TRANSIENT MOTION OF A LINE LOAD ON THE SURFACE OF A TRANSVERSELY ISOTROPIC HALF-SPACE

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Transient waves, generated in a homogeneous transversely isotropic half-space by a line source of an impulsive pressure moving uniformly on the surface of the half-space, have been studied with the help of Laplace transform technique. Closed form solution of the problem has been obtained by using Cagniard's method of inversion of the Laplace transform.

INTRODUCTION

For a few decades, the effect of moving blast waves on the surface of the earth has drawn the attention of many investigators. The problem of distribution of stress produced by a pulse of pressure moving uniformly on the surface of an isotropic elastic half-space has been studied by Sneddon (1952) and later by Cole and Huth (1958), when the motion is steady. Chakraborty (1958) has studied the problem when the medium is transversely isotropic. Recently Dang Dinh Ang (1960) has studied the transient effect of a line load moving uniformly on the surface of an isotropic elastic half-space. The present paper proposes to study the transient effect of a line source of pressure pulse moving uniformly on the surface of an elastic half-space, when the medium is transversely isotropic. In this connection, particular reference may be made to Abubakar's (1961) paper which studies the disturbance produced by a buried line source in a semi-infinite transversely isotropic elastic medium.

Formulation of the problem :

A rectangular Cartesian system of coordinates $O(x, y, z)$ is introduced in the semi-infinite elastic medium so that the axis of symmetry for elastic properties coincides with the z-axis and the semi-infinite medium occupies the space $z \geqslant 0$. We assume that the axis of the moving line load is in the direction of the y-axis and that its initial position coincides with the y-axis. On the assumption that the strength of the load is constant throughout, the problem reduces to one of two dimensions in (x, z).

The strain energy function W in a transversely isotropic medium involving the strain components in the x-z plane only is

$$W = \tfrac{1}{2}c_{11}e_{xx}^2 + \tfrac{1}{2}c_{33}e_{zz}^2 + c_{13}e_{xx}e_{zz} + \tfrac{1}{2}c_{44}e_{zx}^2. \quad \quad (1)$$

Since the strain energy function is positive definite, we have

$$c_{11} > 0,\ c_{33} > 0,\ c_{44} > 0,\ c_{11}c_{33}-c_{13}^2 > 0. \qquad (2)$$

Stress components in terms of displacement components (u, o, w) are given by

$$\left.\begin{aligned} \tau_{xx} &= c_{11}\frac{\partial u}{\partial x}+c_{13}\frac{\partial w}{\partial z} \\ \tau_{zx} &= c_{44}\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right) \\ \tau_{zz} &= c_{33}\frac{\partial w}{\partial z}+c_{13}\frac{\partial u}{\partial x} \end{aligned}\right\}. \qquad (3)$$

Equations of motion in two dimensions in the absence of body forces are

$$\left.\begin{aligned} c_{11}\frac{\partial^2 u}{\partial x^2}+(c_{13}+c_{44})\frac{\partial^2 w}{\partial z\partial x}+c_{44}\frac{\partial^2 u}{\partial z^2} &= \rho\frac{\partial^2 u}{\partial t^2} \\ (c_{13}+c_{44})\frac{\partial^2 u}{\partial z\partial x}+c_{44}\frac{\partial^2 w}{\partial x^2}+c_{33}\frac{\partial^2 w}{\partial z^2} &= \rho\frac{\partial^2 w}{\partial t^2} \end{aligned}\right\}. \qquad (4)$$

Initial and boundary condition:

We assume that a line source of pressure pulse varying as δ-function is moving with uniform velocity v^{-1} in the direction of x-axis on the surface of the medium $z = 0$, the line source initially coinciding with the y-axis. Thus we have on $z = 0$ for all $t \geqslant 0$

$$\left.\begin{aligned} \tau_{zz} &= -P\delta\left(x-\frac{1}{v}t\right), &&\text{when } x > 0 \\ &= 0, &&\text{when } x < 0 \end{aligned}\right\} \qquad (5)$$

$$\tau_{zx} = 0, \qquad \text{for all } x, \qquad (6)$$

where δ denotes Dirac's delta-function.

To the conditions (5) and (6) we must add the condition that the waves may be outgoing.

Now applying Laplace transform

$$\bar{u} = \int_0^\infty u \exp(-pt)\,dt,\ \bar{w} = \int_0^\infty w \exp(-pt)\,dt, \qquad (7)$$

eqn. (4) gives

$$\left.\begin{aligned} c_{11}\frac{\partial^2 \bar{u}}{\partial x^2}+(c_{13}+c_{44})\frac{\partial^2 \bar{w}}{\partial z\partial x}+c_{44}\frac{\partial^2 \bar{u}}{\partial z^2}-\rho p^2\bar{u} &= 0 \\ (c_{13}+c_{44})\frac{\partial^2 \bar{u}}{\partial z\partial x}+c_{44}\frac{\partial^2 \bar{w}}{\partial x^2}+c_{33}\frac{\partial^2 \bar{w}}{\partial z^2}-\rho p^2\bar{w} &= 0 \end{aligned}\right\}. \qquad (8)$$

Let us seek the solution of eqn. (8) in the form

$$\left.\begin{aligned} \bar{u} &= A \exp(-pqz+ikpx) \\ \bar{w} &= B \exp(-pqz+ikpx) \end{aligned}\right\}. \qquad (9)$$

