 2-Recurrent Space

For a non-flat Riemannian space which is 2-recurrent, we have

$$R_{hijk,\,lm} = a_{lm}R_{hijk}, \quad \ldots \quad (5.1)$$

where a_{lm} is some non-zero tensor.

From (5.1), we get

$$R_{hijk,\,lm}-R_{hijk,\,ml}=2a_{[lm]}R_{hijk}. \quad \ldots \quad \ldots \quad \ldots \quad (5.2)$$

Applying Ricci identity to the L.H.S. of (5.2), we get

$$R_{pijk}R^{p}_{hlm}+R_{hpjk}R^{p}_{ilm}+R_{hipk}R^{p}_{jlm}+R_{hijp}R^{p}_{klm}=2a_{[lm]}R_{hijk}. \quad \ldots \quad (5.3)$$

If a_{lm} is a symmetric tensor, then the condition that the Riemannian space be 2-recurrent is

$$R_{pijk}R^{p}_{hlm}+R_{hpjk}R^{p}_{ilm}+R_{hipk}R^{p}_{jlm}+R_{hijp}R^{p}_{klm}=0.$$

The same condition has been obtained by Roter (1964*a*, *b*) but by using Walker's lemmas (1950), whereas we have obtained it in a simpler way.

We shall now consider the case when a_{lm} is non-symmetric. Condition (5.3) can be written as

$$R_{pijk}R^{p}_{hlm}+R_{hpjk}R^{p}_{ilm}-R_{lmpj}R^{p}_{kih}+R_{lmpk}R^{p}_{jih}=2a_{[lm]}R_{hijk}.$$

Using Bianchi identity, we get

$$R_{p[ijk]}R^{p}_{hlm}+R_{hp[jk}R^{p}_{i]lm}-R_{lmp[j}R^{p}_{ki]h}+R_{lmp[k}R^{p}_{ji]h}=2a_{[lm]}R_{h[ijk]}$$

or

$$R_{hp[jk}R^{p}_{i]lm}-R_{lmp[j}R^{p}_{ki]h}+R_{lmp[k}R^{p}_{ji]k}=0, \quad \text{since } R_{h[ijk]}=0$$

or

$$R_{hp[jk}R^{p}_{i]lm}-2R_{lmp[i}R^{p}_{jk]h}=0,$$

that is

$$R_{hpjk}R^{p}_{ilm}+R_{hpki}R^{p}_{jlm}+R_{hpij}R^{p}_{klm}=2\big(R_{lmpi}R^{p}_{jkh}+R_{lmpj}R^{p}_{kih}+R_{lmpk}R^{p}_{ijh}\big).$$

Thus we have the following theorem:

The necessary and sufficient condition that a Riemannian space be 2-recurrent is

$$R_{hp[jk}R^{p}_{i]lm}-2R_{lmp[i}R^{p}_{jk]h}=0.$$

6. n-Recurrent Spaces. Curvature Tensors

The conformal tensor C_{ijkl} is given by

$$C_{ijkl}=R_{ijkl}+\frac{2}{n-2}\{g_{i[k}R_{l]j}-g_{j[k}R_{l]i}\}+\frac{2R}{(n-1)(n-2)}\{g_{j[k}g_{l]i}\}. \quad \ldots \quad (6.1)$$

Differentiating covariantly in succession with respect to $x^{p_1}, x^{p_2}, \ldots x^{p_n}$, we get

$$C_{ijkl,\,p_1p_2\ldots p_n}=R_{ijkl,\,p_1p_2\ldots p_n}+\frac{2}{n-2}\{g_{i[k}R_{l]j,\,p_1p_2\ldots p_n}-g_{j[k}R_{l]i,\,p_1p_2\ldots p_n}\}$$

$$+\frac{2R_{,\,p_1p_2\ldots p_n}}{(n-1)(n-2)}\{g_{j[k}g_{l]i}\}. \quad \ldots \quad \ldots \quad \ldots \quad (6.2)$$

Multiplying (6.1) by $a_{p_1p_2\ldots p_n}$ and subtracting from (6.2), we get

$$C_{ijkl,\,p_1p_2\ldots p_n}-a_{p_1p_2\ldots p_n}C_{ijkl}=R_{ijkl,\,p_1p_2\ldots p_n}-a_{p_1p_2\ldots p_n}R_{ijkl}$$
$$+\frac{2}{(n-1)(n-2)}\{g_{j[k}g_{l]i}\}(R_{,\,p_1p_2\ldots p_n}-a_{p_1p_2\ldots p_n}R)$$
$$-\frac{1}{n-2}\{(R_{jk,\,p_1p_2\ldots p_n}-a_{p_1p_2\ldots p_n}R_{jk})g_{il}-(R_{jl,\,p_1p_2\ldots p_n}-a_{p_1p_2\ldots p_n}R_{jl})g_{ik}$$
$$+(R_{il,\,p_1p_2\ldots p_n}-a_{p_1p_2\ldots p_n}R_{il})g_{jk}-(R_{ik,\,p_1p_2\ldots p_n}-a_{p_1p_2\ldots p_n}R_{ik})g_{jl}\}. \quad (6.3$$

