STABILITY OF A COLLAPSING OR EXPLODING CYLINDRICAL SHELL OF CONDUCTING FLUID IN THE PRESENCE OF A MAGNETIC FIELD

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In this paper we have discussed the stability of a collapsing conducting cylindrical shell of incompressible ideally conducting fluid in the presence of a magnetic field assuming the shell to be of finite thickness. Before any disturbance is imposed on the system, we assume that the shell is moving radially at a uniform speed. We have set the problem as an initial value problem and have considered the stability of the system against the axial disturbances. To avoid cumbersome analysis we have concentrated only on the small wave numbers, and consequently expanded the amplitudes of the perturbations in powers of the wave number. We have evaluated the solution up to the first power of the wave number in both exploding and imploding cases. In the exploding case we find there is overstability in which the frequency of the growing oscillations is proportional to the rate of collapse, the thickness of the shell and the initial magnetic field. In the imploding case there is again instability in which the perturbation increases logarithmically, the growth rate of the instability being proportional to the rate of the collapse and the thickness of the shell. Thus we find that the finite thickness of the shell, the rate of collapse and the initial trapped field have a significant influence on the stability of the shell.

1. Introduction

In an attempt to produce large magnetic fields of the order of megagauss, Linhart (1960), Fowler *et al.* (1960) have discussed the stability of a radially accelerated cylindrical surface of conducting material in the presence of axial or azimuthal magnetic field either situated outside the shell or within the shell. Recently, Harris (1962) has considered the same problem in the presence of azimuthal magnetic field produced by an axial current carried by the shell both for axial and azimuthal disturbances. He has also discussed the problem when the magnetic field is axial and is situated either inside or outside the shell. He evaluates the growth rates for the axial disturbance, azimuthal disturbance and the combination of the two. He predicts the instability for all values of axial and the azimuthal wave numbers and attributes it to the neglect of the finite thickness of the shell.

In the present paper we have considered a cylindrical shell of finite thickness composed of incompressible and ideally conducting fluid in the presence of axial and azimuthal magnetic field prevailing in the shell and in the vacua outside the shell. We have set the problem as an initial value problem and have considered the stability of the system against the axial disturbances. In the case of exploding shell, we find overstability for small values of the wave number, while in the case of imploding shell there is a pure instability in which the disturbances grow logarithmically. We may note that we have carried the calculations in the latter case till the inner vacuum is extinct, for the linearized treatment may not be valid after this stage.

2. Initial State

Let at time $t = 0$, the inner and the outer dimensionless radii of the cylinder be 1 and m; the state of the dimensionless magnetic field at this instant is given by the following:

$$\vec{H} = \begin{bmatrix} \left(0, \frac{1}{r}, H_0\right), & 0 < r < 1 \\ \left(0, \frac{1}{r}, H_1\right), & 1 < r < m \\ \left(0, \frac{1}{r}, H_0\right), & r > m \end{bmatrix}. \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

We note that the postulated azimuthal magnetic field may be realized by a strong axial current.

We shall assume that the fluid is ideally conducting and neglect the displacement currents. Assuming that at the instant $t = 0$ the entire system is at rest, we can show that the electric field and current density are zero everywhere, on both the surfaces of the shell we have an azimuthal surface current given by

$$\vec{J}^* = (0, H_0 - H_1, 0) \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

and the total pressure everywhere is given by

$$P = \tfrac{1}{2}\left(H_0^2 + \frac{1}{r^2}\right). \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (3)$$

Starting from this initial state, we impose a radial velocity v_r on it and calculate the physical and the dynamical state at any time $t > 0$. We recall that in the vacua we have simply to solve the electromagnetic equations, while in the fluid we have to solve the equations of continuity and momentum in conjunction with Maxwell's equations. In order to evaluate the arbitrary constants and functions in these solutions, we have to satisfy the usual electromagnetic conditions and the dynamical condition which we have taken in the form of the equality of the total pressure on both sides of the surfaces. We report below the solutions for this undisturbed unsteady problem on the assumption that the radial velocity of a fluid element at any time t is a function of distance from the axis of the shell:

(*a*) *Exploding case:*

Conducting fluid: $R_i \leqslant r \leqslant R_0$,

where

$$R_i^2 = 1+t \text{ and } R_0^2 = m^2+t;$$

$$\vec{v} = \left(\frac{1}{2r}, 0, 0\right); \quad \vec{H} = \left(0, \frac{r}{r^2-t}, H_1\right);$$

$$\vec{E} = \left(0, \frac{H_1}{2r}, -\frac{1}{2(r^2-t)}\right); \quad \vec{J} = \left(0, 0, -\frac{2t}{(r^2-t)^2}\right);$$

$$p_{\text{plasma}} = \tfrac{1}{2}(H_0^2-H_1^2)+\frac{1}{8R_i^2V_A^2}+\frac{t}{2}-\frac{t}{(r^2-t)^2}-\frac{1}{8r^2V_A^2}.$$

Inner vacuum: $r < R_i$

$$\vec{H} = \left(0, \frac{1+t}{r}, H_0\right),$$

$$\vec{E} = \left[0, \frac{H_0}{2r}, -\tfrac{1}{2}\left(1+\ln\frac{R_i^2}{r^2}\right)\right],$$

$$p_{\text{magnetic}} = \tfrac{1}{2}\left[\left(\frac{1+t}{r}\right)^2+H_0^2\right].$$

Outer vacuum: $r > R_0$

$$\vec{H} = \left(0, \frac{m^2+t}{r}\cdot\frac{1}{m^2}, \alpha(t)\right),$$

$$\vec{E} = \left(0, -\frac{\dot{\alpha}(t)r}{2}+\frac{\dot{\alpha}(t)}{2r}(m^2+t)+\frac{\alpha(t)}{2r}, -\frac{1}{2m^2}\left(1+\ln\frac{m^2+t}{r^2}\right)\right);$$

$$p_{\text{magnetic}} = \tfrac{1}{2}\left[\left(\frac{m^2+t}{r}\cdot\frac{1}{m^2}\right)^2+\alpha^2(t)\right].$$

Inner surface: $r = R_i$

$$\vec{J}^* = (0, H_0-H_1, 0).$$

Outer surface: $r = R_0$

$$\vec{J}^* = (0, \alpha(t)-H_1, 0),$$

where

$$\alpha(t) = \left(H_0^2+\frac{1}{4R_i^2V_A^2}+t-\frac{2t}{m^4}-\frac{1}{4R_0^2V_A^2}\right)^{\frac{1}{2}}. \qquad \text{(4)}$$

V_A = Alfvén velocity for the azimuthal field rendered dimensionless by using the initial velocity of the inner surface of the plasma shell as the typical velocity.

