THE RADIAL PULSATION OF AN INFINITE CYLINDER WITH VARIABLE DENSITY IN FORCE-FREE MAGNETIC FIELD

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The radial pulsations of an infinitely conducting infinite fluid cylinder with variable density under its own gravitation and in a force-free magnetic field have been discussed. A series solution for the displacement function ψ has been obtained for the fundamental mode for different values of the square of the ratio of the Alfven velocity to the sound velocity at the axis. The results have been compared to those of Chakraborty and Ramamoorthy (1960). Relative variation in magnetic field has also been calculated.

1. Introduction

The problem of radial pulsations of a self-gravitating fluid cylinder under adiabatic conditions with the prevailing magnetic field parallel to the axis of the cylinder was initiated by Chandrasekhar and Fermi (1953). Applying variational principle they obtained an integral formula for the frequency of pulsations. Lyttkens (1954) also studied the same problem with a magnetic field varying as the square root of the pressure inside the cylinder. He established that the displacement function remains the same as in zero magnetic field but the frequency of oscillation is given by

$$\frac{\sigma_n^2-\sigma_{on}^2}{\sigma_{on}^2}=\frac{n^2\cdot\gamma}{n^2\gamma-1}H_0^2,$$

where γ is the ratio of the two specific heats and σ_n and σ_{on} are the frequencies for the nth mode with and without magnetic field. Chopra and Talwar (1955) reconsidered the problem with a more general form of magnetic field given by

$$H^2=H_s^2+(H_0^2-H_s^2)(1-x^2),\ x=\frac{r}{R},$$

but the solution obtained by them did not satisfy the boundary condition that the pressure is continuous at the surface $r=R$.

The problem was then studied by Bhatnagar and Nagpaul (1957), Tandon and Talwar (1958), Raju and Talwar (1961) and Chakraborty and Ramamoorthy (1960). Chakraborty and Ramamoorthy (1960) studied the radial pulsations of an infinitely conducting infinite fluid cylinder in a force-free magnetic field. But they have not made any mention of the external magnetic field. In the case considered by them, the change in the total pressure can be made zero only when no toroidal component of the magnetic field exists outside. In this paper we have discussed the pulsations in a force-free field with

$$(H_0)_r = 0,\ (H_0)_\phi = A_1 J_1(ar),\ (H_0)_z = A_1 J_0(ar), \quad \ldots \quad \ldots \quad (1)$$

with variable density

$$\rho = \rho_c \,.\, (1-x^2), \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

where H_0 is the equilibrium magnetic field and ρ_c is the density at the axis.

2. The Equilibrium State

The equilibrium state for a self-gravitating infinitely conducting infinite cylinder of variable density is represented by the following equations:

$$-\text{grad } p - \rho \text{ grad } V = 0, \quad \ldots \quad \ldots \quad \ldots \quad (3)$$

$$\nabla^2 V = 4\pi G\rho. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (4)$$

As $\vec{H}_0$ is a force-free magnetic field, we have

$$\text{curl } \vec{H}_0 = a\vec{H}_0 = 4\pi\vec{J}, \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (5)$$

$$\text{div } \vec{H}_0 = 0,\ \vec{E} = 0. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (6)$$

In the vacuum outside, the magnetic field should satisfy the equations

$$\text{curl } \vec{H}_0^v = 0,\ \text{div } \vec{H}_0^v = 0. \quad \ldots \quad \ldots \quad \ldots \quad (7)$$

The only possible solution admitted by eqns. (7) is

$$(H_0)_r^v = 0,\ (H_0)_\phi^v = \frac{B}{r},\ (H_0)_z^v = A, \quad \ldots \quad \ldots \quad \ldots \quad (8)$$

where A and B are constants and superscript v refers to the outside vacuum.

In the absence of surface currents, we have

$$B = RH_\phi^s,\ A = H_z^s, \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (9)$$

where s refers to the surface of the cylinder.