Substituting the values of $\bar{u}$ and $\bar{w}$ in eqn. (8), we get

$$\left.\begin{aligned} A(c_{44}q^2-c_{11}k^2-\rho)-Bikq(c_{13}+c_{44}) &= 0 \\ -Aikq(c_{13}+c_{44})+B(c_{33}q^2-c_{44}k^2-\rho) &= 0 \end{aligned}\right\} . \qquad (10)$$

A non-zero solution for A and B is possible if

$$c_{33}c_{44}q^4-[c_{33}(c_{11}k^2+\rho)+c_{44}(c_{44}k^2+\rho)-(c_{13}+c_{44})^2k^2]q^2$$
$$+(c_{11}k^2+\rho)(c_{44}k^2+\rho) = 0. \qquad (11)$$

Let q_1^2 and q_2^2 be the roots of eqn. (11). Then q_1^2 and q_2^2 are given by

$$\left.\begin{aligned} 2c_{33}c_{44}q_1^2 \\ 2c_{33}c_{44}q_2^2 \end{aligned}\right\} = Gk^2+\rho H \pm \sqrt{[(Gk^2+\rho H)^2-4c_{33}c_{44}(c_{11}k^2+\rho)(c_{44}k^2+\rho)]}, \qquad (12)$$

where

$$\left.\begin{aligned} G &= c_{11}c_{33}+c_{44}^2-(c_{13}+c_{44})^2 \\ H &= c_{33}+c_{44} \end{aligned}\right\} . \qquad (13)$$

Now solutions of eqn. (8) can be written formally as

$$\bar{u} = \int_{-\infty}^{\infty} [A_1(k) \exp(-pq_1z+ipkx)+A_2(k) \exp(-pq_2z+ipkx)]\, dk \qquad (14)$$

$$\bar{w} = -i\int_{-\infty}^{\infty} \left[\frac{c_{44}q_1^2-c_{11}k^2-\rho}{kq_1(c_{13}+c_{44})} A_1(k) \exp(-pq_1z+ipkx)\right.$$
$$\left.+\frac{c_{44}q_2^2-c_{11}k^2-\rho}{kq_2(c_{13}+c_{44})} A_2(k) \exp(-pq_2z+ipkx)\right] dk, \qquad (15)$$

where the path of integration is along the real axis of the complex k-plane.

If the solutions given by (14) and (15) are to satisfy Sommerfield's radiation condition in space $z \geqslant 0$, we must take Re q_1 and Re $q_2 \geqslant 0$.

Roots of equation (11):

If the expression under the radical sign of eqn. (12) be a perfect square, that is if

$$(G^2-4c_{11}c_{33}c_{44}^2)(H^2-4c_{33}c_{44})-[GH-2c_{33}c_{44}(c_{11}+c_{44})]^2 = 0, \qquad (16)$$

the roots of eqn. (11) are of the form

$$q_{1,2}^2 = \alpha k^2+\rho\beta.$$

This case is similar to the case of isotropic medium of Dang Dinh Ang (1960). Here we shall be concerned with the general case in which the relation (16) does not hold.

Determination of $A_1(k)$ and $A_2(k)$:

Applying Laplace transform to eqn. (3) and substituting the values of $\bar{u}$ and $\bar{w}$, we get

$$\bar{\tau}_{zz} = \frac{ip}{c_{13}+c_{44}}\left[\int_{-\infty}^{\infty}\{m_1(k)A_1(k)\exp(-pq_1z)+m_2(k)A_2(k)\exp(-pq_2z)\}\frac{\exp(ipkx)}{k}\,dk\right] \quad \ldots (17)$$

$$\bar{\tau}_{zx} = -\frac{c_{44}p}{c_{13}+c_{44}}\left[\int_{-\infty}^{\infty}\{n_1(k)A_1(k)\exp(-pq_1z)+n_2(k)A_2(k)\exp(-pq_2z)\}\exp(ipkx)\,dk\right] \quad \ldots (18)$$

where

$$\left.\begin{aligned} m_1(k) &= c_{33}c_{44}q_1^2+(c_{13}^2+c_{13}c_{44}-c_{11}c_{33})k^2-c_{33}\rho \\ m_2(k) &= c_{33}c_{44}q_2^2+(c_{13}^2+c_{13}c_{44}-c_{11}c_{33})k^2-c_{33}\rho \\ n_1(k) &= \frac{1}{q_1}(c_{13}q_1^2+c_{11}k^2+\rho) \\ n_2(k) &= \frac{1}{q_2}(c_{13}q_2^2+c_{11}k^2+\rho) \end{aligned}\right\}. \quad \ldots \quad \ldots (19)$$

Laplace transformation of the boundary condition (5) yields

$$\begin{aligned} [\tau_{zz}]_{z=0} &= -Pv\exp(-pvx), \quad \text{when } x>0 \\ &= 0, \quad \text{when } x<0 \quad \ldots \quad \ldots (20) \end{aligned}$$

$$[\bar{\tau}_{zx}]_{z=0} = 0. \quad \ldots \quad \ldots (21)$$

Comparing eqn. (18) with eqn. (21), we get

$$n_1(k)A_1(k)+n_2(k)A_2(k) = 0. \quad \ldots \quad \ldots (22)$$

In accordance with eqn. (22), let

$$\left.\begin{aligned} n_1(k)A_1(k) &= R(k) \\ n_2(k)A_2(k) &= -R(k) \end{aligned}\right\}. \quad \ldots \quad \ldots (23)$$

