CLOSED PATHS WITH DOUBLE COLLISION IN A PLANE ELLIPTIC RESTRICTED PROBLEM OF THREE BODIES

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(Communicated by N. S. Nagendra Nath, F.N.I.)

(*Received November 4, 1966*)

Here the motion of an infinitesimal mass, under the gravitational field of two finite and unequal masses and moving in ellipses with the centre of masses as the common centre, has been considered. The possibility of double collision with the larger mass has been assumed. The equations of motion are regularized first by the aid of a new set of canonical variables and next the solution is found out by the method of small parameter. Lastly, the convergency of the series representing the path has been studied.

1. INTRODUCTION

In this paper the knowledge of the works of Szebehely and Giacaglia (1964), Duboshin (1964) and Krasinskii (1963) on the elliptic restricted problem of three bodies is assumed. The notations are the same as in my paper (Choudhry 1966).

In the present work the trajectories with double collision with the larger mass in the plane elliptic restricted problem of three bodies are considered. These trajectories are of great importance for astronautics as such orbits are possible in flights to the moon and return to the earth under terrestrial and lunar attractions only. The mathematical problem reduces to the solution of a system of differential equations for the plane restricted problem of three bodies (Szebehely and Giacaglia 1964):

$$\left.\begin{aligned}\frac{d^2x}{df^2}-2\frac{dy}{df}&=\frac{\partial U}{\partial x}\\ \frac{d^2y}{df^2}+2\frac{dx}{df}&=\frac{\partial U}{\partial y}\end{aligned}\right\}, \quad \ldots \quad (1)$$

where f is the true anomaly of the moon round the earth, x and y are the synodic Cartesian rectangular coordinates of the third body in a non-uniformly rotating system (mean motion being equal to unity) and

$$U(x, y) = \left[\tfrac{1}{2}\{(1-\mu)r_1^2+\mu r_2^2\}+\frac{1-\mu}{r_1}+\frac{\mu}{r_2}\right]\Big/(1+e\cos f), \quad \ldots \quad (2)$$

$$r_1^2 = (x-\mu)^2+y^2,\ r_2^2 = (x-\mu+1)^2+y^2.$$

The boundary conditions may be written as

$$x(0) = x(T) = -\mu\,;\; y(0) = y(T) = 0.$$

Since in the required solution, the R.H.S. of (1) reduces to infinity, we shall regularize the solution by using Levi-Civita variables. In regularization I have followed the same method as in Choudhry (1965) for the circular restricted problem of three bodies.

The solution of the boundary value problem (1) and (2) is sought in the form of series in powers of the small parameter μ. For $\mu = 0$, the solution of the corresponding problem of two bodies depends on a single arbitrary parameter ω (the longitude of the perihelion); however, the solution of the boundary value problem is possible only for such values of ω which satisfy certain equations. Having constructed the series formally satisfying (1) and (2), its convergence can be studied by a method similar to one for the theory of periodic solutions (Malkin 1956).

2. Regularization of the Solution

To regularize the solution we reduce the eqns. (1) to canonical form. For this we introduce the variables

$$x_1 = x-\mu,\; y_1 = \dot{x}-y$$

$$x_2 = y, \qquad y_2 = \dot{y}+x-\mu$$

where differentiations with respect to f have been denoted by dots. Taking

$$H = \tfrac{1}{2}(y_1^2+y_2^2+x_1^2+x_2^2)+y_1x_2-x_1y_2-U,$$

the equations of motion (1) may be written as

$$\frac{dx_i}{df} = \frac{\partial H}{\partial y_i},\; \frac{dy_i}{df} = -\frac{\partial H}{\partial x_i} \quad (i = 1, 2). \qquad \ldots \quad \ldots \quad (3)$$

We introduce further variables

$$x_1+ix_2 = (p+iq)^2,\; y_1-iy_2 = (P-iQ)/2(p+iq).$$

Then $y_1\,dx_1+y_2\,dx_2 = P\,dp+Q\,dq$ and so the transformation is canonical and the reduced form of the eqns. (3) are

$$\left.\begin{aligned} \frac{dp}{df} &= \frac{\partial H}{\partial P}, & \frac{dq}{df} &= \frac{\partial H}{\partial Q} \\ \frac{dP}{df} &= -\frac{\partial H}{\partial p}, & \frac{dQ}{df} &= -\frac{\partial H}{\partial q} \end{aligned}\right\}, \qquad \ldots \quad \ldots \quad \ldots \quad (4)$$

where

$$H = \frac{1}{8}\frac{P^2+Q^2}{p^2+q^2}+\tfrac{1}{2}(p^2+q^2)^2+\frac{pq(pP-qQ)}{p^2+q^2}-(p^2-q^2)\,\frac{qP+pQ}{2(p^2+q^2)}-U.$$

Jacobi's integral may be written as

$$H+I+c/2 = 0, \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (5)$$

where

$$I = \int_0^f \left[\tfrac{1}{2}\{(1-\mu)r_1^2+\mu r_2^2\}+\frac{1-\mu}{r_1}+\frac{\mu}{r_2}\right] e \sin f\, df/(1+e\cos f)^2.$$

Putting $df = 4(p^2+q^2)\,d\tau$, our eqns. (4) are transformed to

$$\left.\begin{aligned} \frac{dp}{d\tau} &= \frac{\partial \bar{H}}{\partial P},\ \frac{dP}{d\tau} = -\frac{\partial \bar{H}}{\partial p} \\ \frac{dq}{d\tau} &= \frac{\partial \bar{H}}{\partial Q},\ \frac{dQ}{d\tau} = -\frac{\partial \bar{H}}{\partial q} \end{aligned}\right\}, \quad \dots \quad \dots \quad \dots \quad (6)$$

where

$$\bar{H} = 4(p^2+q^2)(H+c/2)+\int_0^{r_1}\int_0^f \frac{4e\sin f}{(1+e\cos f)^2}\left[\tfrac{1}{2}\{(1-\mu)r_1^2+\mu r_2^2\}+\frac{1-\mu}{r_1}+\frac{\mu}{r_2}\right] df\, dr_1$$

$$= \tfrac{1}{2}(P^2+Q^2)+2(p^2+q^2)^3+2(Pq-Qp)(p^2+q^2)$$

$$-4(p^2+q^2)\left[\frac{1}{1+e\cos f}\left\{\tfrac{1}{2}(p^2+q^2)^2-\mu(p^2-q^2)+\frac{1-\mu}{p^2+q^2}\right.\right.$$

$$\left.\left.+\frac{\mu}{\sqrt{1-2(p^2-q^2)^2+(p^2+q^2)^2}}\right\}-\frac{c}{2}\right]$$

$$+\int_0^{r_1}\int_0^f \frac{4e\sin f\, df}{(1+e\cos f)^2}\left[\tfrac{1}{2}\{(1-\mu)r_1^2+\mu r_2^2\}+\frac{1-\mu}{r_1}+\frac{\mu}{r_2}\right], \quad \dots \quad (7)$$