3. The Equations of Motions

The equation of the pulsation of an infinitely conducting infinite fluid cylinder is given by (Chandrasekhar and Fermi 1953)

$$\frac{d^2\xi}{dt^2} = -\frac{1}{\rho}\frac{d}{dr}\delta p + \frac{4Gm(r)\xi}{r^2} + \frac{1}{4\pi\rho}\left[\left(\text{curl } \vec{H}\right)\times\vec{H}\right], \quad \ldots \quad \ldots \quad (10)$$

$$\vec{H} = \vec{H}_0 + \vec{h}, \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (11)$$

$$\frac{\partial}{\partial m}(2\pi r\xi) = -\frac{\delta\rho}{\rho^2}, \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (12)$$

$$\vec{h} = \text{curl}\left(\vec{\xi}\times\vec{H}\right) + \left(\vec{\xi}.\nabla\right)\vec{H}, \quad \ldots \quad \ldots \quad \ldots \quad (13)$$

where $\xi = \delta r$ is the Lagrangian displacement.

The density of the fluid cylinder is given by eqn. (2) and the components of equilibrium magnetic field in cylindrical coordinates are given by eqn. (1).

Using eqns. (11) and (5) the last term of eqn. (10) gives only the radial component which can be written as

$$\frac{1}{4\pi\rho}\cdot\left[\left(\text{curl }\vec{H}\right)\times\vec{H}\right]=\frac{A_1^2}{4\pi\rho}\left[J_0(ar)\frac{d}{dr}\left\{\frac{J_0(ar)}{r}\frac{d}{dr}(r\xi)\right\}\right.$$

$$\left.+\frac{J_1(ar)}{r}\frac{d}{dr}\left\{rJ_1(ar)\frac{d\xi}{dr}\right\}-\frac{a\xi}{r}J_0(ar)\,.\,J_1(ar)\right], \quad \ldots \quad \ldots \quad (14)$$

where

$$h_\phi=-A_1J_1(ar)\frac{d\xi}{dr},\ h_z=-\frac{A_1J_0(ar)}{r}\frac{d}{dr}(r\xi), \quad \ldots \quad \ldots \quad (15)$$

and

$$\delta p=-\frac{\gamma p}{r}\frac{d}{dr}(r\xi). \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (16)$$

The pulsation eqn. (9) reduces to

$$\rho\left[\frac{d^2\xi}{dt^2}-\frac{4Gm(r)\xi}{r^2}\right]=-\frac{d}{dr}\left[-\frac{\gamma p}{r}\frac{d}{dr}(r\xi)\right]$$

$$+\frac{A_1^2}{4\pi}\left[J_0(ar)\frac{d}{dr}\left\{\frac{J_0(ar)}{r}\frac{d}{dr}(r\xi)\right\}+\frac{J_1(ar)}{r}\frac{d}{dr}\left\{rJ_1(ar)\frac{d\xi}{dr}\right\}\right.$$

$$\left.-\frac{a\xi}{r}J_0(ar)\,.\,J_1(ar)\right], \quad \ldots \quad \ldots \quad (17)$$

where $m(r)$, the mass interior to r per unit length of the cylinder, is given by

$$m(r)=\pi\rho_c^2R^2x^2(1-\tfrac{1}{2}x^2). \quad \ldots \quad \ldots \quad \ldots \quad (18)$$

Also

$$p=\tfrac{1}{12}\pi G\rho_c^2R^2(1-x^2)^2\,.\,(5-2x^2). \quad \ldots \quad \ldots \quad \ldots \quad (19)$$

The change in the magnetic field in the vacuum is governed by the following equations:

$$\text{div }\delta\vec{H}^v=0;\ \text{curl }\delta\vec{H}^v=0, \quad \ldots \quad \ldots \quad \ldots \quad (20)$$

$$\text{curl }\vec{E}^v=-1/c\frac{\partial}{\partial t}\left(\delta\vec{H}^v\right). \quad \ldots \quad \ldots \quad \ldots \quad (21)$$

The only solution of $\delta\vec{H}^v$ which satisfies the above equations is of the form given in eqn. (8), viz.

$$(\delta H^v)_r=0,\ (\delta H^v)_\phi=\frac{B'}{r},\ (\delta H^v)_z=A'. \quad \ldots \quad \ldots \quad (22)$$

Equation (21) leads to three component equations. Substituting for $\delta\vec{H}^v$ in them and integrating we find that $\vec{E}^v$ tends to infinity with r. Therefore $\delta\vec{H}^v$ must vanish because any finite value of it will give rise to an infinite electric field at infinity.