Substituting the values of $A_1(k)$ and $A_2(k)$ in eqn. (17), we get

$$\bar{\tau}_{zz} = \frac{ip}{c_{13}+c_{44}}\left[\int_{-\infty}^{\infty}\frac{R(k)}{k}\left(\frac{m_1(k)}{n_1(k)}\exp(-pq_1z)-\frac{m_2(k)}{n_2(k)}\exp(-pq_2z)\right)\exp(ipkx)\,dk\right]. \quad \ldots (24)$$

Comparing eqn. (24) with eqn. (20), we have

$$\begin{aligned} \frac{ip}{c_{13}+c_{44}}\int_{-\infty}^{\infty}\frac{R(k)}{k}\,\frac{m_1(k)n_2(k)-m_2(k)n_1(k)}{n_1(k)n_2(k)}\exp(ipkx)\,dk &= -Pv\exp(-pvx) \text{ when } x>0 \\ &= 0 \text{ when } x<0. \quad \ldots (25) \end{aligned}$$

By Fourier inversion formula, we get

$$\frac{R(k)}{k}\,\frac{m_1(k)n_2(k)-m_2(k)n_1(k)}{n_1(k)n_2(k)} = \frac{(c_{13}+c_{44})vP}{2\pi}\cdot\frac{1}{p(k-iv)}. \quad \ldots (26)$$

Hence eqns. (23) and (26) yield

$$A_1(k) = \frac{(c_{13}+c_{44})vP}{2\pi}\cdot\frac{n_2(k)}{m_1(k)n_2(k)-m_2(k)n_1(k)}\cdot\frac{k}{p(k-iv)}$$

$$A_2(k) = -\frac{(c_{13}+c_{44})vP}{2\pi}\cdot\frac{n_1(k)}{m_1(k)n_2(k)-m_2(k)n_1(k)}\cdot\frac{k}{p(k-iv)}\cdot \quad .. \quad (27)$$

Substituting the values of $A_1(k)$ and $A_2(k)$ in eqns. (14) and (15) we get

$$\bar{u} = \frac{(c_{13}+c_{44})vP}{2\pi p}\left[\int_{-\infty}^{\infty}\frac{n_2(k)}{\Delta(k)}\,\frac{k}{k-iv}\exp(-pq_1z+ipkx)\,dk - \int_{-\infty}^{\infty}\frac{n_1(k)}{\Delta(k)}\,\frac{k}{k-iv}\exp(-pq_2z+ipkx)\,dk\right]$$

$$\bar{w} = -\frac{ivP}{2\pi p}\left[\int_{-\infty}^{\infty}\frac{c_{44}q_1^2-c_{11}k^2-\rho}{q_1}\,\frac{n_2(k)}{\Delta(k)}\,\frac{1}{k-iv}\exp(-pq_1z+ipkx)\,dk - \int_{-\infty}^{\infty}\frac{c_{44}q_2^2-c_{11}k^2-\rho}{q_2}\,\frac{n_1(k)}{\Delta(k)}\,\frac{1}{k-iv}\exp(-pq_2z+ipkx)\,dk\right], \quad .. \quad (28)$$

where

$$\Delta(k) = m_1(k)n_2(k)-m_2(k)n_1(k). \quad .. \quad .. \quad .. \quad (29)$$

Now substituting the values of $\bar{u}$ and $\bar{w}$ in eqn. (3) we get

$$\bar{\tau}_{zz} = \frac{ivP}{2\pi}\left[\int_{-\infty}^{\infty}\frac{n_2(k)m_1(k)}{\Delta(k)}\,\frac{1}{k-iv}\exp(-pq_1z+ipkx)\,dk - \int_{-\infty}^{\infty}\frac{n_1(k)m_2(k)}{\Delta(k)}\,\frac{1}{k-iv}\exp(-pq_2z+ipkx)\,dk\right] \quad .. \quad .. \quad .. \quad (30)$$

$$\bar{\tau}_{zx} = -\frac{c_{44}vP}{2\pi}\left[\int_{-\infty}^{\infty}\frac{n_2(k)n_1(k)}{\Delta(k)}\,\frac{k}{k-iv}\exp(-pq_1z+ipkx)\,dk - \int_{-\infty}^{\infty}\frac{n_1(k)n_2(k)}{\Delta(k)}\,\frac{k}{k-iv}\exp(-pq_2z+ipkx)\,dk\right] \quad .. \quad .. \quad .. \quad (31)$$

$$\bar{\tau}_{xx} = \frac{ivP}{2\pi}\left[\int_{-\infty}^{\infty}(c_{13}c_{44}q_1^2+c_{11}c_{44}k^2-c_{13}\rho)\frac{n_2(k)}{\Delta(k)}\,\frac{1}{k-iv}\exp(-pq_1z+ipkx)\,dk - \int_{-\infty}^{\infty}(c_{13}c_{44}q_2^2+c_{11}c_{44}k^2-c_{13}\rho)\frac{n_1(k)}{\Delta(k)}\,\frac{1}{k-iv}\exp(-pq_2z+ipkx)\,dk\right]. \quad (32)$$

Singularities of the integrands:

The integrands are four-valued functions. Since we put the restrictions Re $q_1 \geqslant 0$ and Re $q_2 \geqslant 0$, we can make the integrands single-valued on Riemann's surface by introducing suitable cuts joining branch points of the integrands.