We note that during this phase of motion the surface current on the inner boundary of the shell remains unchanged.

(*b*) *Imploding case:*

In the imploding case the solutions are obtained by putting $-t$ for t in the above solutions.

3. Perturbation Equations

At any time t we impose axial perturbations on the system and study its growth in time. It is convenient to measure time from the initial state rather than from the state when the disturbances are imposed. Denoting the perturbations in the velocity, pressure, magnetic field, electric field and the current density by $\tilde{\vec{v}}$, $\tilde{p}$, $\tilde{\vec{E}}$, $\tilde{\vec{H}}$ and $\tilde{\vec{J}}$ we have the following equations determining them in dimensionless form:

Plasma:

$$\frac{\partial \tilde{\vec{v}}}{\partial t}+\left(\tilde{\vec{v}}\,.\,\nabla\right)\vec{v}+\left(\vec{v}\,.\,\nabla\right)\tilde{\vec{v}} = V_A^2\left[-\text{grad}\,\tilde{p}+\text{curl}\,\tilde{\vec{H}}\times\vec{H}+\text{curl}\,\vec{H}\times\tilde{\vec{H}}\right], \qquad (5)$$

$$\frac{\partial \tilde{\vec{H}}}{\partial t} = \text{curl}\left(\vec{v}\times\tilde{\vec{H}}\right)+\text{curl}\left(\tilde{\vec{v}}\times\vec{H}\right), \qquad (6)$$

$$\text{div}\,\tilde{\vec{v}} = 0, \qquad (7)$$

$$\text{div}\,\tilde{\vec{H}} = 0, \qquad (8)$$

$$\tilde{\vec{E}} = -\vec{v}\times\tilde{\vec{H}}-\tilde{\vec{v}}\times\vec{H}. \qquad (9)$$

Inner and outer vacua:

$$\text{div}\,\tilde{\vec{H}} = 0, \qquad (10)$$

$$\text{curl}\,\tilde{\vec{H}} = 0, \qquad (11)$$

$$\text{curl}\,\tilde{\vec{E}} = -\frac{\partial \tilde{\vec{H}}}{\partial t}, \qquad (12)$$

$$\text{div}\,\tilde{\vec{E}} = 0. \qquad (13)$$

For considering the axial disturbances we shall put the perturbations as $\tilde{x} = \hat{x}e^{ikz}$ where z is the axial coordinate rendered dimensionless with respect to the initial inner radius of the plasma.

The boundary conditions satisfied by the perturbations are the following:

$$\vec{n}\cdot\left[\overset{\rightleftarrows}{H}\right]+\overset{\rightleftarrows}{n}\cdot\left[\vec{H}\right]=0, \qquad (14)$$

$$\vec{n}\times\left[\overset{\rightleftarrows}{H}\right]+\overset{\rightleftarrows}{n}\times\left[\vec{H}\right]=\overset{\rightleftarrows}{J}{}^{*}, \qquad (15)$$

$$\vec{n}\times\left[\overset{\rightleftarrows}{E}\right]+\overset{\rightleftarrows}{n}\times\left[\vec{E}\right]=\tilde{u}\left[\vec{H}\right]+u\left[\overset{\rightleftarrows}{H}\right], \qquad (16)$$

$$\vec{n}\cdot\left[\overset{\rightleftarrows}{E}\right]+\overset{\rightleftarrows}{n}\cdot\left[\vec{E}\right]=\tilde{q}^{*}, \qquad (17)$$

$$\vec{n}[\tilde{p}]+\overset{\rightleftarrows}{n}[p]=\overset{\rightleftarrows}{J}{}^{*}\times\bar{\vec{H}}+\vec{J}{}^{*}\times\bar{\overset{\rightleftarrows}{H}}+\tilde{q}^{*}\bar{\vec{E}}+q^{*}\bar{\overset{\rightleftarrows}{E}}, \qquad (18)$$

$$\vec{n}\cdot\overset{\rightleftarrows}{v}+\overset{\rightleftarrows}{n}\cdot\vec{v}=\tilde{u}, \qquad (19)$$

where u stands for the velocity of a boundary and $\tilde{q}^*$ and $\overset{\rightleftarrows}{J}{}^*$ denote the perturbations in the surface charge density and the surface current density respectively. Also a bar over a physical quantity denotes the mean of its value just inside and just outside the surface. It may not be out of place to mention that the equation of a boundary after perturbation is

$$r=R_{i,0}+\delta r_{i,0}e^{ikz} \qquad (20)$$

where δ_r is the radial displacement of the boundary and $R_{i,0}$ stands for the inner radius R_i for the equation of the inner surface of the shell and for R_0 for the equation of the outer surface of the shell. Also from (20) it can be seen that the disturbance in the unit normal at the surface of the shell is

$$\overset{\rightleftarrows}{n}=(0,\ 0,\ -ik\delta re^{ikz})$$

so that

$$\hat{\vec{n}}=(0,\ 0,\ -ik\hat{\delta}r). \qquad (21)$$

4. Solutions of the Problem in the Exploding Case

The equations as they stand are extremely complicated and, therefore, we have solved them for small wave number k corresponding to the large wavelength disturbances which are of particular interest in such problems. Consequently, we set

$$\hat{X}=\hat{X}_0+k\hat{X}_1+k^2\hat{X}_2+\ldots \qquad (22)$$

and evaluate only the first two terms in this expansion.

Without going into the details we record below the zeroth order perturbation equations and their solutions.