4. The Differential Equation

With the help of eqns. (18) and (19), the equation of oscillatory motion (17) reduces to

$$\rho_c(1-x^2)\left[\frac{d^2\xi}{dt^2}-4\pi G\rho_c(1-\tfrac{1}{2}x^2)\xi\right]=\frac{d}{dr}\left[\frac{\pi G\rho_c^2R^2\gamma}{12\,.\,r}(1-x^2)^2\,.\,(5-2x^2)\frac{d}{dr}(r\xi)\right]$$

$$+\frac{A_1^2}{4\pi}\left[J_0(ar)\frac{d}{dr}\left\{\frac{J_0(ar)}{r}\,.\,\frac{d}{dr}(r\xi)\right\}\right.$$

$$\left.+\frac{J_1(ar)}{r}\frac{d}{dr}\left\{rJ_1(ar)\frac{d\xi}{dr}\right\}-\frac{a\xi}{r}J_0(\dot{a}r)\,.\,J_1(ar)\right]. \quad \ldots \quad (23)$$

Assuming that all the physical quantities vary with time as $e^{i\sigma t}$ eqn. (23) becomes

$$-(1-x^2)\left[A-\frac{2x^2}{\gamma}\right]\psi=\frac{d}{dx}\left[\frac{(1-x^2)^2(5-2x^2)}{12\,.\,x}\frac{d}{dx}(x\psi)\right]$$

$$+\lambda\left[\{J_0^2(aRx)+J_1^2(aRx)\}\frac{d^2\psi}{dx^2}+\frac{d}{dx}\left\{\frac{\psi\,.\,J_0^2(aRx)}{x}\right\}\right], \quad \ldots \quad (24)$$

with x as in eqn. (12) and $\psi=\frac{\xi}{R}$,

$$A=\frac{1}{\gamma}\left[\frac{\sigma^2}{\pi G\rho_c}+4\right],\ \lambda=5/48\,.\,\frac{A_1^2}{\pi\gamma p_0}, \quad \ldots \quad \ldots \quad \ldots \quad (25)$$

where p_0 is the pressure at the axis.
Substituting $aR.x=y$ in eqn. (24), we have

$$-(b^2-y^2)\left[Ab^2-\frac{2}{\gamma}y^2\right]\psi=\frac{d}{dy}\left[\frac{(b^2-y^2)^2(5b^2-2y^2)}{12y}\frac{d}{dy}(y\psi)\right]$$

$$+\lambda b^6\left[\left\{J_0^2(y)+J_1^2(y)\right\}\frac{d^2\psi}{dy^2}+\frac{d}{dy}\left\{\frac{\psi\,.\,J_0^2(y)}{y}\right\}\right], \quad \ldots \quad (26)$$

where $b=aR$.

Restricting ourselves to the values of J_0^2 and J_1^2 as given below

$$J_0^2(y)=1-\tfrac{1}{2}y^2+\tfrac{3}{32}y^4\,.-\frac{5}{576}y^6, \quad \ldots \quad \ldots \quad \ldots \quad (27)$$

$$J_1^2(y)=\tfrac{1}{4}y^2-\tfrac{1}{16}y^4+\frac{5y^6}{792}:$$

eqn. (26) may be written in the form

$$y^2\psi''\left[b^6\left(\frac{5}{12}+\lambda\right)-b^4y^2\left(1+\frac{\lambda b^2}{4}\right)+b^2y^4\left(\frac{3}{4}+\frac{\lambda b^4}{32}\right)-y^6\left(\frac{1}{6}+\frac{5\lambda b^6}{2304}\right)\right]$$