The integrands have branch points at points given by

$$\text{(i)}\quad q_{1,2} = 0 \qquad \ldots \qquad (33)$$

$$\text{(ii)}\quad (Gk^2+\rho H)^2-4c_{33}c_{44}(c_{11}k^2+\rho)(c_{44}k^2+\rho) = 0. \qquad \ldots \qquad (34)$$

Case (i). The equation $q_{1,2} = 0$ yields

$$(c_{11}k^2+\rho)(c_{44}k^2+\rho) = 0 \qquad \ldots \qquad (35)$$

which gives the branch points as

$$k = \pm i\sqrt{\frac{\rho}{c_{11}}},\quad \pm i\sqrt{\frac{\rho}{c_{44}}}. \qquad \ldots \qquad (36)$$

Case (ii). Equation (34) can be written as

$$k^4\left(G^2-4c_{33}c_{11}c_{44}^2\right)+2\rho[GH-2c_{33}c_{44}(c_{11}+c_{44})]k^2$$
$$+\rho^2(H^2-4c_{33}c_{44}) = 0. \qquad \ldots \qquad (37)$$

Now we make certain assumptions about the elastic constants of the medium. We assume

$$G^2-4c_{11}c_{33}c_{44}^2 > 0. \qquad \ldots \qquad (38)$$

The value of the expression (38) is zero in isotropic media.

We further assume that

$$[GH-2c_{33}c_{44}(c_{11}+c_{44})]^2-\left(G^2-4c_{33}c_{11}c_{44}^2\right)(H^2-4c_{33}c_{44}) > 0 \qquad \ldots \qquad (39)$$

and

$$GH-2c_{33}c_{44}(c_{11}+c_{44}) > 0, \qquad \ldots \qquad (40)$$

so that the roots of eqn. (37) are purely imaginary. These assumptions are justified because, if eqn. (37) has complex or real roots, the medium admits of harmonic waves whose amplitudes vary as $(A+Bz)\exp(\alpha x+\beta z)$ or as $A+Bz$. Let the branch points arising from case (ii) be denoted by $\pm ik_1$, $\pm ik_2$. Hence the roots of eqn. (11) can be written in the form

$$2c_{33}c_{44}q_{1,2}^2 = Gk^2+\rho H\pm\sqrt{\left(G^2-4c_{11}c_{33}c_{44}^2\right)}\sqrt{\left[\left(k^2+k_1^2\right)\left(k^2+k_2^2\right)\right]}. \qquad \ldots \qquad (41)$$

We also assume that

$$c_{11} > c_{33} > c_{44}. \qquad \ldots \qquad (42)$$

It can be easily verified that

$$G < c_{11}H. \qquad \ldots \qquad (43)$$

Now if q_1 is to be zero at the points $\pm i\sqrt{\frac{\rho}{c_{44}}}$, the value of the expression $Gk^2+\rho H$ must be negative at these points. Thus we have

$$c_{44}H < G. \qquad \ldots \qquad (44)$$

Both the branch points ik_1 and ik_2 are above or below the point $i\sqrt{\frac{\rho}{c_{44}}}$ in the complex k-plane according as $c_{11}c_{44}H^2$ is greater or less than G^2. Now in conformity with the isotropic case, we assume that

$$c_{11}c_{44}H^2 > G^2 \qquad \ldots \qquad (45)$$

so that both the branch points ik_1 and ik_2 are above the point $i\sqrt{\frac{\rho}{c_{44}}}$ in the complex k-plane.

Now we consider the poles of the integrands. These are given by $k-iv = 0$ and $\Delta(k) = 0$. The expression for $\Delta(k)$ can be written as

$$\Delta(k) = \frac{(q_1-q_2)(c_{33}c_{44})^{\frac{1}{2}}(c_{13}+c_{44})}{(c_{44}k^2+\rho)^{\frac{1}{2}}}$$
$$\times\left[\rho(c_{11}k^2+\rho)^{\frac{1}{2}}+\frac{(c_{44}k^2+\rho)^{\frac{1}{2}}}{(c_{33}c_{44})^{\frac{1}{2}}}\left\{(c_{11}c_{33}-c_{13}^2)k^2+\rho c_{33}\right\}\right].$$

The factor $q_1-q_2 = 0$ does not give the poles of the integrands since the equation $q_1^2-q_2^2 = 0$ yields $k = \pm ik_1, \pm ik_2$ which are branch points of the integrands.

The other factor

$$\rho(c_{11}k^2+\rho)^{\frac{1}{2}}+\frac{(c_{44}k^2+\rho)^{\frac{1}{2}}}{(c_{33}c_{44})^{\frac{1}{2}}}\left\{(c_{11}c_{33}-c_{13}^2)k^2+\rho c_{33}\right\} = 0 \quad \ldots \quad (46)$$

yields

$$c_{33}c_{44}\rho^2(c_{11}k^2+\rho)-(c_{44}k^2+\rho)\left\{(c_{11}c_{33}-c_{13}^2)k^2+\rho c_{33}\right\}^2 = 0. \quad \ldots \quad (47)$$

This is a third degree equation in k^2. If we write $k^2 = -\frac{1}{c^2}$, eqn. (47) reduces to

$$f(c^2) = c_{33}c_{44}\rho^2(\rho c^2-c_{11})c^4-(\rho c^2-c_{44})\left\{\rho c_{33}c^2-(c_{11}c_{33}-c_{13}^2)\right\}^2 = 0. \quad (48)$$

Equation (48) corresponds to the equation giving velocity for Rayleigh waves in isotropic case. To find the position of the roots, we note that

$$\left.\begin{aligned} &f(-\infty) = +\infty \\ &f(0) > 0 \\ &f\left(\frac{c_{11}}{\rho}\right) < 0 \\ &f\left(\frac{c_{44}}{\rho}\right) < 0 \\ &f(\infty) < 0 \end{aligned}\right\} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (49)$$

