ON $\Sigma r(n)r(n+a)$

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In 1932, Estermann (1932) obtained an asymptotic formula for the sum $\sum\limits_{n \leqslant x} r(n)r(n+a)$, where $r(n)$ denotes the number of representations of n as a sum of two squares and a is a fixed positive integer. His method is entirely elementary. In this paper, we use an analytic method of Hooley (1957) to find the asymptotic formula for this sum. The method depends on the asymptotic formula for the sum $\sum\limits_{\substack{n \leqslant x \\ n \equiv b \ (\mathrm{mod}\ k)}} r(n)$, with a 'reasonably sharp' error term. The result is in course of publication elsewhere and so is only quoted in this paper as a lemma.

1. Introduction

An unsolved problem for several years was that of finding an asymptotic formula for $\sum\limits_{n \leqslant x} d(n)d_3(n+a)$, where a is a positive integer. Success was accomplished in 1957 by Hooley, who developed a technique for dealing with such sums. In this paper and in another which will appear shortly in the *Canadian Mathematical Bulletin* entitled 'The Circle Problem in an Arithmetic Progression', the method of Hooley is applied to the sum

$$\sum_{n \leqslant x} r(n)r(n+a), \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

where a is a positive integer and $r(n)$ denotes the number of representations of n as a sum of two squares. Originally, this sum was studied by Estermann (1932) using non-analytic techniques. In applying Hooley's method, we require an asymptotic formula for the sum

$$\sum_{\substack{n \leqslant x \\ n \equiv b \ (\mathrm{mod}\ k)}} r(n) \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

which remains valid at least for $k = O(x^{\frac{1}{3}})$. In the above-mentioned paper, 'The Circle Problem in an Arithmetic Progression', we obtain an asymptotic formula for (2) which remains useful as an asymptotic formula essentially up to $k = O(x^{\frac{2}{3}})$ when $(b, k) = 1$. We state this result without proof as Theorem 1. It should also be mentioned that the error term of our asymptotic formula for (2) uses Weil's well-known estimate for the Kloosterman sum.

In the solution of (1), Estermann used imperfect estimates of the Kloosterman sum, as Weil's results were not available until more than a decade later. His error term for (1), for fixed a, is $O\left(x^{\frac{11}{12}+\epsilon}\right)$, which is the error term for (1) obtained in this paper. In a letter communicated to the author, Estermann has suggested that Weil's estimate would sharpen his error term to approximately $O\left(x^{\frac{5}{6}+\epsilon}\right)$, which seems beyond the reach of the method used here.

One consequence of the present method of dealing with (1) is that with minor modifications we can obtain an asymptotic formula for the sum

$$\sum_{n < x} r(pn)r(qn+a), \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \qquad (3)$$

where p, q and a are positive integers. The interest in (3) is that it counts the number of solutions of the Diophantine equation:

$$p(u^2+v^2)-q(z^2+w^2) = pa$$
$$z^2+w^2 \leqslant px,$$

where u, v, z and w are integers.

2. Some Definitions

Throughout this paper, we shall adopt the following conventions:

χ is always the non-principal character modulo 4, and $r(n)$ denotes the number of representations of n as a sum of two squares, expressible in the form

$$r(n) = \sum_{d \mid n} \chi(d).$$

Also, $d(n)$ denotes the number of divisors of n. For brevity, we shall write $\sum_{h<k}$ in place of $\sum_{1 \leqslant h < k}$. Instead of always writing $f = O(g)$, $g > 0$, we shall often use the Vinogradov notation: $f \ll g$, meaning that there exists a positive absolute constant B such that $|f| \leqslant Bg$.

We shall use the following definitions throughout this paper:

1. $a, b, c, \ldots; A, B, C, \ldots; \alpha, \beta, \gamma, \ldots$ are always positive integers, unless otherwise indicated. x is a real number > 1.
2. Whenever we write $a = 2^\alpha A$, it is to be understood that A is odd.
3. $\chi^0(a) = \chi(A)$, where $a = 2^\alpha A$.
4. $\epsilon_a(b) = \begin{cases} 1 \text{ if } \alpha \leqslant \beta \\ 0 \text{ if } \alpha > \beta \end{cases}$, where $a = 2^\alpha A$ and $b = 2^\beta B$.