$$+y\psi'\left[b^6\left(\frac{5}{12}+\lambda\right)-b^4y^2(3+\tfrac{1}{2}\lambda b^2)+b^2y^4\left(\frac{15}{4}+\frac{3\lambda b^4}{32}\right)\right.$$

$$\left.-y^6\cdot\left(\frac{7}{6}+\frac{5\lambda b^6}{576}\right)\right]+\psi\left[-b^6\left(\frac{5}{12}+\lambda\right)+b^4y^2\left(A-1-\frac{\lambda b^2}{2}\right)\right.$$

$$\left.+b^2y^4\cdot\left(\frac{11}{12}+\frac{9}{32}\lambda b^4-A\right)+y^6\left(\tfrac{1}{2}-\frac{25}{576}\lambda b^6\right)\right]=0, \quad \ldots \quad \ldots \quad \ldots \quad (28)$$

where γ has been taken to be 1.5.

In this problem y varies from zero to 1. For the greatest value of y $(=1)$, the values of J_0^2 and J_1^2 are (*see* Watson's Tables 1944)

$$J_0^2 = 0{\cdot}585228, \quad J_1^2 = 0{\cdot}193644,$$

and the values calculated from (27) are

$$J_0^2 = 0{\cdot}58507, \quad J_1^2 = 0{\cdot}193813.$$

The error in J_0 and J_1 from eqn. (27) is not more than 0·078 per cent and 0·088 per cent respectively for $y = 1$ and it will be even less for $y < 1$.

5. The Solution

The solution of the differential eqn. (28) is obtained by the method of series solution as

$$\psi = \sum_{n=0}^{\infty} a_{2n}y^{2n+1}, \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (29)$$

as all the odd coefficients vanish.

The recursion formula to determine different coefficients of the series is given by

$$a_{2n}b^6[2n(2n+2)(\tfrac{5}{12}+\lambda)]-a_{2n-2}b^4\cdot[4n^2+\tfrac{1}{2}\lambda b^2(2n^2-n+1)-A]$$

$$+a_{2n-4}b^2\left[\tfrac{3}{4}(2n+1)\cdot(2n-3)+\tfrac{11}{12}+\frac{\lambda b^4}{8}(n^2-2n+3)\right]$$

$$-a_{2n-6}\left[(n^2-2n-2)+\frac{5\lambda b^6}{1152}-(2n^2-7n+15)\right]=0. \quad \ldots \quad (30)$$

Arranging eqn. (30) in descending powers of n and dividing by $n^2.a_{2n-6}$ and denoting

$$\lim_{n\to\infty}\frac{a_{2n}}{a_{2n-2}}=p, \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (31)$$

and taking the limit as $n \to \infty$, we get

$$f(b^2 \cdot p) = (b^2p)^3 \cdot (\tfrac{5}{3}+4\lambda)-(b^2p)^2(4+\lambda b^2)+(b^2p)\left(3+\frac{\lambda b^4}{8}\right)$$
$$-\left(1+\frac{5\lambda b^6}{576}\right) = 0. \quad \ldots \quad (32)$$

From eqn. (32) we deduce that for every value of λ there corresponds a critical value of b. If b is less than this critical value, the series is divergent. This (λ, b^2) relation is plotted in Fig. 1.