This shows that at least one root of eqn. (48) lies between 0 and $\frac{c_{44}}{\rho}$. Now in analogy with the isotropic case we assume that only one root lies between 0 and $\frac{c_{44}}{\rho}$ and the other two lie between $\frac{c_{11}}{\rho}$ and ∞. Let $c^2 = v_R^2$ be the root that lies between 0 and $\frac{c_{44}}{\rho}$. Since eqn. (47) has been derived from eqn. (46) by squaring, all the roots of eqn. (47) do not satisfy eqn. (46). If we put $k^2 = -\frac{1}{c^2}$ in eqn. (46) we get

$$c^2\left(c^2-\frac{c_{11}}{\rho}\right)^{\frac{1}{2}}+\sqrt{\frac{c_{33}}{c_{44}}}\left(c^2-\frac{c_{44}}{\rho}\right)^{\frac{1}{2}}\left(c^2-\frac{c_{11}c_{33}-c_{13}^2}{\rho c_{33}}\right) = 0. \quad \ldots \quad (50)$$

Now a value of c^2 greater than both $\frac{c_{11}}{\rho}$ and $\frac{c_{44}}{\rho}$ satisfies eqn. (50) on the appropriate Riemann's surface if

$$c^2 < \frac{c_{11}c_{33}-c_{13}^2}{\rho c_{33}},$$

that is, if

$$c^2 < \frac{c_{11}}{\rho}.$$

Hence the roots of eqn. (48) which are greater than $\frac{c_{11}}{\rho}$ do not satisfy eqn. (46). Also eqn. (50) cannot have a root in the interval $\frac{c_{44}}{\rho} < c^2 < \frac{c_{11}}{\rho}$. Hence the singularities of the integrands are

(1) branch points at $k = \pm i\sqrt{\frac{\rho}{c_{11}}},\ \pm i\sqrt{\frac{\rho}{c_{44}}},\ \pm ik_1,\ \pm ik_2$;

(2) poles at $k = iv,\ \pm\frac{i}{v_R}$.

Now the integrals given in eqns. (30), (31), (32) can be transformed into

$$\bar{\tau}_{zz} = -\frac{vP}{\pi}\,\mathrm{Im}\left[\int_0^\infty \frac{n_2(k)m_1(k)}{\Delta(k)}\,\frac{1}{k-iv}\exp(-pq_1z+ipkx)\,dk - \int_0^\infty \frac{m_2(k)n_1(k)}{\Delta(k)}\,\frac{1}{k-iv}\exp(-pq_2z+ipkx)\,dk\right], \quad (51)$$

$$\bar{\tau}_{zx} = -\frac{c_{44}vP}{\pi}\,\mathrm{Re}\left[\int_0^\infty \frac{n_1(k)n_2(k)}{\Delta(k)}\,\frac{k}{k-iv}\exp(-pq_1z+ipkx)\,dk - \int_0^\infty \frac{n_1(k)n_2(k)}{\Delta(k)}\,\frac{k}{k-iv}\exp(-pq_2z+ipkx)\,dk\right], \quad (52)$$

$$\bar{\tau}_{xx} = -\frac{vP}{\pi}\,\mathrm{Im}\left[\int_0^\infty (c_{13}c_{44}q_1^2+c_{11}c_{44}k^2-c_{13}\rho)\,\frac{n_2(k)}{\Delta(k)}\cdot\frac{1}{k-iv}\exp(-pq_1z+ipkx)\,dk - \int_0^\infty (c_{13}c_{44}q_2^2+c_{11}c_{44}k^2-c_{13}\rho)\,\frac{n_1(k)}{\Delta(k)}\cdot\frac{1}{k-iv}\exp(-pq_2z+ipkx)\,dx\right]. \quad (53)$$

A typical integral of the above set is

$$\bar{F} = \mathrm{Im}\left[\int_0^\infty F_1(k)\exp(-pq_1z+ipkx)\,dk + \int_0^\infty F_2(k)\exp(-pq_2z+ipkx)\,dk\right]. \quad (54)$$

Inverse Transform :

To find the inverse transform of the expression $\bar{F}$, we apply Cagniard's method. The method consists in transforming the integral to a form recognizable as a Laplace transform of some function of time. We introduce a new variable

$$t = q_1z - ikx \quad (55)$$

in the first integrand of eqn. (54). Next we deform the path of integration from real k-axis to the curve on which t is real positive.

Now we examine the values of the function q_1 given by eqn. (12) on the imaginary k-axis. At the point $k = 0$, $q_1^2 = \frac{\rho}{c_{44}} > 0$. The expression $Gk^2+\rho H$ in eqn. (12), monotonically decreases as k moves from origin along the imaginary k-axis, it being positive in the segment $\left(-i\sqrt{\frac{\rho H}{G}},\, i\sqrt{\frac{\rho H}{G}}\right)$ and negative outside this range. Now since $\frac{\rho}{c_{11}} < \frac{\rho H}{G} < \frac{\rho}{c_{44}}$, the expression is greater than zero at $\pm i\sqrt{\rho/c_{11}}$ and is less than zero at $\pm i\sqrt{\rho/c_{44}}$. Also the expression $(G^2-4c_{11}c_{33}c_{44}^2)^{\frac{1}{2}}(k^2+k_1^2)^{\frac{1}{2}}(k^2+k_2^2)^{\frac{1}{2}}$ monotonically decreases in the segment $(-i\sqrt{\rho/c_{44}},\, i\sqrt{\rho/c_{44}})$ of the imaginary axis as $|k|$ increases. Hence as $|k|$ increases q_1^2 gradually decreases from the value ρ/c_{44} to zero at $\pm i\sqrt{\rho/c_{44}}$ and then becomes negative. Consequently t is real on the portion of the imaginary k-axis from $-i\sqrt{\rho/c_{44}}$ to $i\sqrt{\rho/c_{44}}$ and becomes complex outside this range. Now when $x > 0$, as k moves along the positive imaginary axis, from the point $k = 0$, t increases from $\sqrt{\frac{\rho}{c_{44}}}\,z$ to a maximum at a point between 0 and $i\sqrt{\frac{\rho}{c_{44}}}$, since $q_1(k)$ gradually decreases while $-ikx$ monotonically increases. Similarly when $x < 0$, t increases as k moves along the negative imaginary axis, having a maximum at a point between $(0, -i\sqrt{\rho/c_{44}})$.