$r_1 = p^2+q^2$, $r_2 = \sqrt{1-2(p^2-q^2)^2+(p^2+q^2)^2}$.

The eqns. (6) are the regularized form of differential equations (Szebehely and Giacaglia 1964). For convenience we use x_1, x_2, x_3, x_4 instead of p, q, P and Q respectively, then the equations of motion are

$$\frac{dx_j}{d\tau} = X_j(x_1, x_2, x_3, x_4, f, c, \mu) \quad (j = 1, 2, 3, 4) \quad \dots \quad \dots \quad (8)$$

$$x_1(0) = x_1(\sigma) = x_2(0) = x_2(\sigma) = 0, \quad \dots \quad \dots \quad \dots \quad (9)$$

where $\tau = \sigma$ when $f = T$. Also let e be of the order of μ, that is $e = e_0\mu$.

3. Limiting Case for $\mu = 0$

Let us consider the solution of the problem (8) and (9) for $\mu = 0$. The equations for $\mu = 0$ have the form:

$$\left.\begin{aligned} \ddot{p}-8(p^2+q^2)\dot{q} &= 12(p^2+q^2)^2p-4cp \\ \ddot{q}+8(p^2+q^2)\dot{p} &= 12(p^2+q^2)^2q-4cq \end{aligned}\right\}. \quad \dots \quad \dots \quad (10)$$

Substituting $p = \rho\sin\theta$, $q = \rho\cos\theta$, the solutions for (10) can be written as

$$\left.\begin{aligned} \rho &= \sqrt{\frac{1}{c}\left(2+\frac{h}{4}\right)}\sin 2an\tau \\ \theta &= \frac{8+h}{16}\{4a\tau-(\sin 4an\tau)/n\}+\omega \end{aligned}\right\}. \quad \dots \quad \dots \quad (11)$$

Thus the solution of the boundary value problem for $\mu = 0$ depends only on two arbitrary constants ω and h. From (11), $\sigma = \pi k/2an$ (k being an integer). Comparing (11) with the solution obtained in Choudhry (1964) we find that $c = 1/a$ where a is the semi-major axis.

4. Perturbations of the First Order

For the perturbations of the first order let us consider the variational equations:

$$\left.\begin{aligned} \delta\ddot{p}-8r\,\delta\dot{q}-16(p\,\delta p+q\,\delta q)(\dot{q}+3pr)-(12r^2-4c)\,\delta p = 0 \\ \delta\ddot{q}+8r\,\delta\dot{p}-16(p\,\delta p+q\,\delta q)(-\dot{p}+3qr)-(12r^2-4c)\,\delta q = 0 \end{aligned}\right\} \quad \ldots \quad (12)$$

which are obtained by substituting $p+\delta p$ and $q+\delta q$ for p and q in (10) and taking the eqns. (10) into account. The obvious solutions for (12) are

$$\delta p_1 = q, \quad \delta p_2 = \dot{p}, \quad \delta p_3 = p+fq, \quad \delta p_4 = q \cot 2an\tau+\frac{2}{n}\dot{p}\tau-\frac{qf}{n}$$

$$\delta q_1 = -p, \quad \delta q_2 = \dot{q}, \quad \delta q_3 = q-fp, \quad \delta q_4 = -p \cot 2an\tau+\frac{2}{n}\dot{q}\tau+\frac{pf}{n}$$

where $f = 4a\tau-\dfrac{\sin 4an\tau}{n}$.

We shall try to find out the general solution of the eqns. (8) and (9) in the form of a series in powers of μ,

$$x_j = \sum_{j=0}^{\infty} x_j^{(i)}\mu^i \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (13)$$

and also assume that

$$c = c_0+\sum_{i=1}^{\infty} c_i\mu^i, \quad c_0 = \frac{1}{a} = \frac{\pi^2k^2}{4\sigma^2} \quad (k = 1, 2, \ldots).$$

Let $\mu = 0$. Then

$$x_1^{(0)} = p, \quad x_2^{(0)} = q, \quad x_3^{(0)} = \dot{x}_1^{(0)}-2x_2^{(0)}r, \quad x_4^{(0)} = \dot{x}_2^{(0)}+2x_1^{(0)}r. \quad \ldots \quad (14)$$

Functions $x_j^{(i)}$ $(i > 0)$ satisfy the system of linear differential equations:

$$\frac{dx_j^{(i)}}{d\tau} = \sum_{\alpha=1}^{4} p_{j\alpha}x_\alpha^{(i)}+F_j^{(i)}(x_1^{(0)}, \ldots, x^{(i-1)}, c_0, \ldots, c_{i-1}, f, 0)+\phi_j c_i \quad \ldots \quad (15)$$

$$\left.\begin{aligned} p_{j\alpha} &= \frac{\partial X_j(x_1^{(0)}, x_2^{(0)}, x_3^{(0)}, x_4^{(0)}, c_0, f, 0)}{\partial x_\alpha^{(0)}} \\ \phi_j &= \frac{\partial X_j}{\partial c_0} \end{aligned}\right\} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (16)$$

$$(j = 1, 2, 3, 4; \ \alpha = 1, 2, 3, 4).$$

In particular,

$$\phi_1 = \frac{\partial X_1}{\partial c_0} = 0 = \frac{\partial X_2}{\partial c_0} = \phi_2, \quad \phi_3 = \frac{\partial X_3}{\partial c_0} = -4x_1^{(0)}, \quad \phi_4 = \frac{\partial X_4}{\partial c_0} = -4x_2^{(0)}$$

and $F_j^{(i)}$ includes all other terms. For $i = 1$, especially

$$F_j^{(i)} = \left.\frac{\partial X_j(x_1^{(0)}, x_2^{(0)}, x_3^{(0)}, x_4^{(0)}, c_0, \mu)}{\partial \mu}\right|_{\mu=0} .$$

