ON ENTIRE FUNCTIONS REPRESENTED BY DIRICHLET SERIES

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An equality connecting $M(\sigma, f^p)$, $\mu(\sigma)$ and $\lambda_{\nu(\sigma)}$ for functions of zero order has been obtained in Theorem 1 by imposing a condition on the coefficients a_n. In the 2nd theorem an inequality for $M(\sigma, f^p)$ and $\mu(\sigma)$ has been established for functions of finite order. Both the results include as particular cases the results obtained by Kamthan.

§ 1. INTRODUCTION: Let

$$f(s) = \sum_{n=1}^{\infty} a_n \exp(s\lambda_n), \quad (s = \sigma + it),$$

represent an entire Dirichlet function where

$$0 \leqslant \lambda_0 < \lambda_1 < \ldots < \lambda_n \to \infty, \qquad \ldots \quad (1.1)$$

$$\overline{\lim_{n\to\infty}} (\lambda_{n+1} - \lambda_n) = \frac{d}{h}, \quad h > 0, d < \infty \qquad \ldots \quad (1.2)$$

and

$$\overline{\lim_{n\to\infty}} \frac{n}{\lambda_n} = D < \infty. \qquad \ldots \quad (1.3)$$

Let $M(\sigma)$, $\mu(\sigma)$ and $\lambda_{\nu(\sigma)}$ stand as usual. In this paper we are interested to establish certain inequalities involving $M(\sigma)$, $\mu(\sigma)$ and $\lambda_{\nu(\sigma)}$. Both of the results, which we state and prove below, generalize the theorems proved by Kamthan (1964). Our first theorem deals with a class of entire Dirichlet function of order (R) zero and the second one relates to the general order $(R)\rho$, $0 < \rho < \infty$.

§ 2. THEOREM 1: If $M(\sigma, f^p)$ represents the maximum modulus of the pth derivative of $f(s)$ and relations (1.1), (1.2)

and
$$\phi^2 \frac{\log \left|\frac{a_{n-1}}{a_n}\right|}{\lambda_n - \lambda_{n-1}} \leqslant \frac{\log \left|\frac{a_n}{a_{n+1}}\right|}{\lambda_{n+1} - \lambda_n}, \qquad \ldots \quad (2.1)$$

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where $\phi > 1$ hold good, then

$$\lim_{\sigma\to\infty} \frac{M(\sigma, f^p)}{\mu(\sigma)\lambda^p_{\nu(\sigma)}} = 1.$$

PROOF: Let

$$\chi_n = \frac{\log \left|\frac{a_{n-1}}{a_n}\right|}{\lambda_n - \lambda_{n-1}}. \qquad (2.2)$$

So from (2.1) we now get

$$\phi^2\chi_n \leqslant \chi_{n+1}, \quad \text{for all } n \text{ and any } \phi > 1.$$

Let the maximum of $|f^p(s)|$ occur at $Rl(s) = \sigma$, then

$$M(\sigma, f^p) \leqslant \sum_{n=1}^{N-1} \lambda_n^p |a_n| e^{\sigma\lambda_n} + \lambda_N^p |a_N| e^{\sigma\lambda_N} + \sum_{N+1}^{\infty} \lambda_n^p |a_n| e^{\sigma\lambda_n}.$$

Put $\sigma = \phi\chi_N$, then $\phi\chi_N$ lies in between χ_N and χ_{N+1} and so

$$M(\phi\chi_N, f^p) \leqslant \sum_{n=1}^{N-1} \lambda_n^p |a_n| \exp(\phi\chi_N\lambda_n) + \sum_{N+1}^{\infty} \lambda_n^p |a_n| \exp(\phi\chi_N\lambda_n) + \lambda_N^p \mu(\phi\chi_N)$$
$$= \Sigma_1 + \Sigma_2 + \lambda_N^p \mu(\phi\chi_N), \quad \text{(say)}.$$

Now

$$\Sigma_1 < \lambda_N^p |a_{N-1}| \exp(\phi\chi_N\lambda_{N-1}) + \lambda_N^p |a_{N-2}| \exp(\phi\chi_N\lambda_{N-2}) + \ldots \text{ to } (N-1) \text{ terms}$$
$$= \lambda_N^p \mu(\phi\chi_N) \left[\left|\frac{a_{N-1}}{a_N}\right| \exp[\phi\chi_N(\lambda_{N-1} - \lambda_N)] + \left|\frac{a_{N-2}}{a_N}\right| \exp[\phi\chi_N(\lambda_{N-2}\lambda_N)] \right.$$
$$\left. + \ldots \text{ to } (N-1) \text{ terms} \right].$$