Now to find the position of the point on the imaginary k-axis where t is maximum, we put $k = i\eta$. Differentiating the expression for t in eqn. (55), we get

$$\phi(\eta) = \frac{dt}{d\eta} = -\frac{\eta[Gq_1^2+c_{11}(c_{44}\eta^2-\rho)+c_{44}(c_{11}\eta^2-\rho)]}{q_1[2c_{33}c_{44}q_1^2-\rho H+G\eta^2]}z+x. \quad .. \quad (56)$$

When $x > 0$, we have

$$\phi(0) > 0$$

$$\phi(\sqrt{\rho/c_{44}}) < 0, \text{ since } q_1 \to 0+0 \text{ as } \eta \to \sqrt{\rho/c_{44}}-0.$$

$$\phi(\sqrt{\rho/c_{11}}) = -\frac{G(c_{11}H-G)-c_{11}c_{33}c_{44}(c_{11}-c_{44})}{\sqrt{c_{33}c_{44}}(c_{11}H-G)^{\frac{3}{2}}}z+x.$$

Writing $f(x, z) = -\frac{G(c^{\mathrm{I}}{}_1H-G)-c_{11}c_{33}c_{44}(c_{11}-c_{44})}{\sqrt{c_{33}c_{44}}(c_{11}H-G)^{\frac{3}{2}}}z+x, \quad .. \quad .. \quad .. \quad (57)$

the value of η corresponding to the maximum of t lies in the intervals $\sqrt{\rho/c_{11}} < \eta < \sqrt{\rho/c_{44}}$ or $0 < \eta < \sqrt{\rho/c_{11}}$ according as $f(x, z) \gtrless 0$. Let the variable t attain its maximum value t' at the point $k = i\eta'$. Therefore t' is given by

$$t' = q'z+\eta' x, \quad .. \quad .. \quad ... \quad .. \quad (58)$$

where q' denotes the value of q_1 at the point $k = i\eta'$. The determination of t as a function of x and z is very cumbersome as it involves the solution of a twelve-degree equation in t.

By a similar method, it can be proved that when x is less than zero, the value of η corresponding to the maximum value of t lies in the intervals $-\sqrt{\rho/c_{44}} < \eta < -\sqrt{\rho/c_{11}}$ or $-\sqrt{\rho/c_{11}} < \eta < 0$ according as

$$-\frac{G(c_{11}H-G)-c_{11}c_{33}c_{44}(c_{11}-c_{44})}{c_{33}^{\frac{1}{2}}c_{44}^{\frac{1}{2}}(c_{11}H-G)^{\frac{3}{2}}}z+|x| \gtrless 0.$$

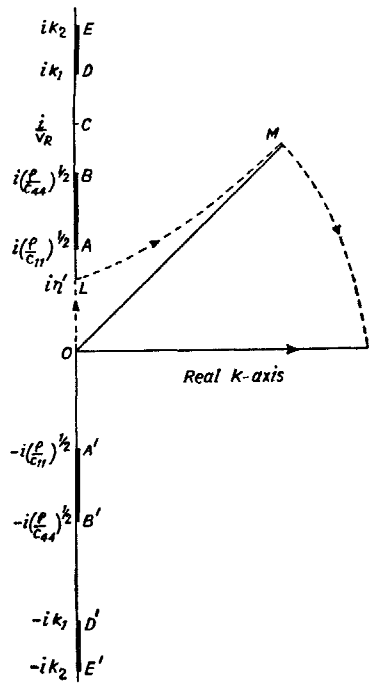

FIG. 1. Path of the integration of the function $F_1(k)$ in the complex k-plane in the case $x > 0$ and $f(x, z) < 0$.

To examine the case when $t \to \infty$ through real values, we observe that when k is large,

$$t \approx \left[\frac{G+\left(G^2-4c_{11}c_{33}c_{44}^2\right)^{\frac{1}{2}}}{2c_{33}c_{44}}\right]^{\frac{1}{2}} . kz-ikx. \qquad (59)$$

If we write $k = \xi+i\eta$, t is real on the line

$$\eta\left[\frac{G+\left(G^2-4c_{11}c_{33}c_{44}^2\right)^{\frac{1}{2}}}{2c_{33}c_{44}}\right]z-\xi x = 0. \qquad (60)$$

Thus when x is greater than zero, the path on which t is real positive is a curve in the first quadrant of the complex k-plane having an asymptote making an angle $\tan^{-1}\left[\frac{x}{z}\left\{\frac{2c_{33}c_{44}}{G+(G^2-4c_{11}c_{33}c_{44}^2)^{\frac{1}{2}}}\right\}^{\frac{1}{2}}\right]$ with the real k-axis. Similarly, when $x < 0$, the path for real t is confined to the fourth quadrant of the complex k-plane having an asymptote making an angle $\tan^{-1}\left[\frac{x}{z}\left\{\frac{2c_{33}c_{44}}{G+(G^2-4c_{11}c_{33}c_{44}^2)^{\frac{1}{2}}}\right\}^{\frac{1}{2}}\right]$ with the real k-axis.