5. $D_a(m) = \chi^0(m+a)\epsilon_{m+a}(m) - \chi^0(3m+a)\epsilon_{3m+a}(m)$.

6. $$\epsilon_{\alpha,\beta} = \begin{cases} 1 \text{ if } \beta < \alpha-1 \\ -1 \text{ if } \beta = \alpha-1 \\ 0 \text{ if } \beta > \alpha-1. \end{cases}$$

7. $\delta_{\alpha,\beta}$ is Kronecker's delta.

8. $$t_\alpha = \begin{cases} 0 \text{ if } \alpha = 0 \\ 1-\dfrac{3}{2^\alpha} \text{ if } \alpha \geqslant 1. \end{cases}$$

9. $$H_k(b) = \sum_{q \mid k} \chi(q) c_q(b)/q,$$

where $c_q(b)$ is the Ramanujan sum.

10. $H_k^*(b) = [1+\chi^0(b)\epsilon_{4b}(k)]H_k(b)$.

11. $$A(a) = \sum_{n \geqslant 1} \frac{\chi^2(n)}{n^2} c_n(a).$$

3. Some Preliminary Results

THEOREM 1. For $k = O(x^{\frac{2}{3}})$, then

$$\sum_{\substack{n \leqslant x \\ n \equiv b \ (\mathrm{mod}\ k)}} r(n) = \frac{\pi}{4} H_k^*(b) \frac{x}{k} + O(x^{\frac{2}{3}} k^{-\frac{1}{2}} \ln (k+1)(b,\ k)^{\frac{1}{2}} d(k)).$$

LEMMA 1. If $a = 2^\alpha A$ and $m = 2^\mu M$, then

$$D_a(m) = 2\chi^0(m)\epsilon_{\alpha,\mu}.$$

The proof is immediate if we consider three cases: $\alpha = 0$; $\alpha \geqslant 1$ and $\mu = 0$; $\alpha \geqslant 1$ and $\mu \geqslant 1$. In the last case, consider subcases: $\alpha < \mu+1$ and $\alpha \geqslant \mu+1$.

LEMMA 2. If $a = 2^\alpha A$ and n is an odd positive integer, then

$$\sum_{m \leqslant x} \frac{D_a(mn)}{m} = \pi\chi(n) \begin{cases} 0 & \text{if } \alpha = 0 \\ -\frac{1}{2}+O\left(\dfrac{1}{x}\right) & \text{if } \alpha = 1 \\ 1+O\left(\dfrac{\ln x}{x}\right) & \text{if } x < 2^{\alpha-1} \text{ and } \alpha > 1 \\ 1-\dfrac{3}{2^\alpha}+O\left(\dfrac{\ln x}{x}\right) & \text{if } x \geqslant 2^{\alpha-1} \text{ and } \alpha > 1. \end{cases}$$

PROOF. For each $m \leqslant x$, write $m = 2^\mu M$. If $\alpha = 0$, the result is obvious, since $\epsilon_{\alpha,\mu} = 0$ for all $\mu \geqslant 0$. If $\alpha = 1$, then

$$\epsilon_{1,\mu} = \begin{cases} -1 \text{ if } \mu = 0 \\ 0 \text{ if } \mu > 0 \end{cases}$$

so that

$$\sum_{m \leqslant x} \frac{D_a(mn)}{m} = -2\chi(n) \sum_{M \leqslant x} \frac{\chi(M)}{M}$$

$$= -\frac{\pi}{2}\chi(n) + O\left(\frac{1}{x}\right).$$

If $\alpha > 1$, then

$$\sum_{m \leqslant x} \frac{D_a(mn)}{m} = 2\chi(n) \sum_{0 \leqslant \mu \leqslant \frac{\ln x}{\ln 2}} \frac{\epsilon_{\alpha,\mu}}{2^\mu} \sum_{M \leqslant x/2^\mu} \frac{\chi(M)}{M}$$

$$= \frac{\pi}{2}\chi(n) \sum_{0 \leqslant \mu \leqslant \frac{\log x}{\log 2}} \frac{\epsilon_{\alpha,\mu}}{2^\mu} + O\left(\frac{1}{x} \sum_{0 \leqslant \mu \leqslant \frac{\log x}{\log 2}} |\epsilon_{\alpha,\mu}|\right).$$