The functions $x_j^{(i)}$ must satisfy the boundary conditions as well:

$$x_1^{(i)}(0) = x_2^{(i)}(0) = x_1^{(i)}(\sigma) = x_2^{(i)}(\sigma) = 0. \qquad (17)$$

Knowing the solutions of the system in variation (12), we can find out all the linearly independent solutions of the homogeneous system:

$$dy_j/d\tau = \sum_{\alpha=1}^{4} p_{j\alpha} y_\alpha \qquad (18)$$

whose obvious solutions may be written as

$$\left.\begin{array}{ll} y_{1i} = \delta p_i & y_{2i} = \delta q_i \\ y_{3i} = \delta \dot{p}_i - 2r\, \delta q_i - 2q\, \delta r_i, & y_{4i} = \delta \dot{q}_i + 2r\, \delta p_i + 2p\, \delta r_i \end{array}\right\}. \qquad (19)$$

The last two expressions are obtained by varying $x_3^{(0)}$ and $x_4^{(0)}$ given by (14).

Let us now pass on to the solution for $x_j^{(1)}$. For this we find out a particular solution of the differential equations (15) for $i = 1$. Using the method of variation of the arbitrary constants, we look for the particular solutions in the form:

$$x_k^{(1)} = \sum_{i=1}^{4} \alpha_i y_{ki}. \qquad (20)$$

Differentiating on both sides,

$$\begin{aligned} \frac{dx_k^{(1)}}{d\tau} &= \sum_{i=1}^{4} \dot{\alpha}_i y_{ki} + \sum_{i=1}^{4} \alpha_i \dot{y}_{ki} \\ &= \sum_{i=1}^{4} \dot{\alpha}_i y_{ki} + \sum_{i=1}^{4} \alpha_i \left[\sum_{\alpha=1}^{4} p_{k\alpha} y_{\alpha i}\right] \\ &= \sum_{i=1}^{4} \dot{\alpha}_i y_{ki} + \sum_{\alpha=1}^{4} p_{k\alpha} \left[\sum_{i=1}^{4} \alpha_i y_{\alpha i}\right] \\ &= \sum_{i=1}^{4} \dot{\alpha}_i y_{ki} + \sum_{\alpha=1}^{4} p_{k\alpha} x_\alpha^{(i)}. \qquad (21) \end{aligned}$$

From (15) and (21),

$$\begin{aligned} \frac{dx_k^{(1)}}{d\tau} &= \sum_{\alpha=1}^{4} p_{k\alpha} x_\alpha^{(1)} + F_k^{(1)}(x_1^{(0)}, x_2^{(0)}, x_3^{(0)}, x_4^{(0)}, c_0, f) + \phi_k c_1 \\ &= \sum_{i=1}^{4} \dot{\alpha}_i y_{ki} + \sum_{\alpha=1}^{4} p_{k\alpha} x_\alpha^{(i)}, \end{aligned}$$

whence

$$\sum_{i=1}^{4} \dot{\alpha}_i y_{ki} = F_k^{(1)} + c_1\phi_k. \qquad (22)$$

Here

$$F_1^{(1)} = \frac{\partial}{\partial\mu}[P+2q(p^2+q^2)]_{\mu=0} = 0 = F_2^{(1)},$$

$$F_3^{(1)} = -16x_1^3 + \frac{8x_1}{r_2} + \frac{8rx_1 - 8r^2x_1}{r_2^3} - 16e_0x_1r^2 \cos f - 8x_1\int_0^f 4e_0 \sin f\left(\frac{1}{r}+\frac{r^2}{2}\right) df,$$

$$F_4^{(1)} = 16x_2^3 + \frac{8x_2}{r_2} + \frac{-8rx_2 - 8r^2x_2}{r_2^3} - 16e_0x_2r^2 \cos f - 8x_2\int_0^f 4e_0 \sin f\left(\frac{1}{r}+\frac{r^2}{2}\right) df.$$

Let

$$z_{1i} = -y_{3i},\; z_{2i} = -y_{4i},\; z_{3i} = y_{1i},\; z_{4i} = y_{2i}. \qquad (23)$$

It is well known for the canonical system, the quantities z_{ki} constitute the matrix of solutions for the system of equations, conjugate to the system (18), i.e. for equations

$$\frac{dz_j}{d\tau} = -\sum_{\alpha=1}^{4} p_{\alpha j} z_\alpha. \qquad (24)$$

The basic properties of the conjugate system are that the expression $A_{ij} = \sum_{k=1}^{4} z_{ki}y_{kj}$ does not depend on τ. Now from the definition $A_{ij} = -A_{ji}$. Multiplying (22) by z_{kj} and summing for k from 1 to 4 we get

$$\sum_{i=1}^{4} \dot{\alpha}_i A_{ji} = f_j, \qquad (25)$$

where

$$f_j = \sum_{k=1}^{4} (F_k^{(1)} + c_1\phi_k) z_{kj}.$$

Since the coefficients A_{ij} do not depend on τ, for their calculation let $\tau = 0$. Using the formulas (11), the matrix of the system (22) for $\tau = 0$ is obtained:

$$\begin{Vmatrix} 0 & \sqrt{8}\sin\omega & 0 & \sqrt{2a}\cos\omega \\ 0 & \sqrt{8}\cos\omega & 0 & -\sqrt{2a}\sin\omega \\ \sqrt{8}\cos\omega & 0 & \sqrt{8}\sin\omega & 0 \\ \sqrt{8}\sin\omega & 0 & \sqrt{8}\cos\omega & 0 \end{Vmatrix}$$