Plasma:

Zeroth order equations:

$$\frac{\partial \hat{v}_{r0}}{\partial t}+\frac{1}{2r}\frac{\partial \hat{v}_{r0}}{\partial r}+\hat{v}_{r0}\frac{\partial}{\partial r}\left(\frac{1}{2r}\right) = V_A^2\left[-H_1\frac{\partial \hat{H}_{z0}}{\partial r}-\frac{H_\theta}{r}\frac{\partial}{\partial r}(r\hat{H}_{\theta 0})-\frac{\hat{H}_{\theta 0}}{r}\frac{\partial}{\partial r}(rH_\theta)-\frac{\partial \hat{p}_0}{\partial r}\right], \quad (23)$$

$$\frac{\partial \hat{v}_{\theta 0}}{\partial t}+\frac{1}{2r}\frac{\partial \hat{v}_{\theta 0}}{\partial r}+\frac{\hat{v}_{\theta 0}}{2r^2} = V_A^2\left[\frac{\hat{H}_{r0}}{r}\frac{\partial}{\partial r}(rH_\theta)\right], \quad (24)$$

$$\frac{\partial \hat{v}_{z0}}{\partial t}+\frac{1}{2r}\frac{\partial \hat{v}_{z0}}{\partial r} = 0, \quad (25)$$

$$\frac{\partial \hat{H}_{r0}}{\partial t} = 0, \quad (26)$$

$$\frac{\partial \hat{H}_{\theta 0}}{\partial t} = -\frac{\partial}{\partial r}\left(\frac{\hat{H}_{\theta 0}}{2r}\right)-\frac{\partial}{\partial r}(H_\theta \hat{v}_{r0}), \quad (27)$$

$$\frac{\partial \hat{H}_{z0}}{\partial t} = -\frac{1}{2r}\frac{\partial}{\partial r}\hat{H}_{z0}-H_1\frac{\partial}{\partial r}(r\hat{v}_{r0}), \quad (28)$$

$$\frac{1}{r}\frac{\partial}{\partial r}(r\hat{v}_{r0}) = 0, \quad (29)$$

$$\frac{1}{r}\frac{\partial}{\partial r}(r\hat{H}_{r0}) = 0. \quad (30)$$

We note that equations (26) and (30) are equivalent and thus we can omit one of them.

Zeroth order solutions:

We have solved the equations (23) to (27) and the solutions obtained are recorded below:

$$\hat{v}_{r0} = \frac{F_0(t)}{r},$$

$$\hat{v}_{\theta 0} = \frac{BV_A^2 r(r^2-2t)}{(r^2-t)^2}+\frac{1}{r}f_1(r^2-t),$$

$$\hat{v}_{z0} = f(r^2-t),$$

$$\hat{H}_{r0} = \frac{B}{r},$$

$$\hat{H}_{\theta 0} = \frac{2r\int^t F_0(t)\,dt}{(r^2-t)^2}+rh(r^2-t),$$

$$\hat{H}_{z0} = g(r^2-t),$$

$$\hat{p}_0 = \phi_0(t) - \frac{1}{V_A^2}\left[F_0'(t)\ln r + \frac{F_0(t)}{2r^2}\right] - H_1 g(r^2-t)$$

$$-\left\{\frac{1}{(r^2-t)^2}+\frac{2t}{(r^2-t)^3}\right\}\int^t F_0(t)\,dt - \frac{r^2 h(r^2-t)}{(r^2-t)}$$

$$-\int^r \frac{2rh(r^2-t)}{(r^2-t)^2}\,dr.$$

Also from eqn. (9) we can obtain the components of the electric field as given below:

$$\hat{E}_{r0} = -\left[\frac{BV_A^2 H_1 r(r^2-2t)}{(r^2-t)^2} + \frac{H_1}{r} f_1(r^2-t) - \frac{r}{r^2-t} f(r^2-t)\right],$$

$$\hat{E}_{\theta 0} = \left[\frac{1}{2r} g(r^2-t) + \frac{H_1 F_0(t)}{r}\right],$$

$$\hat{E}_{z0} = -\left[\frac{\int^t F_0(t)\,dt}{(r^2-t)^2} + \tfrac{1}{2}h(r^2-t) + \frac{F_0(t)}{(r^2-t)}\right], \qquad \ldots \quad \ldots \quad \ldots \quad (31)$$

where $F_0(t)$ and $\phi_0(t)$ are arbitrary constants of integration which are functions of t and B is a pure constant. Also

$$f(r^2-t),\ f_1(r^2-t),\ g(r^2-t) \text{ and } h(r^2-t)$$

are the arbitrary functions of the argument.

Zeroth order solutions for the inner vacuum:

We have solved the eqns. (10) to (13) and have obtained the following set of zeroth order solutions for the vacuum $r < R_i$:

$$\left.\begin{aligned} \hat{H}_{r0} &= 0, & \hat{E}_{r0} &= 0, \\ \hat{H}_{\theta 0} &= 0, & \hat{E}_{\theta 0} &= 0, \\ \hat{H}_{z0} &= 0, & \hat{E}_{z0} &= f_2(t), \end{aligned}\right\} \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (32)$$

where $f_2(t)$ is an arbitrary function of t.

Zeroth order solutions for the outer vacuum:

From the same set of equations we deduce the following set of solutions for the outer vacuum $r > R_0$:

$$\left.\begin{aligned} \hat{H}_{r0} &= \frac{\phi_2(t)}{r}, & \hat{E}_{r0} &= \frac{f_3(t)}{r}, \\ \hat{H}_{\theta 0} &= 0, & \hat{E}_{\theta 0} &= 0, \\ \hat{H}_{z0} &= 0, & \hat{E}_{z0} &= 0, \end{aligned}\right\} \qquad \ldots \quad \ldots \quad \ldots \quad (33)$$

where $\phi_2(t)$ and $f_3(t)$ are arbitrary functions of t.