The paths are shown in Figs. 1 and 2.

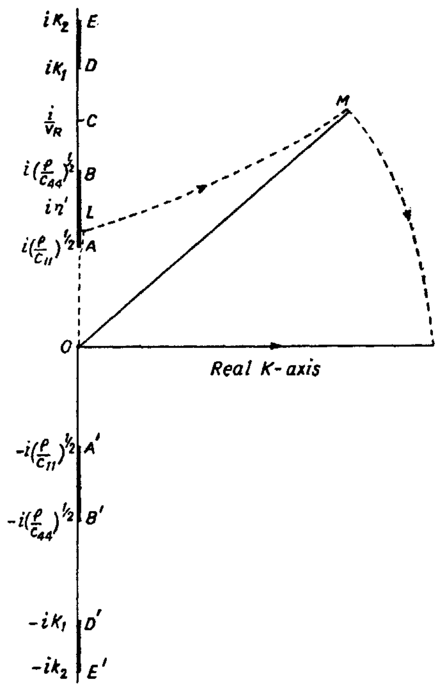

FIG. 2. Path of the integration of the function $F_1(k)$ in the complex k-plane in the case $x > 0$ and $f(x, z) > 0$.

Now we consider the second integral of eqn. (54). In this case we put

$$t = q_2 z - ikx. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (61)$$

At the point $k = 0$, $q_2 = \sqrt{\rho/c_{33}}$ and hence $t = \sqrt{\frac{\rho}{c_{33}}}\,z$. As k moves along the imaginary axis from the origin (either in the positive or negative direction), q_2 decreases from the value $\sqrt{\rho/c_{33}}$ to zero at the point $\pm i\sqrt{\rho/c_{11}}$ and then becomes imaginary. Hence t is real on the portion of imaginary axis between

$-i\sqrt{\rho/c_{11}}$ and $i\sqrt{\rho/c_{11}}$ and has a maximum at a point $k = i\eta''$ in the segment $(O, i\sqrt{\rho/c_{11}})$ when $x > 0$ and at the point $k = -i\eta''$ in the segment $(-i\sqrt{\rho/c_{11}}, O)$ when $x < 0$. Let the point $k = i\eta''$ correspond to the value t'' of t. Proceeding exactly as in the case of the first integral, the path on which t is real can be constructed. It is shown in Fig. 3.

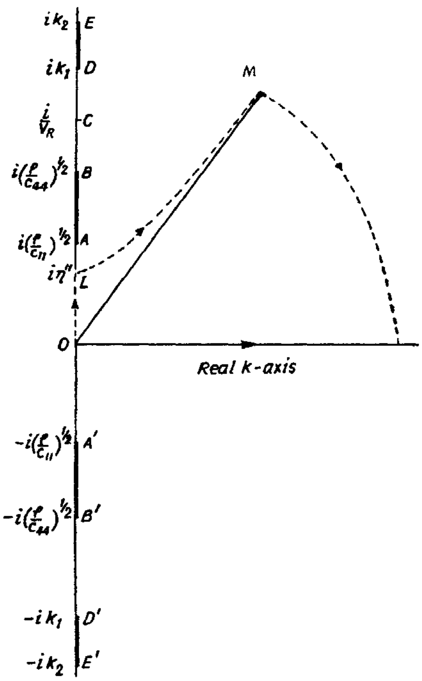

FIG. 3. Path of the integration of the function $F_2(k)$ in the complex k-plane in the case $x > 0$.

Now the integrals can be transformed according to the requirements of Cagniard's method. Let us consider the first integral of eqn. (54) when $x > 0$. Since the integral, taken along the arc of a large circle with centre at the origin, joining the real k-axis to the path for which t is real, tends to zero, we get from Cauchy's integral theorem

$$\int_0^\infty F_1(k) \exp(-pq_1 z + ipkx)\, dk = \int_{\text{curve } OLM} F_1(k) \exp(-pq_1 z + ipkx)\, dk$$

$$= \int_{\sqrt{\rho/c_{44}z}}^{t'} F_1[k(t)] \frac{dk}{dt} \exp(-pt)\, dt + \int_{t'}^{\infty} F_1[k(t)] \frac{dk}{dt} \exp(-pt)\, dt$$

$$+ H(\eta' - v)\pi i R \exp(-p(q_0 z + vx)), \qquad \qquad (62)$$

where R denotes $\lim_{k \to iv} F_1(k)\,(k-iv)$, q_0 denotes the value of q_1 at the point $k = iv$ and $H(z)$ represents Heaviside's unit function. Here in analogy with the isotropic case, we have assumed that the expression $\frac{dt}{dk}$ [given by eqn. (56)] is not zero on the transformed path of integration except at t'.