Forming the matrix $\| A_{ji} \|$ we get

$$\| A_{ji} \| = \begin{Vmatrix} 0 & 0 & 0 & -4\sqrt{a} \\ 0 & 0 & 8 & 0 \\ 0 & -8 & 0 & 0 \\ 4\sqrt{a} & 0 & 0 & 0 \end{Vmatrix}$$

Thus we have the following system of differential equations:

$$\left.\begin{aligned} 4\sqrt{a}\dot{\alpha}_4 &= x_2 F_3^{(1)} - x_1 F_4^{(1)}, \quad 8\dot{\alpha}_3 = \dot{x}_1 F_3^{(1)} + \dot{x}_2 F_4^{(1)}, \\ -8\dot{\alpha}_2 &= (x_1 + f x_2) F_3^{(1)} + (x_2 - f x_1) F_4^{(1)}, \\ 4\sqrt{a}\dot{\alpha}_1 &= \left(x_2 \cot 2an\tau + \frac{2}{n}\dot{x}_1 \tau - x_2 \frac{f}{n}\right) F_3^{(1)} \\ &\quad + \left(-x_1 \cot 2an\tau + \frac{2}{n}\dot{x}_2 \tau + x_1 \frac{f}{n}\right) F_4^{(1)} \end{aligned}\right\} \quad \ldots \quad (26)$$

Consequently,

$$x_k^{(1)} = \sum_{i=1}^{4} (\alpha_i + a_i^{(1)}) y_{ki},$$

where α_i are calculated by the formulas (26), $a_i^{(1)}$ are arbitrary constants. Let us choose the constants so that the following boundary conditions:

$$x_1^{(1)}(0) = x_2^{(1)}(0) = x_1^{(1)}(\sigma) = x_2^{(1)}(\sigma) = 0 \quad \ldots \quad \ldots \quad (27)$$

are satisfied.

If in (26) the limits of integration be taken from 0 to τ, then for $\tau = 0$, all $\alpha_i = 0$. Putting $\tau = 0$, we have

$$x_1^{(1)}(0) = a_2^{(1)}\sqrt{8} \sin \omega + a_4^{(1)}\sqrt{2a} \cos \omega = 0$$

$$x_2^{(1)}(0) = a_2^{(1)}\sqrt{8} \cos \omega - a_4^{(1)}\sqrt{2a} \sin \omega = 0$$

whence $a_2^{(1)} = a_4^{(1)} = 0$. Putting $\tau = \sigma$, we have

$$\left.\begin{aligned} -\alpha_2(\sigma) \sin\left(\omega + \frac{\pi k}{2n}\right) - \alpha_4(\sigma)\sqrt{2a} \cos\left(\omega + \frac{\pi k}{2n}\right) = 0 \\ -\alpha_2(\sigma) \cos\left(\omega + \frac{\pi k}{2n}\right) + \alpha_4(\sigma)\sqrt{2a} \sin\left(\omega + \frac{\pi k}{2n}\right) = 0 \end{aligned}\right\} \quad \ldots \quad (28)$$

and so $\alpha_2(\sigma) = \alpha_4(\sigma) = 0$. Under this $a_1^{(1)}$, $a_3^{(1)}$ remain undetermined.

Thus in order that the system of eqns. (15) for $i = 1$ may have a solution satisfying the conditions (17), the necessary and sufficient conditions are

$$P_1(\omega, h) = \int_0^\sigma \left[x_2 F_3^{(1)} - x_1 F_4^{(1)} \right] d\tau$$

$$= 16 \int_0^\sigma x_1 x_2 r \left(\frac{1}{r_2^3} - 1 \right) d\tau = 0 \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (29)$$

$$P_2(\omega, h, c) = \int_0^\sigma \left[(x_1+fx_2)F_3^{(1)}+(x_2-fx_1)F_4^{(1)}\right] d\tau$$

$$= \int_0^\sigma 4r\left[4(x_2^2-x_1^2)+\frac{2}{r_2}+\frac{2(x_1^2-x_2^2)-2r^2}{r_2^2}-c_1\right] d\tau$$

$$-\int_0^\sigma \left[16e_0 r^3 \cos f+8r\int_0^f 4e_0 \sin f\left(\frac{1}{r}+\frac{r^2}{2}\right) df\right] d\tau = 0. \quad (30)$$

Passing to the variables x, y, f and putting $h = 0$, we have

$$\int_0^T y\left(\frac{1}{r_2^3}-1\right) df = 0 \quad \cdots \quad (31)$$

$$c_1\tau+\int_0^T \left[4x-\frac{2}{r_2}-\frac{2(r^2-x)}{r_2^3}\right] df-2\int_0^T y\left(\frac{1}{r_2^3}-1\right) f\, df$$

$$+\int_0^T \left[4e_0 r^2 \cos f+8\int_0^f \sin f'\left(\frac{1}{r}+\frac{r^2}{2}\right) df'\right] df = 0. \quad \cdots \quad (32)$$

From (31) we get ω and, putting this value in (32), we get c_1, the correction for Jacobi's constant for $\mu \neq 0$.

5. Formation of the Formal Solution

Under the conditions (31), (32) the solution of the eqns. (15) for $i = 1$ satisfying the conditions (17) will depend upon two arbitrary constants $a_1^{(1)}$, $a_1^{(3)}$ which remain undetermined. These arbitrary constants are determined from the conditions of solvability of the boundary value problem for equations of the second approximation. The quantity c_2 is also determined with them. The arbitrary constants appearing in the second approximation are determined from the conditions of solvability of the boundary problem for equations of the third approximation and so on. We use here transformations used in the theory of the periodic solution (Malkin 1956).