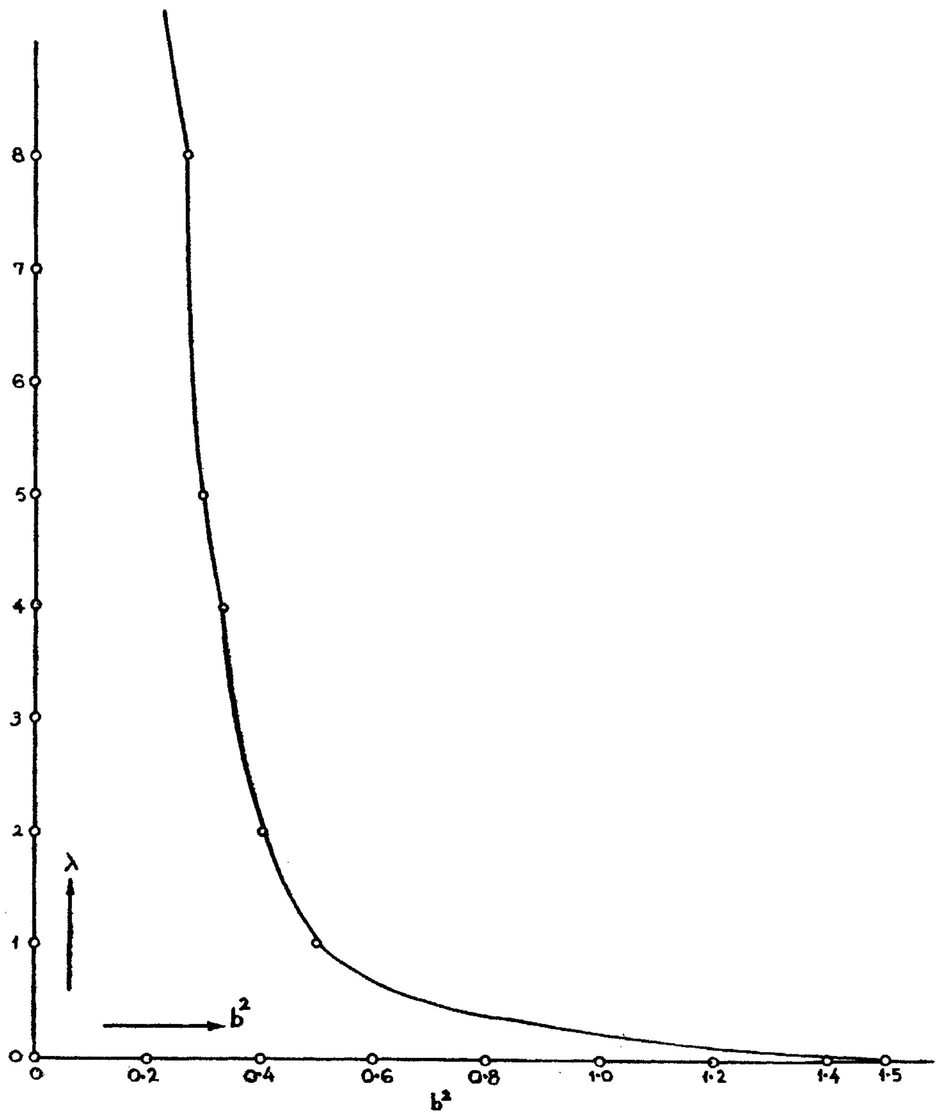

FIG. 1. The curve showing the convergence of series (32).

On the right of the curve the series (29) is always convergent and on the left it is divergent.

6. The Boundary Conditions

As the total pressure across the surface should be continuous, we must have

$$-\left[\left(\frac{\gamma p}{r}+\frac{H_z^2}{4\pi r}\right)\frac{d}{dr}(r\xi)+\frac{H_\phi^2}{4\pi}\frac{d\xi}{dr}\right]_{r=R}=\left[\frac{\xi H_\phi^v}{4\pi}\frac{d}{dr}\cdot H_\phi^v\right]_{r=R}. \quad \text{.. (33)}$$

Equation (33), after certain simplifications, reduces to

$$\left[\frac{d\psi}{dy}+l\right]_{y=aR}=0, \quad \text{.. .. .. .. (34)}$$

$$l=b\frac{L^2-1}{L^2+1},\ b=aR, \quad \text{.. .. .. (35)}$$

$$L^2=\frac{J_0^2(a\,.\,R)}{J_1^2(a\,.\,R)}. \quad \text{.. .. .. .. (36)}$$

If $l=0$, i.e. $L=1$ (which obviously implies the equality of the two components of magnetic field on the boundary), the boundary condition reduces to

$$\frac{d\psi}{dy}=0, \quad \text{.. .. .. (37)}$$

which is the condition used by Chakraborty and Ramamoorthy (1960).