From (1.2) and (2.2) by simple calculations we obtain

$$\Sigma_1 < \lambda_N^p \mu(\phi\chi_N)[\exp\{(1-\phi)L\chi_N\} + \exp\{2(1-\phi)L\chi_N\} + \ldots \text{ to } (N-1) \text{ terms}],$$

where $L = h - \epsilon$,

$$< \lambda_N^p \mu(\phi\chi_N)[a + a^2 + \ldots \text{ to } \infty],$$

where $a = \exp\{(1-\phi)L\chi_N\} < 1$,

i.e.

$$\Sigma_1 < \lambda_N^p \mu(\phi\chi_N) \frac{a}{1-a}. \qquad (2.3)$$

Now

$$\Sigma_2 = \lambda_{N+1}^p \mu(\phi\chi_N) \left|\frac{a_{N+1}}{a_N}\right| \exp\{(\lambda_{N+1} - \lambda_N)\phi\chi_N\}$$
$$+ \lambda_{N+2}^p \mu(\phi\chi_N) \left|\frac{a_{N+2}}{a_N}\right| \exp\{(\lambda_{N+2} - \lambda_N)\phi\chi_N\} + \ldots \text{ to } \infty.$$

Therefore by (1.2) and (2.2) we have

$$\Sigma_2 < \lambda_{N+1}^p \mu(\phi\chi_N) \exp\{(\phi-\phi^2)L\chi_N\}$$

$$+\lambda_{N+2}^p \mu(\phi\chi_N) \exp\{(\phi-\phi^2)2L\chi_N\}+\dots \text{ to } \infty$$

$$< \mu(\phi\chi_N)[\lambda_{N+1}^p a+\lambda_{N+2}^p a^2+\dots \text{ to } \infty].$$

Again by (1.2), where $M = d+\epsilon$, we get

$$\Sigma_2 < \mu(\phi\chi_N)[(M+\lambda_N)^p a+(2M+\lambda_N)^p a^2+\dots \text{ to } \infty].$$

Now the series

$$(M+\lambda_N)^p a+(2M+\lambda_N)^p a^2+\dots \text{ to } \infty$$

$$= \sum_{r=0}^{p} \left\{{}^pC_{p-r} M^r \lambda_N^{p-r}(a+2^r a^2+3^r a^3+\dots \text{ to } \infty)\right\}$$

$$< \sum_{r=0}^{p} \left\{{}^pC_{p-r} M^r \lambda_N^{p-r}((2^r a)+(2^r a)^2+(2^r a)^3+\dots \text{ to } \infty)\right\}$$

$$= \sum_{r=0}^{p} \left({}^pC_{p-r} M^r \lambda_N^{p-r} \frac{2^r a}{1-2^r a}\right).$$

Hence we have

$$\Sigma_2 < \lambda_N^p \mu(\phi\chi_N) \sum_{r=0}^{p} \left({}^pC_{p-r} \frac{M^r}{\lambda_N^r} \frac{2^r a}{1-2^r a}\right). \quad \dots \quad \dots \quad (2.4)$$

So

$$M(\phi\chi_N, f^p) < \lambda_N^p \mu(\phi\chi_N)\left[1+\frac{a}{1-a}+\sum_{r=0}^{p}\left({}^pC_{p-r} \frac{M^r}{\lambda_N^r} \frac{2^r a}{1-2^r a}\right)\right].$$

Therefore

$$\varlimsup_{\sigma\to\infty} \frac{M(\sigma, f^p)}{\mu(\sigma)\lambda_{\nu(\sigma)}^p} \leqslant 1. \quad \dots \quad \dots \quad \dots \quad \dots \quad (2.5)$$

Again

$$M(\sigma, f^p) > \lambda_n^p\, |a_n| \exp(\sigma\lambda_n).$$

When

$$\chi_n < \sigma < \chi_{n+1}$$

$$\mu(\sigma) = |a_n| \exp(\sigma\lambda_n), \quad \lambda_{\nu(\sigma)} = \lambda_n.$$

Therefore

$$M(\sigma, f^p) > \lambda_{\nu(\sigma)}^p \mu(\sigma).$$

Hence

$$\varliminf_{\sigma\to\infty} \frac{M(\sigma, f^p)}{\mu(\sigma)\lambda_{\nu(\sigma)}^p} \geqslant 1. \quad \dots \quad \dots \quad \dots \quad \dots \quad (2.6)$$

From (2.5) and (2.6) we obtain the result.