The lemma easily follows by considering two cases: $x < 2^{\alpha-1}$ and $x \geqslant 2^{\alpha-1}$.

LEMMA 3.

$$\sum_{\lambda \leqslant x} \frac{\chi(\lambda)}{\lambda} H_\lambda(a) = \frac{\pi}{4} A(a) + O\left(d(a)\frac{\ln x}{x}\right).$$

PROOF,

$$\sum_{\lambda \leqslant x} \frac{\chi(\lambda)}{\lambda} H_\lambda(a) = \sum_{q \leqslant x} \frac{\chi^2(q)}{q^2} c_q(a) \sum_{m \leqslant x/q} \frac{\chi(m)}{m}$$

$$= \frac{\pi}{4} \sum_{q \leqslant x} \frac{\chi^2(q)}{q^2} c_q(a) + O\left(\frac{1}{x} \sum_{q \leqslant x} \frac{1}{q} |c_q(a)|\right)$$

$$= \frac{\pi}{4} A(a) - \frac{\pi}{4} \sum_{q > x} \frac{\chi^2(q)}{q^2} c_q(a) + O\left(d(a)\frac{\ln x}{x}\right),$$

since

$$\sum_{q \leqslant x} \frac{1}{q} |c_q(a)| \leqslant \sum_{q \leqslant x} \frac{1}{q} \sum_{d|q,\, d|a} d$$

$$\leqslant \sum_{d|a} \sum_{q \leqslant x/d} \frac{1}{q}$$

$$\ll d(a) \ln x.$$

Similarly,

$$\sum_{q > x} q^{-2} c_q(a) \ll \frac{d(a)}{x},$$

which completes the proof.

LEMMA 4.

$$\sum_{\lambda \leqslant x} \chi(\lambda) H_\lambda(a) = O(d(a) \ln x).$$

4. The Main Result

Theorem 2. Suppose a is fixed and $a = 2^{\alpha}A$. Then

$$\sum_{n \leqslant x} r(n)r(n+a) = \frac{\pi^2}{16}(1+t_{\alpha})A(a)x + O\left(x^{\frac{11}{12}} \ln^2 x\right).$$

Proof.

$$\sum_{n \leqslant x} r(n)r(n+a) = \sum_{n \leqslant x} r(n+a)\left[\sum_{\substack{\lambda\nu = n \\ \lambda \leqslant x^{\frac{1}{2}}}} \chi(\lambda) + \sum_{\substack{\lambda\nu = n \\ \nu \leqslant x^{\frac{1}{2}}}} \chi(\lambda) + \sum_{\substack{\lambda\nu = n \\ \lambda,\, \nu \leqslant x^{\frac{1}{2}}}} \chi(\lambda)\right]$$

$$= \sum_{\lambda \leqslant x^{\frac{1}{2}}} \chi(\lambda) \sum_{\substack{n \leqslant x+a \\ n \equiv a \ (\mathrm{mod}\ \lambda)}} r(n) + \sum_{\nu \leqslant x^{\frac{1}{2}}} \left[\sum_{\substack{n \leqslant x+a \\ n \equiv \nu + a \ (\mathrm{mod}\ 4\nu)}} r(n)\right.$$

$$\left. - \sum_{\substack{n \leqslant x+a \\ n \equiv 3\nu + a \ (\mathrm{mod}\ 4\nu)}} r(n)\right] - \sum_{\lambda \leqslant x^{\frac{1}{2}}} \chi(\lambda) \sum_{\substack{n \leqslant \lambda x^{\frac{1}{2}} + a \\ n \equiv a \ (\mathrm{mod}\ \lambda)}} r(n)$$