Let us assume that we have determined all functions $x_j^{(i)}$ $(j = 1, 2, 3, 4)$ up to the kth order, which satisfy the eqns. (15) and the boundary conditions (17); also let all the constants c_i $(i = 1, 2, \ldots k)$ be determined. Let us enumerate the solutions of the equations in variation in such a way that

$$y_{j1} = \frac{\partial x_j}{\partial \omega}, \; y_{j2} = \frac{\partial x_j}{\partial h}.$$

Functions $x_j^{(i)}$ are of the form

$$x_j^{(i)} = a_1^{(i)} y_{j1}+a_2^{(i)} y_{j2}+\bar{x}_j^{(i)},$$

where $\bar{x}_j^{(i)}$ is a particular solution, satisfying the boundary conditions and $a_1^{(i)}$, $a_2^{(i)}$ are certain constants. The quantities $a_1^{(i)}$, $a_2^{(i)}$ are taken to be determined from the conditions of solvability of the boundary value problem of the $(k+1)$th approximation. Functions $F_j^{(k+1)}$ have the following form:

$$F_s^{(k+1)} = \tfrac{1}{2} \sum_{j=1}^{2} \sum_{\alpha_1\beta=1}^{4} \left(\frac{\partial^2 X_s}{\partial x_\alpha \partial x_\beta}\right)(x_\alpha^{(1)} y_{\beta j} + x_\beta^{(1)} y_{\alpha j}) a_j^{(k)}$$
$$+ \sum_{j=1}^{2} \sum_{\alpha=1}^{4} \left[\left(\frac{\partial F_s^{(1)}}{\partial x_\alpha}\right) + c_1 \left(\frac{\partial \phi_s}{\partial x_\alpha}\right)\right] y_{\alpha j} a_j^{(k)} + R_s^{(k+1)},$$

where the brackets signify that the derivatives are taken for the values of the parameters $h = 0$, $\omega = \omega^*$, $\mu = 0$ and ω^*, c_1 satisfy (31), (32). $R_s^{(k+1)}$ is a known function not depending on $a_j^{(k)}$, c_{k+1}. This may be written as

$$F_s^{(k+1)} = \sum_{j=1}^{2} \sum_{\alpha_1\beta=1}^{4} \left(\frac{\partial^2 X_s}{\partial x_\alpha \partial x_\beta}\right) x_\alpha^{(1)} y_{\beta j} a_j^{(k)}$$
$$+ \sum_{j=1}^{2} \sum_{\alpha=1}^{4} \left[\left(\frac{\partial F_s^{(1)}}{\partial x_\alpha}\right) + c_1 \left(\frac{\partial \phi_s}{\partial x_\alpha}\right)\right] y_{\alpha j} a_j^{(k)} + R_s^{(k+1)}.$$

Let us denote the parameters ω, h by h_1 and h_2 respectively, then $y_{\beta j} = \dfrac{\partial x_\beta}{\partial h_j}$ $(j = 1, 2)$ and $F_s^{(k+1)}$ has the form

$$F_s^{(k+1)} = \sum_{j=1}^{2} \sum_{\alpha_1\beta=1}^{4} \left[\frac{\partial p_{s\alpha}}{\partial h_j} x_\alpha^{(1)} + \frac{\partial (F_s^{(1)} + c_1 \phi_s)}{\partial h_j}\right] a_j^{(k)} + R_s^{(k+1)},$$

for $k > 1$, $F_s^{(k+1)}$ are linear relative to $a_j^{(k)}$, but for $k = 1$ it contains quadratic terms as well, since the constants $a_j^{(k)}$ are involved in $x_\alpha^{(1)}$ and it is given as

$$F_s^{(2)} = \sum_{\alpha=1}^{4} \sum_{i,j=1}^{2} \frac{\partial p_{s\alpha}}{\partial h_j} y_{\alpha i} a_i^{(1)} a_j^{(1)}$$
$$+ \sum_{j=1}^{2} \left[\sum_{\alpha=1}^{4} \frac{\partial p_{s\alpha}}{\partial h_j} x_\alpha^{(1)} + \frac{\partial (F_s + c_1 \phi_s)}{\partial h_j}\right] a_j^{(1)} + R_s^{(2)}.$$

Let us consider now the expression

$$P_n^{(k+1)} = \int_0^\sigma \sum_{\beta=1}^{4} (F_\beta^{(k+1)} + c_{k+1} \phi_\beta + R^{(k+1)}) z_{\beta n} d\tau \quad (n = 1, 2), \quad \ldots \quad (33)$$

where z_{j1}, z_{j2} are the solutions of the conjugate systems, corresponding to the solutions y_{j1}, y_{j2} of the system in variation. Similar to § 3 for $k = 0$, it can be easily shown that $P_n^{(k+1)} = 0$ is the necessary and sufficient condition for the solvability of our boundary value problem for the differential equations of the $(k+1)$th approximation. We note that $x_\beta^{(1)}$ $(\beta = 1, 2, 3, 4)$ satisfy the equations

$$\frac{dx_\beta^{(1)}}{d\tau} = \sum_{\alpha=1}^{4} p_{\beta\alpha} x_\alpha^{(1)} + F_\beta^{(1)} + c_1\phi_\beta . \qquad \text{(34)}$$

Differentiating both sides w.r.t. h_j,

$$\frac{d}{d\tau}\left(\frac{\partial x_\beta^{(1)}}{\partial h_j}\right) = \sum_{\alpha=1}^{4} p_{\beta\alpha}\frac{\partial x_\alpha^{(1)}}{\partial h_j} + \sum_{\alpha=1}^{4}\frac{\partial p_{\beta\alpha}}{\partial h_j} x_\alpha^{(1)} + \frac{\partial(F_\beta^{(1)} + c_1\phi_\beta)}{\partial h_j}. \qquad \text{(35)}$$

Equations $P_1^{(k+1)} = 0$ and $P_2^{(k+1)} = 0$ may be written as

$$\int_0^\sigma \sum_{\beta=1}^{4} (F_\beta^{(k+1)} + c_{k+1}\phi_\beta + R^{(k+1)}) z_{\beta n}\, d\tau = 0 \quad (n = 1, 2),$$

i.e.