Zeroth order boundary conditions:

$$(\hat{H}_{r0})_i = 0,$$
$$(\hat{H}_{r0})_0 = 0,$$
$$(\hat{E}_{z0})_0-(\hat{E}_{z0})_i = u[(\hat{H}_{\theta 0})_i-(\hat{H}_{\theta 0})_0],$$
$$(\hat{E}_{\theta 0})_i-(\hat{E}_{\theta 0})_0 = \hat{u}_0[H_{zi}-H_{z0}]+u[(\hat{H}_{z0})_i-(\hat{H}_{z0})_0],$$
$$(\hat{E}_{r0})_i-(\hat{E}_{r0})_0 = 0,$$
$$(\hat{v}_{r0})_i = \frac{d}{dt}(\delta\hat{r}_0),$$
$$(\hat{p}_0)_i = \tfrac{1}{2}[(H_{zi}+H_{z0})\{(\hat{H}_{z0})_0-(\hat{H}_{z0})_i\} - \{(\hat{H}_{\theta 0})_i-(\hat{H}_{\theta 0})_0\}(H_{\theta i}+H_{\theta 0}) + \{(\hat{H}_{z0})_i+(\hat{H}_{z0})_0\}(H_{z0}-H_{zi})], \quad \ldots \quad \ldots \quad (34)$$

where the suffixes i and 0 stand for the value (34) of a physical parameter inside and outside the shell respectively. It may be mentioned that while writing (34) we have made use of the fact that the jump in the θ-components of the initial magnetic fields at the surfaces of the shell is zero.

After making use of the conditions (34) we deduce the following values of the constants of integration and the arbitrary functions occurring in the solutions of the equations of the system:

$$(\delta\hat{R}_0)_0 = 0\,;\ (\delta\hat{R}_i)_0 = 0\,;\ f_3(t) = 0\,;\ B = 0\,;\ f_2(t) = 0$$
$$\phi_0(t) = 0\,;\ h(r^2-t) = f_1(r^2-t) = f(r^2-t) \equiv 0\,;\ F_0(t) = 0 \quad \ldots \quad (35)$$

and $\qquad \phi_2(t) = 0$

where $(\delta\hat{R}_0)_0$ and $(\delta\hat{R}_i)_0$ are the zeroth order displacements of the external and the internal boundaries respectively. We shall now write down the zeroth order solutions in the three regions of the system as follows:

Plasma:

$$\vec{v}_0 = (0,0,0)\,; \qquad \hat{p}_0 = -H_1 g(r^2-t)\,e^{ikz},$$
$$\vec{H}_0 = (0,0,g(r^2-t))\,e^{ikz}\,; \quad \vec{E}_0 = \left(0, \frac{1}{2r}g(r^2-t), 0\right)e^{ikz}, \quad \ldots \quad (36)$$

Inner vacuum:

$$\vec{H}_0 = (0,0,0)\,; \qquad \vec{E}_0 = (0,0,0). \quad \ldots \quad \ldots \quad \ldots \quad (37)$$

Outer vacuum

$$\vec{H}_0 = (0,0,0)\,; \qquad \vec{E}_0 = (0,0,0). \quad \ldots \quad \ldots \quad \ldots \quad (38)$$

It may be noted that the solutions in the imploding case are same except that t is to be changed into $-t$. Later on, we shall see from (51) that $g(r^2-t) \equiv 0$ so that all the zeroth order perturbations are zero as they should be.

5. First Order Equations and their Solutions in the Three Regions of the System

Plasma: The set of the first order equations after feeding the set of the zeroth order solutions is

$$\frac{\partial \hat{v}_{r1}}{\partial t}+\frac{1}{2r}\frac{\partial \hat{v}_{r1}}{\partial r}+\hat{v}_{r1}\frac{\partial}{\partial r}\left(\frac{1}{2r}\right) = V_A^2\left[-H_1\frac{\partial \hat{H}_{z1}}{\partial r} - \frac{H_\theta}{r}\frac{\partial}{\partial r}(r\hat{H}_{\theta 1}) - \frac{\hat{H}_{\theta 1}}{r}\frac{\partial}{\partial r}(rH_\theta) - \frac{\partial \hat{p}_1}{\partial r}\right], \quad (39)$$

$$\frac{\partial \hat{v}_{\theta 1}}{\partial t}+\frac{1}{2r}\frac{\partial \hat{v}_{\theta 1}}{\partial r}+1/2r^2\hat{v}_{\theta 1} = V_A^2\frac{\hat{H}_{r1}}{r}\frac{\partial}{\partial r}(rH_\theta), \quad (40)$$

$$\frac{\partial \hat{v}_{z1}}{\partial t}+\frac{1}{2r}\frac{\partial \hat{v}_{z1}}{\partial r} = iH_1V_A^2 g(r^2-t), \quad (41)$$

$$\frac{\partial \hat{H}_{r1}}{\partial t} = \frac{i}{2r}g(r^2-t), \quad (42)$$

$$\frac{\partial \hat{H}_{\theta 1}}{\partial t} = -\frac{\partial}{\partial r}\left(\frac{\hat{H}_{\theta 1}}{2r}\right) - \frac{\partial}{\partial r}(H_\theta \hat{v}_{r1}), \quad (43)$$

$$\frac{\partial \hat{H}_{z1}}{\partial t} = -\frac{1}{2r}\frac{\partial \hat{H}_{z1}}{\partial r} - H_1\frac{\partial}{\partial r}(r\hat{v}_{r1}), \quad (44)$$

$$\frac{1}{r}\frac{\partial}{\partial r}(r\hat{v}_{r1}) = 0, \quad (45)$$

$$\frac{1}{r}\frac{\partial}{\partial r}(r\hat{H}_{r1})+ig(r^2-t) = 0. \quad (46)$$