$$= \Sigma_1 + \Sigma_2 - \Sigma_3, \text{ say.} \qquad \qquad (4)$$

Estimation of Σ_3: By Theorem 1, we immediately obtain

$$\sum\nolimits_3 \ll x^{\frac{1}{2}} \left| \sum_{\lambda \leqslant x^{\frac{1}{2}}} \chi(\lambda) H_{\lambda}(a) \right| + \left| \sum_{\lambda \leqslant x^{\frac{1}{2}}} \frac{\chi(\lambda)}{\lambda} H_{\lambda}(a) \right| + x^{\frac{1}{3}} \sum_{\lambda \leqslant x^{\frac{1}{2}}} \lambda^{\frac{1}{6}} \ln \lambda \, d(\lambda)$$

since the only contribution to the main term is for λ odd so that $H^{*}_{\lambda}(a) = H_{\lambda}(a)$. By Lemmas 3 and 4, using $A(a) \ll 1$, we obtain

$$\sum\nolimits_3 \ll x^{\frac{11}{12}} \ln^2 x.$$

Estimation of Σ_1: Applying Theorem 1 and Lemma 3 to Σ_1, we easily obtain

$$\sum\nolimits_1 = \frac{\pi^2}{16} A(a)x + O\left(x^{\frac{11}{12}} \ln^2 x\right).$$

Estimation of Σ_2: Following the same procedure as above, we obtain

$$\sum\nolimits_2 = \frac{\pi}{16}(x+a) \sum\nolimits_4 + O\left(x^{\frac{11}{12}} \ln^2 x\right)$$

where

$$\sum\nolimits_4 = \sum_{\nu < x^{\frac{1}{2}}} \frac{1}{\nu} \left[H^{*}_{4\nu}(\nu + a) - H^{*}_{4\nu}(3\nu + a)\right].$$

For any positive integers ν, m, then $H_{4\nu}(m\nu + a) = H_{\nu}(a)$. Hence,

$$H^{*}_{4\nu}(\nu + a) - H^{*}_{4\nu}(3\nu + a) = D_a(\nu) H_{\nu}(a)$$

so that

$$\sum_4 = \sum_{\nu < x^{\frac{1}{3}}} \frac{D_a(\nu)}{\nu} H_\nu(a)$$

$$= \sum_{n < x^{\frac{1}{3}}} \frac{\chi(n)}{n^2} c_n(a) \sum_{m < x^{\frac{1}{3}}/n} \frac{D_a(mn)}{m}.$$

If $\alpha = 0$, then $\Sigma_4 = 0$.
If $\alpha = 1$, then

$$\sum_4 = -\frac{\pi}{2} A(a) + O\left(\frac{\ln x}{x^{\frac{1}{3}}}\right).$$

If $\alpha > 1$, then

$$\sum_4 = \pi\left(1 - \frac{3}{2^\alpha}\right) A(a) + O\left(\frac{\ln x}{x^{\frac{1}{3}}}\right).$$

Consequently,

$$\sum_2 = \frac{\pi^2}{16} t_\alpha A(a) x + O\left(x^{\frac{11}{12}} \ln^2 x\right)$$

and so the theorem follows from (4) and the above results.

5. Some Remarks

(i) It is easy to show that

$$\frac{\pi^2}{16}(1+t_\alpha)A(a) = \tfrac{1}{2}\left[\sigma_{-1}(a) - \sigma_{-1}\left(\frac{a}{2}\right) + \sigma_{-1}\left(\frac{a}{4}\right)\right]$$

where

$$\sigma_{-1}(b) = \sum_{d \mid b} \frac{1}{d}$$

if b is an integer, and $\sigma_{-1}(b) = 0$ if b is not an integer. From this result, Theorem 2 takes the form obtained by Estermann.

(ii) The best error term this method has produced in Theorem 2 is

$$O\left(x^{\frac{8}{9}+\epsilon}\right).$$

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