If V_n is n-Ricci-recurrent, then

$$R_{ij,\,p_1p_2\ldots p_n}=a_{p_1p_2\ldots p_n}R_{ij}.$$

Therefore (6.3) becomes

$$C_{ijkl,\,p_1p_2\ldots p_n}-a_{p_1p_2\ldots p_n}C_{ijkl}=R_{ijkl,\,p_1p_2\ldots p_n}-a_{p_1p_2\ldots p_n}R_{ijkl}.$$

Conversely, if (6.3) is satisfied, then

$$-\frac{1}{n-2}\{(R_{jk,\,p_1p_2\ldots p_n}-a_{p_1p_2\ldots p_n}R_{jk})g_{il}-(R_{jl,\,p_1p_2\ldots p_n}-a_{p_1p_2\ldots p_n}R_{jl})g_{ik}$$
$$+(R_{il,\,p_1p_2\ldots p_n}-a_{p_1p_2\ldots p_n}R_{il})g_{jk}-(R_{ik,\,p_1p_2\ldots p_n}-a_{p_1p_2\ldots p_n}R_{ik})g_{jl}\}$$
$$+\frac{2}{(n-1)(n-2)}\{g_{j[k}g_{l]i}\}(R_{,\,p_1p_2\ldots p_n}-a_{p_1p_2\ldots p_n}R)=0.$$

Transvecting by g^{il}, we get

$$R_{jk,\,p_1p_2\ldots p_n}-a_{p_1p_2\ldots p_n}R_{jk}=0.$$

Thus a necessary and sufficient condition that a V_n may be n-Ricci-recurrent is

$$C_{ijkl,\,p_1p_2\ldots p_n}-a_{p_1p_2\ldots p_n}C_{ijkl}=R_{ijkl,\,p_1p_2\ldots p_n}-a_{p_1p_2\ldots p_n}R_{ijkl}.$$

The Weyl projective tensor is given by

$$W_{ijkl}=R_{ijkl}-\frac{2}{n-1}(g_{i[l}R_{k]j}). \quad \ldots \quad \ldots \quad \ldots \quad (6.4)$$

Differentiating covariantly in succession with respect to $x^{p_1}, x^{p_2}, \ldots, x^{p_n}$, we get

$$W_{ijkl,\,p_1p_2\ldots p_n}=R_{ijkl,\,p_1p_2\ldots p_n}-\frac{2}{n-1}(g_{i[l}R_{k]j,\,p_1p_2\ldots p_n}). \quad \ldots \quad (6.5)$$

Subtracting (6.4) multiplied by $a_{p_1p_2\ldots p_n}$ from (6.5), we get

$$W_{ijkl,\,p_1p_2\ldots p_n}-a_{p_1p_2\ldots p_n}W_{ijkl}=R_{ijkl,\,p_1p_2\ldots p_n}-a_{p_1p_2\ldots p_n}R_{ijkl}$$
$$-\frac{1}{n-1}\{(R_{jk,\,p_1p_2\ldots p_n}-a_{p_1p_2\ldots p_n}R_{jk})g_{il}-(R_{jl,\,p_1p_2\ldots p_n}-R_{jl}a_{p_1\ldots p_n})g_{ik}\}.$$
$$\ldots \quad (6.5a)$$

Since $R_{ij,\,p_1p_2\ldots p_n}=a_{p_1p_2\ldots p_n}R_{ij}$, it follows from (6.5a) that

$$W_{ijkl,\,p_1p_2\ldots p_n}-a_{p_1p_2\ldots p_n}=R_{ijkl,\,p_1p_2\ldots p_n}-a_{p_1p_2\ldots p_n}R_{ijkl}. \qquad (6.6)$$

Conversely, if (6.6) holds, then (6.5a) gives

$$(R_{jk,\,p_1p_2\ldots p_n}-a_{p_1p_2\ldots p_n}R_{jk})g_{il}-(R_{jl,\,p_1p_2\ldots p_n}-a_{p_1p_2\ldots p_n}R_{jl})g_{ik}=0.$$

Multiplying by g^{il}, we get

$$R_{jk,\,p_1p_2\ldots p_n}-a_{p_1p_2\ldots p_n}R_{jk}=0.$$

Thus a necessary and sufficient condition that a V_n be n-Ricci-recurrent is

$$W_{ijkl,\,p_1p_2\ldots p_n}-a_{p_1p_2\ldots p_n}W_{ijkl}=R_{ijkl,\,p_1p_2\ldots p_n}-a_{p_1p_2\ldots p_n}R_{ijkl}.$$

If

$$W_{ijkl,\,p_1p_2\ldots p_n}-a_{p_1p_2\ldots p_n}W_{ijkl}=0$$

then

$$R_{ijkl,\,p_1p_2\ldots p_n}-a_{p_1p_2\ldots p_n}R_{ijkl}=0.$$

Hence a necessary and sufficient condition that an n-Ricci-recurrent space be n-recurrent is that its Weyl projective tensor satisfies the identity

$$W_{ijkl,\,p_1p_2\ldots p_n}-a_{p_1p_2\ldots p_n}W_{ijkl}=0.$$

7. n-Symmetric Spaces

A Riemannian space of n dimensions whose curvature tensor satisfies the relation

$$R_{hijk,\,m_1m_2\ldots m_n}=0$$

will be called n-symmetric.