§ 3. THEOREM 2: Let

$$f(s) = \sum_{n=1}^{\infty} a_n \exp(s\lambda_n), \quad (s = \sigma + it),$$

represent a Dirichlet entire function of order $(R)\rho$, $(0 < \rho < \infty)$ where conditions (1.1), (1.2) for lower limit only, and (1.3) are satisfied. Then for all $\sigma \geqslant \sigma_0$ and $\epsilon > 0$

$$M(\sigma, f^p) < \mu(\sigma) \exp[\{(p+1)\rho+\epsilon\}\sigma].$$

PROOF:

Now

$$M(\sigma, f^p) \leqslant \sum_{n=1}^{\infty} \lambda_n^p |a_n| \exp(\sigma\lambda_n) \quad \ldots \quad \ldots \quad \ldots \quad (3.1)$$

is true for all σ

and
$$|a_n| \exp(\sigma\lambda_n) \leqslant M(\sigma),$$

i.e.
$$|a_n| < \exp\{\exp((\rho+\epsilon_1)\sigma) - \sigma\lambda_n\} \quad \ldots \quad \ldots \quad \ldots \quad (3.2)$$

for all $\sigma \geqslant \sigma_0$, $\epsilon_1 > 0$, and all 'n'.

Hence the above inequality (3.2) must be satisfied for the minimum value of the R.H.S. corresponding to σ. Simple calculations show that the minimum value occurs when

$$\sigma = \frac{1}{\rho+\epsilon_1} \log\left(\frac{\lambda_n}{\rho+\epsilon_1}\right). \quad \ldots \quad \ldots \quad \ldots \quad (3.3)$$

Write (3.1) as

$$M(\sigma, f^p) < \sum_{n=1}^{N} \lambda_n^p |a_n| \exp(\sigma\lambda_n) + \sum_{N+1}^{\infty} \lambda_n^p |a_n| \exp(\sigma\lambda_n) = \Sigma_1 + \Sigma_2, \quad \text{(say)},$$

where N is at our choice to be determined later.

By (3.2) and (3.3) we get

$$\Sigma_2 \leqslant \sum_{N+1}^{\infty} \lambda_n^p \exp\left(\frac{\lambda_n}{\rho+\epsilon_1}\right)$$

$$= \sum_{N+1}^{\infty} \lambda_n^p \left\{\frac{e(\rho+\epsilon_1)\exp((\rho+\epsilon_1)\sigma)}{\lambda_n}\right\}^{\lambda_n/(\rho+\epsilon_1)}$$

$$\left(\text{Since } \frac{(\rho+\epsilon_1)\exp((\rho+\epsilon_1)\sigma)}{\lambda_n} = 1\right),$$

i.e.
$$\Sigma_2 < \sum_{N+1}^{\infty} \lambda_n^p \left\{\frac{e(\rho+\epsilon_1)\exp(\rho+\epsilon_1)\sigma}{\lambda_N}\right\}^{\lambda_n/(\rho+\epsilon_1)}$$

So choose

$$\lambda_N = (\rho+\epsilon_1) \exp\{(\rho+\epsilon_1)\sigma+k\}, \quad k \geqslant 2. \qquad \ldots \qquad \ldots \quad (3.4)$$

Then

$$\Sigma_2 < \sum_{N+1}^{\infty} \left\{ \lambda_n^p \exp\left[(1-k)\lambda_n/(\rho+\epsilon_1)\right] \right\}.$$

Hence

$$M(\sigma, f^p) < \lambda_N^p \mu(\sigma) N + \sum_{N+1}^{\infty} \left\{ \lambda_n^p \exp\left[(1-k)\lambda_n/(\rho+\epsilon_1)\right] \right\}.$$

As $\dfrac{\log \lambda_n}{\lambda_n} \to 0$, we may write

$$\lambda_n^p < \exp\left[\lambda_n/k(\rho+\epsilon_1)\right] \quad \text{for all } n > N.$$

Therefore

$$M(\sigma, f^p) < N\mu(\sigma)\lambda_N^p + \sum_{N+1}^{\infty} \exp\left[\left(\frac{1}{k}+1-k\right)\frac{\lambda_n}{\rho+\epsilon_1}\right].$$

But the series

$$\sum_{N+1}^{\infty} \exp\left[\left(\frac{1}{k}+1-k\right)\frac{\lambda_n}{\rho+\epsilon_1}\right)\right] \to 0 \quad \text{as } n \to \infty.$$

So

$$M(\sigma, f^p) < N\mu(\sigma)\lambda_N^p + o(1).$$

Again by relations (1.3) and (3.4) we get

$$M(\sigma, f^p) < \mu(\sigma)(D+\epsilon_2)(\rho+\epsilon_1)^{p+1} \exp\{(\rho+\epsilon_1)\sigma+k\}(p+1)+o(1)$$
$$= \mu(\sigma) \exp\{(p+1)(\rho+\epsilon_1)\sigma\} O(1) + o(1).$$

Hence the theorem.

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Reference

Kamthan, P. K. (1964). *Mathematica jap.*, **9**, 79–82.