$$a_1^{(k)} \int_0^\sigma \sum_{\beta=1}^{4}\left[\sum_{\alpha=1}^{4}\frac{\partial p_{\beta\alpha}}{\partial h_1} x_\alpha^{(1)} + \frac{\partial(F_\beta^{(1)} + c_1\phi_\beta)}{\partial h_1}\right] z_{\beta 1}\, d\tau$$

$$+a_2^{(k)} \int_0^\sigma \sum_{\beta=1}^{4}\left[\sum_{\alpha=1}^{4}\frac{\partial p_{\beta\alpha}}{\partial h_2} x_\alpha^{(1)} + \frac{\partial(F_\beta^{(1)} + c_1\phi_\beta)}{\partial h_2}\right] z_{\beta 1}\, d\tau$$

$$+c_{k+1} \int_0^\sigma \sum_{\beta=1}^{4}\left[\sum_{\alpha=1}^{4}\frac{\partial p_{\beta\alpha}}{\partial h_3} x_\alpha^{(1)} + \frac{\partial(F_\beta^{(1)} + c_1\phi_\beta)}{\partial h_3}\right] z_{\beta 1}\, d\tau = \int_0^\sigma -\Sigma R^{(k+1)} z_{\beta n}\, d\tau$$

(h_3 being taken equal to c_1),

i.e.

$$\left.\begin{aligned} A_{11}a_1^{(k)} + A_{12}a_2^{(k)} + A_{13}c_{k+1} = B_1^{(k)} \\ A_{21}a_1^{(k)} + A_{22}a_2^{(k)} + A_{23}c_{k+1} = B_2^{(k)} \end{aligned}\right\}, \qquad \text{(36)}$$

where

$$A_{nm} = \int_0^\sigma \sum_{\beta=1}^{4}\left[\sum_{\alpha=1}^{4}\frac{\partial p_{\beta\alpha}}{\partial h_m} x_\alpha^{(1)} + \frac{\partial\left(F_\beta^{(1)} + c_1\phi_\beta\right)}{\partial h_m}\right] z_{\beta n}\, d\tau$$

$$= \int_0^\sigma \sum_{\beta=1}^{4}\left[\frac{d}{d\tau}\left(\frac{\partial x_\beta^{(1)}}{\partial h_m}\right) - \sum_{\alpha=1}^{4} p_{\beta\alpha}\frac{\partial x_\alpha^{(1)}}{\partial h_m}\right] z_{\beta n}\, d\tau \quad \text{(using (35))}.$$

Integrating by parts and taking into account that $z_{\beta n}$ satisfy the conjugate system of equations, we have

$$A_{nm} = \sum_{\beta=1}^{4}\frac{\partial x_\beta^{(1)}}{\partial h_m} z_{\beta n}\Bigg|_0^\sigma - \int_0^\sigma \sum_{\beta=1}^{4}\frac{\partial x_\beta^{(1)}}{\partial h_m}\left[\frac{dz_{\beta n}}{d\tau} + \sum_{\gamma=1}^{4} p_{\gamma\beta} z_{\beta n}\right] d\tau$$

$$= \sum_{\beta=1}^{4}\frac{\partial x_\beta^{(1)}}{\partial h_m} z_{\beta n}\Bigg|_0^\sigma = \left\{\frac{\partial}{\partial h_m}\left(\sum_{\beta=1}^{4} x_\beta^{(1)} z_{\beta n}\right) - \sum_{\beta=1}^{4} x_\beta^{(1)}\frac{\partial z_{\beta n}}{\partial h_m}\right\}\Bigg|_0^\sigma .$$

Now by virtue of the fact that

$$x_1^{(1)}(0) = x_1^{(1)}(\sigma) = x_2^{(1)}(0) = x_2^{(1)}(\sigma) = 0 \quad \text{and} \quad z_{\beta n}(0) = 0 = z_{\beta n}(\sigma)$$

we have

$$\frac{\partial z_{\beta n}(0)}{\partial h_m} = \frac{\partial z_{\beta n}(\sigma)}{\partial h_m} = 0 \quad (\beta = 3, 4;\ m, n = 1, 2)$$

and so $A_{nm} = \dfrac{\partial}{\partial h_m}\left(\sum_{\beta=1}^{4} x_\beta^{(1)} z_{\beta n}\right)$.

On the other hand, we have

$$\frac{d}{d\tau}\sum_{\beta=1}^{4} x_\beta^{(1)} z_{\beta n} = \sum_{\beta=1}^{4}\left(p_{\beta\alpha}x_\alpha^{(1)} + F_\beta^{(1)} + c_1\phi_\beta\right)z_{\beta n}$$

$$- \sum_{\beta=1}^{4} x_\beta^{(1)} \sum_{\gamma=1}^{4} p_{\gamma\beta}\, z_{\gamma n} = \sum_{\beta=1}^{4}\left(F_\beta^{(1)} + c_1\phi_\beta\right)z_{\beta n}$$

and finally,

$$A_{nm} = \frac{\partial}{\partial h_m}\int_0^\sigma \sum_{\beta=1}^{4}\left(F_\beta^{(1)} + c_1\phi_\beta\right)z_{\beta n}\, d\tau = \frac{\partial P_n(h_1, h_2)}{\partial h_m}.$$

Quantities A_{13} and A_{23} are easily calculated by the relation

$$A_{13} = \int_0^\sigma \left[-4x_1\frac{\partial x_1}{\partial \omega} - 4x_2\frac{\partial x_2}{\partial \omega}\right] d\tau = \int_0^\sigma [-4x_1x_2 + 4x_1x_2]\, d\tau = 0$$

$$A_{23} = \int_0^\sigma \left[-4x_1\frac{\partial x_1}{\partial h} - 4x_2\frac{\partial x_2}{\partial h}\right] d\tau$$

$$= \int_0^\sigma [-4x_1(x_1 + fx_2) - 4x_2(x_2 - fx_1)]\, d\tau = -4\int_0^\sigma r\, d\tau = -T.$$

Besides eqns. (36) we have one more equation given by

$$H + I + c/2 = 0.$$

As $H+I$ preserves the constant value, so we may put $\tau = 0$. We then get

$$\tfrac{1}{2}\left[\dot{x}_1^2(0) + \dot{x}_2^2(0)\right] = 4(1-\mu),$$

as $I = 0$ when $\tau = 0$. So writing the terms with μ^k we have

$$\dot{x}_1^{(k)}(0)\dot{x}_1(0) + \dot{x}_2^{(k)}(0)\dot{x}_2(0) = B_3^{(k)},$$

where B_3^k depends only on $\dot{x}_j^{(i)}(0)$ $(i < k)$. Substituting here the expression for $x_j^{(k)}$, we find that

$$a_1^{(k)}\left[\dot{y}_{11}(0)\dot{x}_1(0) + \dot{y}_{21}(0)\dot{x}_2(0)\right] + a_2^{(k)}\left[\dot{y}_{12}(0)\dot{x}_1(0) + \dot{y}_{22}(0)\dot{x}_2(0)\right] = B_3^{(k)}.$$

Coefficients for $a_1^{(k)}$ are equal to $\dot{x}_1(0)\dot{x}_2(0) - \dot{x}_1(0)\dot{x}_2(0) = 0$ and for $a_2^{(k)}$ these are equal to $\{\dot{x}_1(0)\}^2 + \{\dot{x}_2(0)\}^2 = 8$.