Equation (37) with the help of (1) gives

$$(1+lb)+a_2b^2(3+lb)+a_4b^4(5+lb)+\ldots=0, \quad \text{.. .. (38)}$$

where a_2, a_4, a_6 . . . are the coefficients of series solution (29).

The other condition to be satisfied is

$$\psi=0,\ \text{at}\ x=0. \quad \text{.. .. .. .. (39)}$$

7. Evaluation of Frequency and Numerical Data

For a given value of b the eqn. (38) can be treated as $f(A)=0$ because the coefficients a_2, a_4, etc., contain $A\left[=\frac{1}{\gamma}\left[\frac{\sigma^2}{\pi G\rho_c}+4\right]\right]$. We have solved the eqn. (38) graphically.

Chakraborty and Ramamoorthy (1960) have derived this relation for a uniform density. In Fig. 2 we have plotted the variation of frequency of oscillation with respect to λ (the ratio of the Alfven velocity to the sound velocity) for $b=1$. The Chakraborty and Ramamoorthy curve is also given for comparison.

To give an idea of the convergence of the series for the displacement function ψ, viz.

$$\psi = y - \frac{A-1-\lambda b^2}{8b^2(\frac{5}{12}+\lambda)} \cdot y^3$$
$$+\frac{\frac{(1+\lambda b^2-A)}{8(\frac{5}{12}+\lambda)} \cdot \left(16-A+\frac{7}{2}\lambda b^2\right)-\left(\frac{14}{3}-A+\frac{3\lambda b^4}{8}\right)}{24b^4(\frac{5}{12}+\lambda)} y^5 - \dots, \qquad (40)$$

we give in the following the calculated boundary condition and series for $b = 1$, $\lambda = 4$, 8, 12 and 16.