For a recurrent space K_n, we have

$$R_{hijk,\,l_1}=\lambda_{l_1}R_{hijk}. \qquad (7.1)$$

Differentiating covariantly with respect to x^{l_2}, we get

$$R_{hijk,\,l_1l_2}=(\lambda_{l_1,\,l_2}+\lambda_{l_1}\lambda_{l_2})R_{hijk}.$$

Hence every K_n is 2-recurrent provided that

$$\lambda_{l_1,\,l_2}+\lambda_{l_1}\lambda_{l_2}\neq 0.$$

Differentiating (7.1) covariantly $(n-1)$ times, we get

$$R_{hijk,\,l_1l_2\ldots l_n}=(\lambda_{l_1,\,l_2\ldots l_n}+\lambda_{l_1,\,l_2\ldots l_{n-1}}\lambda_{l_n}+\lambda_{l_1,\,l_2\ldots l_{n-2}l_n}\lambda_{l_{n-1}}+\ldots+\lambda_{l_1}\lambda_{l_2}\ldots\lambda_{l_n})R_{hijk}.$$

Hence every K_n is n-recurrent if

$$(\lambda_{l_1,\,l_2\ldots l_n}+\lambda_{l_1,\,l_2\ldots l_{n-1}}\lambda_{l_n}+\ldots\lambda_{l_1}\lambda_{l_2}\ldots\lambda_{l_n})\neq 0.$$

If $(\lambda_{l_1,\,l_2\ldots l_n}+\lambda_{l_1,\,l_2\ldots l_{n-1}}\lambda_{l_n}+\lambda_{l_1,\,l_2\ldots l_{n-2}l_n}\lambda_{l_{n-1}}\ldots\lambda_{l_1}\lambda_{l_2}\lambda_{l_n}=0$, *then the space is n-symmetric.*

8. Conformally Symmetric Spaces

If the space is conformally symmetric (Chaki and Gupta 1963), then $C^h_{ijk,l} = 0$ contracting, we get $C^h_{ijk,h} = 0$.
But

$$C^h_{ijk,h} = \frac{n-3}{n-2}\left\{R_{ij,k} - R_{ik,j} + \frac{1}{2(n-1)}(g_{ik}R_{,j} - g_{ij}R_{,k})\right\} = 0.$$

Therefore

$$R_{ij,k} - R_{ik,j} = \frac{1}{2(n-1)}\left[g_{ij}R_{,k} - g_{ik}R_{,j}\right].$$

Differentiating covariantly with respect to x^m, we get

$$R_{ij,km} - R_{ik,jm} = \frac{1}{2(n-1)}\left[g_{ij}R_{,km} - g_{ik}R_{,jm}\right].$$

Since $R_{ij,km} = a_{km}R_{ij}$ for some tensor a_{km}, we get

$$a_{km}R_{ij} - a_{jm}R_{ik} = \frac{1}{2(n-1)}\left[g_{ij}a_{km}R - g_{ik}a_{jm}R\right]$$

or

$$a_{km}\left[R_{ij} - \frac{R}{2(n-1)}g_{ij}\right] - a_{jm}\left[R_{ik} - \frac{R}{2(n-1)}g_{ik}\right] = 0. \qquad .. \quad (8.1)$$

Transvecting by g^{ij}, we get

$$a_{km}\left[\frac{R(n-2)}{2(n-1)}\right] - a_{jm}\left[R^j_k - \frac{R}{2(n-1)}\delta^j_k\right] = 0.$$

Transvecting again by g^{km}, we get

$$a_{jm}R^{jm}\frac{R}{2} - a_{jm}R^{jm} = 0.$$

Therefore

$$(R^{jm} - \tfrac{1}{2}Rg^{jm})a_{jm} = 0$$

is a necessary condition that a 2-Ricci-recurrent space be conformally symmetric.

For an Einstein space,

$$R_{ij} = \frac{R}{n}g_{ij}.$$

Therefore (8.1) gives

$$a_{km}R\left[\frac{n-2}{2(n-1)n}g_{ij}\right] - Ra_{jm}\left[\frac{n-2}{2(n-1)n}g_{ik}\right] = 0$$

or

$$R[a_{km}g_{ij} - a_{jm}g_{ik}] = 0.$$

Transvecting by g^{ij}, we get

$$R(n-1)a_{ij} = 0.$$

But both a_{ij} and R are non-zero. Hence a conformally symmetric 2-Ricci-recurrent space cannot be an Einstein space.

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