We have solved the eqns. (39) to (46) and the solutions obtained are as follows:

$$\hat{v}_{r1} = \frac{F_1(t)}{r},$$

$$\hat{H}_{r1} = \frac{c}{r} - \frac{i}{r}\int^r rg(r^2-t)\,dr,$$

$$\hat{v}_{\theta 1} = \frac{g_3(r^2-t)}{r} - \frac{4V_A^2}{r(r^2-t)^2}\int^r \frac{\left\{c-i\int^r rg(r^2-t)\,dr\right\}}{(r^2-c_1)^{-1}}\,r\,dr,$$

where after integration we have to put $c_1 = r^2-t$,

$$\hat{H}_{z1} = g_4(r^2-t),$$

$$\hat{H}_{\theta 1} = \frac{2r\int^t F_1(t)\,dt}{(r^2-t)^2} + rg_1(r^2-t),$$

$$\hat{v}_{z1} = \left\{ iH_1V_A^2 g(r^2-t) \right\} t + g_2(r^2-t),$$

$$\hat{p}_1 = \phi_1(t) - \frac{1}{V_A^2}\left\{ F_1'(t)\ln r + \frac{F_1(t)}{2r^2} \right\} - H_1 g_4(r^2-t) - \left\{ \frac{1}{(r^2-t)^2} + \frac{2t}{(r^2-t)^3} \right\} \int^t F_1(t)\,dt$$

$$- \frac{r^2 g_1(r^2-t)}{(r^2-t)} - \int^r \frac{2rg_1(r^2-t)}{(r^2-t)^2}\,dr.$$

Also from eqn. (9) the components of the electric field are

$$\hat{E}_{r1} = -(H_1\hat{v}_{\theta 1} - H_\theta \hat{v}_{z1}),$$

$$\hat{E}_{\theta 1} = \left(\frac{1}{2r}\hat{H}_{z1} + H_1\hat{v}_{r1} \right),$$

$$\hat{E}_{z1} = -\left(\frac{1}{2r}\hat{H}_{\theta 1} + H_\theta \hat{v}_{r1} \right), \qquad \ldots \ldots \ldots \quad (47)$$

where $F_1(t)$ and $\phi_1(t)$ are arbitrary functions of integration and c is a pure constant. Also g_1, g_2, g_3 and g_4 are the arbitrary functions of the argument.

First order solutions in the inner vacuum :

Solving the eqns. (10)–(13) we obtain the following first order solutions for the vacuum $r < R_i$

$$\left.\begin{aligned} \hat{H}_{r1} &= 0, & \hat{E}_{r1} &= 0, \\ \hat{H}_{\theta 1} &= 0, & \hat{E}_{\theta 1} &= -\frac{i\psi'(t)r}{2}, \\ \hat{H}_{z1} &= i\psi(t), & \hat{E}_{z1} &= f_4(t), \end{aligned}\right\} \qquad \ldots \ldots \ldots \quad (48)$$

where $\psi(t)$ and $f_4(t)$ are arbitrary functions of t.

First order solutions in the outer vacuum :

Solving the same set of equations we have the following set of solutions for the outer vacuum $r > R_0$:

$$\left.\begin{aligned} \hat{H}_{r1} &= 0, & \hat{E}_{r1} &= \frac{f_5(t)}{r}, \\ H_{\theta 1} &= 0, & \hat{E}_{\theta 1} &= \frac{f_1(t)}{r}, \\ \hat{H}_{z1} &= 0, & \hat{E}_{z1} &= f_7(t), \end{aligned}\right\} \qquad \ldots \ldots \ldots \quad (49)$$

where $f_1(t)$, $f_5(t)$ and $f_7(t)$ are arbitrary functions of the argument.

First order boundary conditions:

$$(\hat{H}_{r1})_i = 0,$$

$$(\hat{H}_{r1})_0 = 0,$$

$$(\hat{E}_{z1})_0-(\hat{E}_{z1})_i = u[(\hat{H}_{\theta 1})_i-(\hat{H}_{\theta 1})_0],$$

$$(\hat{E}_{\theta 1})_i-(\hat{E}_{\theta 1})_0 = \hat{u}_1[H_{zi}-H_{z0}]+u[(\hat{H}_{z1})_i-(\hat{H}_{z1})_0],$$

$$(\hat{E}_{r1})_i-(\hat{E}_{r1})_0 = 0,$$

$$(\hat{v}_{r1})_i = \frac{d}{dt}\left(\hat{\delta r}_1\right),$$

$$(\hat{p}_1)_i = \tfrac{1}{2}[(H_{zi}+H_{z0})\{(\hat{H}_{z1})_0-(\hat{H}_{z1})_i\} -(H_{\theta i}+H_{\theta 0})\{(\hat{H}_{\theta 1})_i-(\hat{H}_{\theta 1})_0\} +(H_{z0}-H_{zi})\{(\hat{H}_{z1})_i+(\hat{H}_{z1})_0\}]. \quad \ldots \quad (50)$$

We note that the above conditions (50) are simplified on account of the fact $(\hat{\delta R}_i)_0 = (\hat{\delta R}_0)_0 = 0$ got earlier. After applying the boundary conditions (50) we get the following set of equations which enable us to determine the arbitrary constants and functions in the solution:

$$g(r^2-t) \equiv 0\,;\ f_5(t) = 0\,;\ C = 0, \quad \ldots \quad (51)$$

$$g_1(r^2-t) = g_2(r^2-t) = g_3(r^2-t) \equiv 0, \quad \ldots \quad (52)$$

$$\psi'(t)+\frac{\psi(t)}{R_i^2} = \frac{2iH_0F_1(t)}{R_i^2}, \quad \ldots \quad (53)$$

$$\frac{F_1(t)}{R_i}-\frac{x}{2R_i^2} = \frac{dx}{dt}, \quad \ldots \quad (54)$$

$$\frac{F_1(t)}{R_0}-\frac{y}{2R_0^2} = \frac{dy}{dt}, \quad \ldots \quad (55)$$

$$iH_0\psi(t) = \phi_1(t)-\frac{1}{V_A^2}\left\{F_1'(t)\ln R_i+\frac{F_1(t)}{2R_i^2}\right\} +\int^t F_1(t)\,dt+\frac{x}{4R_i^3V_A^2}, \quad \ldots \quad (56)$$

$$0 = \phi_1(t)-\frac{1}{V_A^2}\left\{F_1'(t)\ln R_0+\frac{F_1(t)}{2R_0^2}\right\} +\frac{1}{m^4}\int^t F_1(t)\,dt+\frac{y}{4R_0^3V_A^2}, \quad \ldots \quad (57)$$

$$f_4(t) = -F_1(t), \quad \ldots \quad (58)$$

$$f_7(t) = -\frac{F_1(t)}{m^2}, \quad \ldots \quad (59)$$

$$f_1(t) = \alpha(t)F_1(t)+\dot{\alpha}(t)R_0y, \quad \ldots \quad (59a)$$

where $(\delta R_i)_1 = x$ and $(\delta R_0)_1 = y$.