Finally we have a system of three linear equations with three unknowns:

$$\left.\begin{aligned}&\frac{\partial P_1}{\partial \omega} a_1^{(k)} + \frac{\partial P_1}{\partial h} a_2^{(k)} = B_1^{(k)}\\&\frac{\partial P_2}{\partial \omega} a_1^{(k)} + \frac{\partial P_2}{\partial h} a_2^{(k)} - Tc_{k+1} = B_2^{(k)}\\&8a_2^{(k)} = B_3^{(k)}\end{aligned}\right\}. \qquad (37)$$

The determinant of this system is equal to $8T\frac{\partial P_1}{\partial \omega}$. If ω^* is a simple root of the equation $P_1(\omega, h) = 0$, this determinant is a non-zero quantity. It remains to examine the case $k = 1$, when $P_n^{(2)} = 0$ contains quadratic terms in $a_j^{(1)}$. These terms are involved in $P_n^{(2)}$ in the following manner:

$$A_n = \sum_{j, m=1}^{4} A_{jmn} a_j^{(1)} a_m^{(1)}, \quad A_{jmn} = \int_0^\sigma \left(\sum_{\alpha, \beta=1}^{m} \frac{\partial P_{\beta\alpha}}{\partial h_j} y_{\alpha m} z_{\beta n} \right) d\tau.$$

It is easy to see that $A_{jmn} = 0$. For this we follow the same process as was done for the coefficients A_{mn} and we take into account the fact that $y_{\alpha m}$ satisfy the variational equations whence

$$A_{jmn} = \frac{\partial}{\partial h_j} \left(\sum_{\beta=1}^{4} y_{\beta m} z_{\beta n} \right) \Bigg|_0^\sigma = \text{const.} \Bigg|_0^\sigma = 0.$$

The linear part of the equations for $k = 1$ coincides with the linear part of the eqns. (37) for $k > 1$.

Thus the quantities $a_1^{(k)}$, $a_2^{(k)}$, c_{k+1} satisfy the linear system of equations whose homogeneous part does not depend on the number k and the corresponding determinant is not equal to zero.

In this way, using the above method, we obtained series satisfying the eqns. (8) and boundary conditions (9), where the coefficients are determined uniquely. If we now somehow prove the existence of the solution of the problem (8), (9), then the convergence will follow.

6. Existence of the Solution of the Problem (8), (9)

Let $x_j = x_j(\tau, \gamma_1, \gamma_2, \gamma_3, \gamma_4, c, \mu)$ be the general solution of the system of eqns. (8) such that

$$x_j(0, \gamma_1, \gamma_2, \gamma_3, \gamma_4, c, \mu) = \gamma_j. \qquad (38)$$

For the particular solution satisfying the boundary conditions (9) we put $\tau = 0$, and get immediately $\gamma_1 = \gamma_2 = 0$. For $\tau = \sigma$, we get the following system of equations:

$$\left.\begin{aligned}x_1(\sigma, 0, 0, \gamma_3, \gamma_4, c, \mu) = 0\\x_2(\sigma, 0, 0, \gamma_3, \gamma_4, c, \mu) = 0\end{aligned}\right\}. \qquad (39)$$

Also

$$H(\gamma_3, \gamma_4, c, \mu) = 0 \qquad (40)$$

which is obtained from $H+c/2+I = 0$ when $\tau = 0$. For $\mu = 0$, there are solutions depending on two arbitrary constants ω and h, consequently, for $\mu = 0$

$$x_1(\sigma, 0, 0, \gamma_0^{(3)}, \gamma_4^{(0)}, c_0, 0) = x_2(\sigma, 0, 0, \gamma_3^{(0)}, \gamma_4^{(0)}, c_0, 0) = 0. \quad .. \quad (41)$$

By virtue of (39) and (41), we have

$$\left.\frac{\partial x_1(\sigma)}{\partial \gamma_j}\right|_{\mu=0} = \left.\frac{\partial x_2(\sigma)}{\partial \gamma_j}\right|_{\mu=0} = 0 \quad (j = 3, 4). \quad .. \quad .. \quad .. \quad (42)$$

Here $c_0 = 4\sigma^2/\pi^2k^2$ ($k = 1, 2, \ldots$). The eqn. (40) for $\mu = 0$ has the form

$$\tfrac{1}{2}\left\{\left(\gamma_3^{(0)}\right)^2+\left(\gamma_4^{(0)}\right)^2\right\} = 4. \quad .. \quad .. \quad .. \quad (43)$$

We shall show now that for sufficiently small values of μ, under certain restrictions, the eqns. (39), (40) determine γ_3, γ_4, c as functions of μ. Let

$$\beta_1 = \gamma_3-\gamma_3^{(0)}, \quad \beta_2 = \gamma_4-\gamma_4^{(0)}, \quad c_1 = c-c_0.$$