Since q_1 and q_2 are both real on the segment $(O, i\sqrt{\rho/c_{11}})$, the imaginary part of the first integral of the right-hand side of eqn. (62) is zero if $\eta' < \sqrt{\rho/c_{11}}$, i.e. if $f(x, z) < 0$ and it becomes

$$\operatorname{Im} \int_T^{t'} F_1(k) \frac{dk}{dt} \exp(-pt)\, dt \text{ if } \eta' > \sqrt{\rho/c_{11}}, \text{ i.e. if } f(x, z) > 0,$$

where

T, the value of t at the point $k = i\sqrt{\rho/c_{11}}$, is given by

$$T = \left[\frac{(c_{11}H - G)\rho}{c_{11}c_{33}c_{44}}\right]^{\frac{1}{2}} z + \sqrt{\rho/c_{11}}\, x. \qquad \ldots \quad \ldots \quad \ldots \quad (63)$$

Hence the first integral of eqn. (54) can be written as

$$\operatorname{Im} \int_0^{\infty} F_1(k) \exp(-pq_1 z + ipkx)\, dk$$

$$= \begin{cases} \operatorname{Im} \int_{k'}^{\infty} F_1[k(t)] \frac{dk}{dt} \exp(-pt)\, dt + \operatorname{Im} H(\eta' - v)\pi i R \exp(-p(q_0 z + vx)) \\ \qquad \text{when } f(x, z) < 0 \\ \operatorname{Im} \int_T^{t'} F_1[k(t)] \frac{dk}{dt} \exp(-pt)\, dt + \operatorname{Im} \int_{t'}^{\infty} F_1[k(t)] \frac{dk}{dt} \exp(-pt)\, dt \\ \qquad\qquad + \operatorname{Im} H(\eta' - v)\pi i R \exp(-p(q_0 z + vx)) \\ \qquad \text{when } f(x, z) > 0. \end{cases} \quad \ldots \quad \ldots \quad (64)$$

By an exactly similar method, the second integral of eqn. (54) reduces to

$$\operatorname{Im} \int_0^{\infty} F_2(k) \exp(-pq_2 z + ipkx)\, dk$$

$$= \operatorname{Im} \int_{t''}^{\infty} F_2[k(t)] \frac{dk}{dt} \exp(-pt)\, dt$$

$$+ \operatorname{Im} H(\eta'' - v)\pi i R' \exp(-p(q_0' z + vx)), \qquad \ldots \quad \ldots \quad \ldots \quad (65)$$

where R' represents $\lim_{k \to iv} F_2(k)\,.\,(k-iv)$ and q_0' represents the value of q_2 at the point $k = iv$.

Hence the inverse transform of eqn. (54) is obtained as

$$F = \begin{cases} H(t-t') \operatorname{Im} F_1[k(t)]\dfrac{dk}{dt}+H(t-t'') \operatorname{Im} F_2[k(t)]\dfrac{dk}{dt} \\ \quad +H(\eta'-v) \operatorname{Im} \pi i R\delta(t-vx-q_0 z) \\ \quad +H(\eta''-v) \operatorname{Im} \pi i R'\delta\big(t-vx-q_0' z\big), \\ \qquad \text{when } f(x, z) < 0 \\ H(t-t') \operatorname{Im} F_1[k(t)]\dfrac{dk}{dt}+H(t-t'') \operatorname{Im} F_2[k(t)]\dfrac{dk}{dt} \\ \quad +\{H(t-T)-H(t-t')\} \operatorname{Im} F_1[k(t)]\dfrac{dk}{dt} \\ \quad +H(\eta'-v) \operatorname{Im} \pi i R\delta(t-vx-q_0 z)+H(\eta'-v) \operatorname{Im} \pi i R'\delta\big(t-vx-q_0' z\big) \\ \qquad \text{when } f(x, z) > 0. \end{cases} \quad (66)$$

In $F_1[k(t)]$, the relation between k and t is given by $t = q_1 z - ikx$ while in $F_2[k(t)]$ it is given by $t = q_2 z - ikx$. The two k's appearing in $F_1(k)$ and $F_2(k)$ are the two roots of

$$\left[(t+ikx)^2 - \frac{Gk^2+\rho H}{2c_{33}c_{44}} z^2\right]^2 - z^4[(Gk^2+\rho H)^2 - 4c_{33}c_{44}(c_{11}k^2+\rho)(c_{44}k^2+\rho)] = 0, \quad (67)$$

having positive real parts.

In the case $x < 0$, the expression for F is similar to the one obtained in eqn. (66), the terms containing Dirac's δ-function being absent.

For the case $z = 0$ eqn. (55) becomes $t = ikx$, so that the path on which t is real coincides with the imaginary k-axis. Proceeding exactly as above we get a similar result for F with an additional term

$$\operatorname{Im} \pi i \lim_{k \to \frac{i}{v_R}} \left(k - \frac{i}{v_R}\right) [F_1(k)+F_2(k)]\delta\left(t - \frac{x}{v_R}\right),$$

which corresponds to Rayleigh wave.

Equation (66) shows that in the region $f(x, z) < 0$ the motion is the superposition of two waves represented by the first and the second terms, arriving at the point (x, z) at times t' and t''. These times correspond to the arrival times of the P- and S-waves for an isotropic medium. However, for large velocities of the moving load, there is a discontinuous change (for $x > 0$) in the motion due to the terms involving Dirac's delta-functions. This change is due to the arrival of a 'super wave front' at the point (x, z).

If $f(x, z) > 0$, the main difference is that one wave represented by the third term arrives at time T, earlier than t'; in the isotropic case this may be explained as due to the conversion of the S-wave into the P-wave at the boundary. The phenomena have been interpreted in terms of results for an isotropic case, since the expressions are too complicated to be capable of direct interpretation.

In fact, when τ_{zz} is prescribed on the boundary $z = 0$ by

$$\tau_{zz} = -Pf\left(x - \frac{1}{v}t\right), \text{ when } x > 0$$
$$= 0, \text{ when } x < 0,$$

we can determine the stress components from the corresponding stress components given by eqn. (66) with the help of convolution theorem.

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