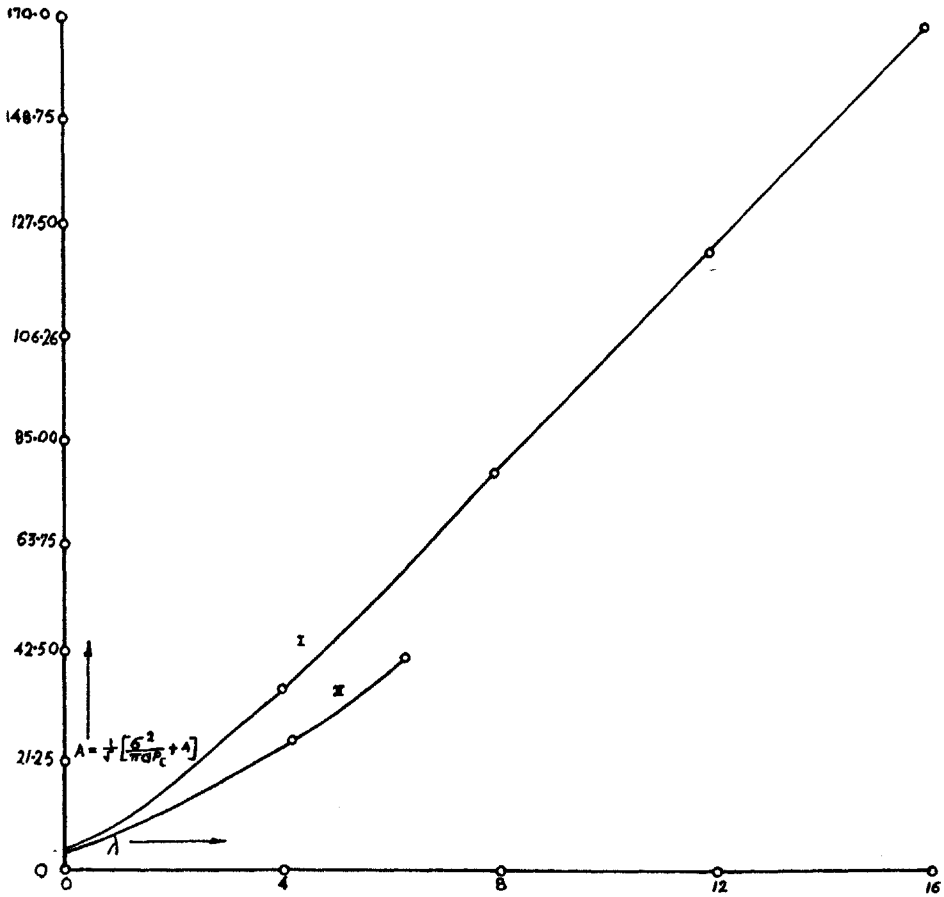

FIG. 2. The curves A as a function of λ when $b = 1$. The curves labelled I and II refer to our case and Chakraborty and Ramamoorthy (1960) case.

The boundary condition for $b = 1$ is

$$1{\cdot}50295+3{\cdot}50295a_2+5{\cdot}50295a_4+7{\cdot}50295a_6+\dots = 0. \qquad (41)$$

The series for $\lambda = 4$, 8, 12 and 16 respectively are

$$\psi = y - 0{\cdot}884207y^3 + 0{\cdot}335797y^5 - 0{\cdot}012638y^7 + 0{\cdot}0226375y^9 + 0{\cdot}0026631y^{11} + 0{\cdot}0024909y^{13} - 0{\cdot}00021424y^{15} - 0{\cdot}00028365y^{17} + 0{\cdot}00002087y^{19} - \ldots \quad (42)$$

$$\psi = y - 1{\cdot}045812y^3 + 0{\cdot}538574y^5 - 0{\cdot}115964y^7 - 0{\cdot}00067007y^9 + 0{\cdot}00703597y^{11} - 0{\cdot}000243706y^{13} - 0{\cdot}00038584y^{15} + 0{\cdot}000051041y^{17} + \ldots \quad (43)$$