Then the equations $x_1(\sigma) = x_2(\sigma) = 0$ can be written in the form

$$x_1(\sigma, 0, 0, \gamma_3, \gamma_4, c, \mu) = \left(\frac{\partial x_1}{\partial \gamma_3}+U_1\right)\beta_1+\left(\frac{\partial x_1}{\partial \gamma_4}+U_2\right)\beta_2+\left(\frac{\partial x_1}{\partial c}+U_3\right)c_1+\left(\frac{\partial x_1}{\partial \mu}+U_4\right)\mu$$

$$x_2(\sigma, 0, 0, \gamma_3, \gamma_4, c, \mu) = \left(\frac{\partial x_2}{\partial \gamma_3}+V_1\right)\beta_1+\left(\frac{\partial x_2}{\partial \gamma_4}+V_2\right)\beta_2+\left(\frac{\partial x_2}{\partial c}+V_3\right)c_1+\left(\frac{\partial x_2}{\partial \mu}+V_4\right)\mu$$

where U_1, U_2, U_3, U_4, V_1, V_2, V_3, V_4 are analytic functions of β_1, β_2, c, μ reducing to zero for $\mu = \beta_1 = \beta_2 = c_1 = 0$ and the derivatives have been calculated for $\mu = 0$, $c = c_0$. Let us divide the eqns. (40) by μ and let μ tend to zero. Putting $\nu = \lim \frac{c_1}{\mu}$ and taking into account (42), we find that

$$\left.\begin{aligned} \Phi_1\left(\gamma_3^{(0)}, \gamma_4^{(0)}, \nu\right) &= \left.\frac{\partial x_1(\sigma)}{\partial c}\right|_{\mu=0} \nu + \left.\frac{\partial x_1(\sigma)}{\partial \mu}\right|_{\mu=0} = 0 \\ \Phi_2\left(\gamma_3^{(0)}, \gamma_4^{(0)}, \nu\right) &= \left.\frac{\partial x_2(\sigma)}{\partial c}\right|_{\mu=0} \nu + \left.\frac{\partial x_2(\sigma)}{\partial \mu}\right|_{\mu=0} = 0 \end{aligned}\right\} \quad .. \quad .. \quad (44)$$

We combine with these equations the equation

$$\Phi_3 = \left\{\gamma_3^{(0)}\right\}^2+\left\{\gamma_4^{(0)}\right\}^2 = 8.$$

If $\gamma_3^{(0)}$, $\gamma_4^{(0)}$, ν satisfy these equations and if $\frac{\partial(\Phi_1, \Phi_2, \Phi_3)}{\partial\left(\gamma_3^{(0)}, \gamma_4^{(0)}, \nu\right)} \neq 0$, then the eqns. (39), (40) determine γ_3, γ_4, c as functions of μ for small μ. Thus the existence of the solution can be proved. Proceeding as in Krasinskii (1963), we find that

$$\Phi_1 = -\frac{\sqrt{2}}{4}T\nu \sin\alpha+\frac{\sqrt{2}}{4}(\cos\alpha+T\sin\alpha)I_1+\frac{\sin\alpha}{\sqrt{8}}I_2 = 0, \quad .. \quad (45)$$

$$\Phi_2 = -\frac{\sqrt{2}}{4}T\nu \cos\alpha+\frac{\sqrt{2}}{4}(-\sin\alpha+T\cos\alpha)I_1+\frac{\cos\alpha}{\sqrt{8}}I_2 = 0, \quad .. \quad (46)$$

where

$$I_1 = \int_0^\sigma \left[x_2F_3^{(1)} - x_1F_4^{(1)}\right] d\tau,$$

$$I_2 = \int_0^\sigma \left[(x_1+fx_2)F_3^{(1)} + (x_2-fx_1)F_4^{(1)}\right] d\tau.$$

After a simple manipulation it is seen that

$$I_1 = I_1[\gamma_3^{(0)}(\omega, h), \gamma_4^{(0)}(\omega, h)] \equiv P_1(\omega, h) = 0,$$

$$I_2 - \nu T = I_2[\gamma_3^{(0)}(\omega, h), \gamma_4^{(0)}(\omega, h)] - \nu T \equiv P_2(\omega, h, \nu) = 0.$$

Using (45) and (46),

$$\frac{\partial(\Phi_1, \Phi_2, \Phi_3)}{\partial(\gamma_3^{(0)}, \gamma_4^{(0)}, \nu)} = \frac{\partial(\Phi_1, \Phi_2, \Phi_3)}{\partial(P_1, P_2, \Phi_3)} \cdot \frac{\partial(P_1, P_2, \Phi_3)}{\partial(\omega, h, \nu)} \cdot \frac{\partial(\omega, h, \nu)}{\partial(\gamma_3^{(0)}, \gamma_4^{(0)}, \nu)}$$

$$= \frac{T}{4\sigma^2}\left[\frac{\partial P_1(\omega, 0)}{\partial \omega}\right]_{\omega = \omega^*} \neq 0$$

which follows from § 4. Hence the conditions (31) and (32), which are the necessary and sufficient conditions for the formation of the formal series, are found to be the necessary and sufficient conditions for the existence of the solution of our problem as well.

Since the series representing the solution of the problem is determined uniquely, the convergence of these series becomes obvious.

Acknowledgement

I shall take this opportunity to express my heartiest thanks to Dr. N. S. Nagendra Nath, F.N.I., for his help in the publication of the paper.

References

Choudhry, R. K. (1964). Transformation of Birkhoff's rings into themselves and rotational numbers. *Proc. natn. Inst. Sci. India*, A **30**, 689–706.

——— (1965). Birkhoff's rings in the restricted problem of three bodies in a three-dimensional coordinate system. *Proc. Indian Acad. Sci.*, A **62**, 252–270.

——— (1966). Periodic solution of the third kind in the elliptic restricted problem of three bodies and the instability of the generating solution (Russian). *Byull. Inst. teor. Astr., Leningr.*, **10**, No. 8 (121), 523–536.

Duboshin, G. N. (1964). Nebesnaya mekhanika. Analiticheskie i kachestvennye metody. Fizmatgiz, Moskva.

Krasinskii, G. A. (1963). Zamknutye traektorii v ploskoĭ krugovoĭ ogranichennoĭ zadache trekh tel. *Byull. Inst. teor. Astr., Leningr.*, **9**, No. 2 (105), 154–167.

Malkin, I. G. (1956). Some problems of theory of non-linear oscillations (Russian). Moscow.

Szebehely, V., and Giacaglia, G. E. O. (1964). On the elliptic restricted problem of three bodies. *Am. Astr. J.*, **69**, 230–235.