We shall take (53), (54), (55), (56) and (57) for determining the values of $\phi_1(t)$, $F_1(t)$, x, y and $\psi(t)$.
From (53) we have

$$i\psi(t) = \frac{-2H_0 \int^t F_1(t)\,dt}{R_i^2} + \frac{il_1}{R_i^2}$$

$$= \frac{-2H_0 Z_1}{R_i^2} + \frac{il_1}{R_i^2}, \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (60)$$

where l_1 is an arbitrary constant of integration and $Z_1 = \int^t F_1(t)\,dt$.

Similarly, from (53) to (54), we have

$$x = \frac{Z_1}{R_i} + \frac{l_2}{R_i}, \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (61)$$

$$y = \frac{Z_1}{R_0} + \frac{l_3}{R_0}, \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (62)$$

where l_2 and l_3 are arbitrary constants of integration.

Substituting (60) to (62) in the result got by subtracting (57) from (56), we have

$$\frac{d^2Z_1}{dt^2} - \frac{\bar{a}^2}{(1+t)(m^2+t)\ln\left(\dfrac{m^2+t}{1+t}\right)} \cdot \frac{dZ_1}{dt}$$

$$+2\left[\frac{\bar{a}^2(m^2+2t+1)}{4(m^2+t)^2(1+t)^2} + \frac{2V_z^2}{(1+t)^2} + V_A^2\left(1-\frac{1}{m^4}\right)\right] \cdot \frac{Z_1}{\ln\left(\dfrac{m^2+t}{1+t}\right)}$$

$$= 2\left[\frac{l_3}{4(m^2+t)^2} - \frac{l_2}{4(1+t)^2} + \frac{iV_A^2 H_0 l_1}{(1+t)}\right] \cdot \frac{1}{\ln\left(\dfrac{m^2+t}{1+t}\right)}, \qquad \ldots \quad (63)$$

where

$$V_z^2 = H_0^2 V_A^2$$

and

$$\bar{a}^2 = m^2 - 1 = R_0^2 - R_i^2.$$

Exploding case :

In this case t can take any value > 0, hence we have solved (63) asymptotically (Ince 1956; Tricomi 1960) when $t \to \infty$ and its general solution is as follows:

$$
\begin{aligned}
Z_1 = {} & D_1\xi^{\frac{1}{2}} \exp\left\{\frac{m_1}{3}\,\xi^3+(\alpha_2)_1\xi-\frac{\lambda_1}{2\xi^2}-\frac{\lambda_2}{4\xi^4}-\dots\right\} \\
& \times\left[1+\frac{(\alpha_2)_1^2+4\omega_3}{2m_1}\cdot\frac{1}{\xi}+O\left(\frac{1}{\xi^2}\right)\right] \\
& +D_2\xi^{\frac{1}{2}} \exp\left\{\frac{m_2}{3}\,\xi^3+(\alpha_2)_2\xi-\frac{\lambda_1}{2\xi^2}-\frac{\lambda_2}{4\xi^4}-\dots\right\} \\
& \times\left[1+\frac{(\alpha_2)_2^2+4\omega_3}{2m_2}\cdot\frac{1}{\xi}+O\left(\frac{1}{\xi^2}\right)\right] \\
& +\xi^{\frac{1}{2}} \exp\left\{\frac{m_2}{3}\,\xi^3+(\alpha_2)_2\xi-\frac{\lambda_1}{2\xi^2}-\frac{\lambda_2}{4\xi^4}-\dots\right\} \\
& \times\left[1+\frac{(\alpha_2)_2^2+4\omega_3}{2m_2}\cdot\frac{1}{\xi}+O\left(\frac{1}{\xi^2}\right)\right] \\
& \times\int^{\xi} \exp\left\{\frac{m_1}{3}\,\xi^3+(\alpha_2)_1\xi+\frac{\lambda_1}{2\xi^2}+\dots\right\} \\
& \times\frac{\left[\beta_1+\beta_1\left\{\dfrac{(\alpha_2)_1^2+4\omega_3}{2m_1}\right\}\dfrac{1}{\xi}+O\left(\dfrac{1}{\xi^2}\right)\right]d\xi}{(m_2-m_1)\xi^{\frac{5}{2}}+\left[\dfrac{m_1}{2m_2}\left\{(\alpha_2)_2^2+4\omega_3\right\}+(\alpha_2)_1^2-(\alpha_2)_2^2-\dfrac{m_2}{2m_1}\left\{(\alpha_2)_1^2+4\omega_3\right\}\right]\xi^{\frac{3}{2}}+\dots} \\
& -\xi^{\frac{1}{2}} \exp\left\{\frac{m_1}{3}\,\xi^3+(\alpha_2)_1\xi-\frac{\lambda_1}{2\xi^2}-\frac{\lambda_2}{4\xi^4}-\dots\right\} \\
& \times\left[1+\frac{(\alpha_2)_1^2+4\omega_3}{2m_1}\cdot\frac{1}{\xi}+O\left(\frac{1}{\xi^2}\right)\right]\cdot\int^{\xi} \exp\left\{\frac{m_2}{3}\,\xi^3+(\alpha_2)_2\xi+\frac{\lambda_1}{2\xi^2}+\dots\right\} \\
& \times\frac{\left[\beta_1+\beta_1\left\{\dfrac{(\alpha_2)_2^2+4\omega_3}{2m_2}\right\}\dfrac{1}{\xi}+O\left(\dfrac{1}{\xi^2}\right)\right]d\xi}{(m_2-m_1)\xi^{\frac{5}{2}}+\left[\dfrac{m_1}{2m_2}\left\{(\alpha_2)_2^2+4\omega_3\right\}+(\alpha_2)_1^2-(\alpha_2)_2^2-\dfrac{m_2}{2m_1}\left\{(\alpha_2)_1^2+4\omega_3\right\}\right]\xi^{\frac{3}{2}}+\dots}
\end{aligned}
\qquad (64)
$$