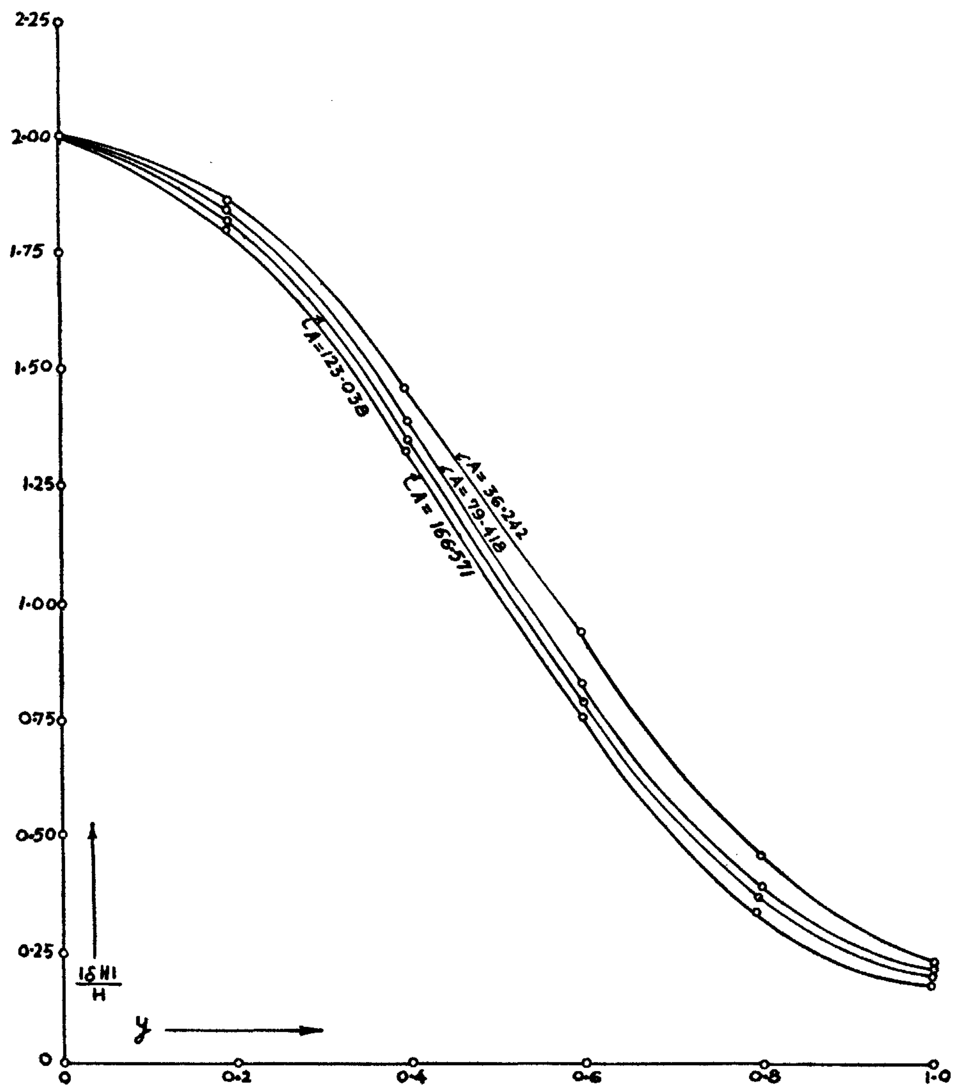

FIG. 3. The curves $\frac{|\delta H|}{H}$ as a function of y for values of A.

$$\psi = y - 1{\cdot}07765y^3 + 0{\cdot}623886y^5 - 0{\cdot}168873y^7 + 0{\cdot}0178014y^9 + 0{\cdot}0048842y^{11}$$
$$-0{\cdot}0010327y^{13} - 0{\cdot}00010246y^{15} + 0{\cdot}0000701y^{17} + \ldots \qquad (44)$$

$$\psi = y - 1{\cdot}138866y^3 + 0{\cdot}669055y^5 - 0{\cdot}199164y^7 + 0{\cdot}030014y^9 + 0{\cdot}00244568y^{11}$$
$$-0{\cdot}001186y^{13} + 0{\cdot}000036067y^{15} + 0{\cdot}00006411y^{17} \ldots \qquad (45)$$

We have also observed that for a fixed λ the frequency decreases with increase in b.

The relative change in the magnetic field $\left|\frac{\delta\vec{H}}{\vec{H}}\right|$ with the help of eqns. (1) and (15) can be written as

$$\left|\frac{\delta\vec{H}}{\vec{H}}\right| = b \, . \left[\frac{(L_1^2+1)\psi'^2 + L_1^2\psi^2/y^2 + 2L_1^2\psi\psi'/y}{L_1^2+1}\right]^{\frac{1}{2}}, \qquad (46)$$

where

$$L_1^2 = \frac{J_0^2(y)}{J_1^2(y)}; \quad \psi' = \frac{d\psi}{dy}. \qquad (47)$$

Fig. 3 represents the variation of $\left|\frac{\delta\vec{H}}{\vec{H}}\right|$ with respect to $y = x = \frac{r}{R}$.

8. Conclusions

It is seen from the curves that the frequency of pulsation increases with λ, the square of the ratio of the Alfven velocity to the sound velocity. This ratio also defines the dominance of the magnetic pressure over the fluid pressure at the axis. From numerical calculation it is also found that the frequency decreases as b increases with same λ. The variation of b corresponds to the rate of change of magnetic field along the radius of the configuration for a fixed radius of the cylinder.

The relative change in the magnetic field $\left|\frac{\delta\vec{H}}{\vec{H}}\right|$ is maximum at the axis and tends to a finite value at the boundary $r = R$ of the cylinder.

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