where D_1 and D_2 are constants of integration. Also

$$\lambda_1 = -\frac{m^2+1}{2},$$

$$\lambda_2 = \frac{5m^4+2m^2+5}{12},$$

$$\dots\dots\dots\dots\dots\dots\dots,$$

$$\omega_1 = \frac{2V_A^2}{\bar{a}^2}\left(1-\frac{1}{m^4}\right) > 0,$$

$$\omega_2 = \frac{2V_A^2}{\bar{a}^2}\left(1-\frac{1}{m^4}\right)\left(\frac{m^2+1}{2}\right) > 0,$$

$$\omega_3 = \frac{2}{\bar{a}^2}\left\{2V_Z^2 - V_A^2\left(1-\frac{1}{m^4}\right)\left(\frac{m^4-2m^2+1}{12}\right)\right\},$$

$$\ldots\ldots\ldots\ldots\ldots\ldots,$$

$$m_{1,2} = \pm 2i\omega_1^{\frac{1}{2}},$$

$$(\alpha_2)_{1,2} = \pm i\omega_2\omega_1^{-\frac{1}{2}},$$

$$\beta_1 = \frac{2i}{\bar{a}^2} V_A^2 H_0 l_1,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots,$$

and $t = \xi^2$.

From (64) we can write down the value of $F_1(t)$ as follows:

$$F_1(t) = \tfrac{1}{2}\xi^{\frac{3}{2}}\Bigg[D_1 \exp\left\{\frac{m_1}{3}\xi^3+(\alpha_2)_1\xi-\frac{\lambda_1}{2\xi^2}-\ldots\right\}$$
$$\times\left\{m_1+\frac{(\alpha_2)_1^2+4\omega_3}{2}\cdot\frac{1}{\xi}+O\left(\frac{1}{\xi^2}\right)\right\}$$
$$+D_2 \exp\left\{\frac{m_2}{3}\xi^3+(\alpha_2)_2\xi-\frac{\lambda_1}{2\xi^2}-\ldots\right\}$$
$$\times\left\{m_2+\frac{(\alpha_2)_2^2+4\omega_3}{2}\cdot\frac{1}{\xi}+O\left(\frac{1}{\xi^2}\right)\right\}$$
$$+\exp\left\{\frac{m_2}{3}\xi^3+(\alpha_2)_2\xi-\frac{\lambda_1}{2\xi^2}-\ldots\right\}\left\{m_2+\frac{(\alpha_2)_2^2+4\omega_3}{2}\cdot\frac{1}{\xi}+O\left(\frac{1}{\xi^2}\right)\right\}\left(\int^{\xi}\frac{R_1Z_{1,1}}{W}\,d\xi\right)$$
$$-\exp\left\{\frac{m_1}{3}\xi^3+(\alpha_2)_1\xi-\frac{\lambda_1}{2\xi^2}-\ldots\right\}\left\{m_1+\frac{(\alpha_2)_1^2+4\omega_3}{2}\cdot\frac{1}{\xi}+O\left(\frac{1}{\xi^2}\right)\right\}\left(\int^{\xi}\frac{R_1Z_{1,2}}{W}\,d\xi\right)\Bigg], \quad \ldots (65)$$

where

$$W = \text{Wronskian}\,[Z_{1,1}, Z_{1,2}],$$

$$Z_{1,1} = \xi^{\frac{1}{2}} \exp\left\{\frac{m_1}{3}\xi^3+(\alpha_2)_1\xi-\frac{\lambda_1}{2\xi^2}-\ldots\right\}$$
$$\times\left[1+\frac{(\alpha_2)_1^2+4\omega_3}{2m_1}\cdot\frac{1}{\xi}+O\left(\frac{1}{\xi^2}\right)\right],$$

$$Z_{1,2} = \xi^{\frac{1}{2}} \exp\left\{\frac{m_2}{3}\xi^3+(\alpha_2)_2\xi-\frac{\lambda_1}{2\xi^2}-\ldots\right\}$$
$$\times\left[1+\frac{(\alpha_2)_2^2+4\omega_3}{2m_2}\cdot\frac{1}{\xi}+O\left(\frac{1}{\xi^2}\right)\right],$$

and

$$R_1 = \beta_1 + \frac{\beta_2}{\xi^2} + O\left(\frac{1}{\xi^3}\right).$$

Thus we can be able to find all arbitrary constants of integration.

Conclusions.—We see that the disturbances in the plasma pressure and the plasma velocity grow with time. They manifest in the form of unbounded oscillations with their amplitudes varying as time and frequencies proportional to

$$N \equiv 2\left[2\frac{V_A^2}{\bar{a}^2}\left(1-\frac{1}{m^4}\right)\right]^{\frac{1}{2}}.$$

The perturbations in the plasma magnetic field die as they manifest in the form of damped oscillations. The perturbations in the plasma electric field are again harmonic with unbounded amplitudes and having the same frequency as the plasma pressure disturbances and plasma velocity disturbances.

Again we see that there are no perturbations in the outer vacuum magnetic field up to the order of approximation we have worked out the problem. The disturbances in the electric field in this vacuum again appear in the form of unbounded oscillations with a frequency proportional to N. Similarly, it can be seen that in the inner vacuum the magnetic field perturbations die with time. There occurs the so-called Landau-damping in these perturbations as their amplitudes are inversely proportional to time. The perturbations in the electric field in this vacuum again appear in the form of unbounded oscillations of which the frequency is proportional to N. Thus we see that the instability which occurs in the system is in fact in the form of over-stability. The frequency of such growing oscillations is proportional to the thickness of the shell through $\bar{a}^2$ the initial magnetic field and the rate of explosion through V_A^2 and to the initial radii of the shell through m.

6. Collapsing Case

In this case R_i will tend to zero when the inner vacuum of the shell is extinct. This means that $t \to 1$ and $R_0^2 \to \bar{a}^2$. We can show that here the following equation determines Z_1:

$$\frac{d^2Z_1}{dt^2} + \frac{\bar{a}^2}{(1-t)(m^2-t)\ln\left(\dfrac{m^2-t}{1-t}\right)} \cdot \frac{dZ_1}{dt}$$

$$+2\left[V_A^2\left(1-\frac{1}{m^4}\right) + \frac{\bar{a}^2(m^2-2t+1)}{4(m^2-t)^2(1-t)^2} + \frac{2V_s^2}{(1-t)^2}\right] \cdot \frac{Z_1}{\ln\left(\dfrac{m^2-t}{1-t}\right)}$$

$$= 2\left[\frac{l_3}{4(m^2-t)^2} - \frac{l_2}{4(1-t)^2} + \frac{iV_A^2H_0l_1}{(1-t)}\right] \cdot \frac{1}{\ln\left(\dfrac{m^2-t}{1-t}\right)}. \qquad (66)$$

We shall effect the transformation

$$t = 1-\xi, \quad \text{where } \xi \to 0 \quad \text{as } t \to 1,$$

in order to study how the disturbance behaves when the inner vacuum is becoming extinct.

The transformed equation is

$$\frac{d^2Z_1}{d\xi^2} - \frac{\bar{a}^2}{\{(m^2-1)+\xi\}\xi} \cdot \frac{1}{\ln\left(\dfrac{m^2-1+\xi}{\xi}\right)} \frac{dZ_1}{d\xi}$$

$$+2\left[V_A^2\left(1-\frac{1}{m^4}\right)+\frac{\bar{a}^2(2m^2+2\xi-\bar{a}^2-2)}{4(m^2-1+\xi)^2\xi^2}+\frac{2V_2^2}{\xi}\right]\frac{Z_1}{\ln\left(\dfrac{m^2-1+\xi}{\xi}\right)}$$

$$= 2\left[\frac{l_3}{4(m^2-1+\xi)^2}-\frac{l_2}{4\xi^2}+\frac{iV_A^2H_0l_1}{\xi}\right]\cdot\frac{1}{\ln\left(\dfrac{m^2-1+\xi}{\xi}\right)}. \quad \ldots \quad (67)$$

Again put $\xi = (m^2-1)\dfrac{1}{\nu} = \dfrac{\bar{a}^2}{\nu}$ so that $\nu \to \infty$ as $\xi \to 0$, and we may apply the asymptotic methods.

This transformation reduced the eqn. (67) to the following form:

$$\frac{d^2Z_1}{d\nu^2}+\left[\frac{2}{\nu}+\frac{1}{(\nu+1)\ln(\nu+1)}\right]\frac{dZ_1}{d\nu}$$

$$+2\left[V_A^2\left(1-\frac{1}{m^4}\right)\frac{\bar{a}^4}{\nu^4}+\frac{\nu+2}{4\nu(\nu+1)^2}+\frac{V_z^2\bar{a}^2}{\nu^3}\right]\frac{Z_1}{\ln(1+\nu)}$$

$$= 2\left[\frac{l_3}{4\nu^2(\nu+1)^2}-\frac{l_2}{4\nu^2}+\frac{iV_A^2H_0l_1}{\nu^3}\right]\frac{1}{\ln(1+\nu)}. \quad \ldots \quad \ldots \quad (68)$$

Retaining the dominating terms only in the coefficients of (68) we have

$$\frac{d^2Z_1}{d\nu^2}+\frac{2}{\nu}\cdot\frac{dZ_1}{d\nu}+\frac{1}{2\nu^2}\cdot\frac{Z_1}{\ln\nu} = -\frac{l_2}{2\nu^2\ln\nu} \quad \ldots \quad \ldots \quad (69)$$

and putting $Z_1 = Z_1'-l_2$ it reduces to

$$\frac{d^2Z_1'}{d\nu^2}+\frac{2}{\nu}\frac{dZ_1'}{d\nu}+\frac{1}{2\nu^2}\cdot\frac{Z_1'}{\ln\nu} = 0. \quad \ldots \quad \ldots \quad \ldots \quad (70)$$

This can be further reduced by putting $\nu = e^{\xi_1}$:

$$\frac{d^2Z_1'}{d\xi_1^2}+\frac{dZ_1'}{d\xi_1}+\frac{Z_1'}{2\xi_1} = 0 \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (71)$$

where

$$\xi_1 \to \infty \text{ when } \nu \to \infty.$$

Equation (71) admits the following asymptotic solution:

$$Z_1 = A'\left\{\ln\left(\frac{\bar{a}^2}{1-t}\right)\right\}^{-\frac{1}{4}}\left[1+\frac{3}{4}\,\frac{1}{\ln\left(\frac{\bar{a}^2}{1-t}\right)}-\frac{45}{32}\cdot\frac{1}{\ln\left(\frac{\bar{a}^2}{1-t}\right)^2}+\ldots\right]$$

$$+B'\left(\frac{\bar{a}^2}{1-t}\right)^{-1}\left\{\ln\left(\frac{\bar{a}^2}{1-t}\right)\right\}^{\frac{1}{4}}\left[1+\frac{1}{4}\,\frac{1}{\ln\left(\frac{\bar{a}^2}{1-t}\right)}-\frac{3}{32}\cdot\frac{1}{\ln\left(\frac{\bar{a}^2}{1-t}\right)^2}+\ldots\right]-l_2, \quad (72)$$

where A' and B' are arbitrary constants.

Conclusions.—We see that the disturbances in the plasma pressure become large as the shell collapses and at the same time the disturbances in the velocity of the shell also grows. The growth rate of this instability is proportional to $\left\{\ln\left(\frac{\bar{a}^2}{1-t}\right)\right\}^{\frac{1}{4}} \equiv \bar{N}$. The perturbation in the electric field inside the shell also grows with the same rate.

In the vacua outside the shell, the perturbations in the electric field grow as $\bar{N}$ as $t \to 1$. Thus, we conclude that the collapsing shell is unstable and the growth-rate is proportional to the thickness of the shell through $\bar{a}^2$ and to the rate of collapse through the dimensionless time t.

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