ON A MIXED PROBLEM OF THERMOELASTICITY

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This paper is concerned with the evaluation of the stresses and displacements in an isotropic, elastic, semi-infinite disc due to arbitrary heating of two equal strips on its boundary while the remainder is insulated, a constant pressure being applied on the strip. It is shown that the normal surface displacement is not affected by heating except on the pressed portions and the region between them. The stresses, apart from the component required to maintain the body in a state of plane strain, are independent of the temperature.

INTRODUCTION

The solution of boundary value problems of elastostatics involving temperature changes for an isotropic, elastic, semi-infinite disc has been carried out by various methods. Melan and Parkus (1953) studied the effect of a concentrated heat source inside the disc while Sen (1951) considered that of a nucleus of thermoelastic strain in a semi-infinite solid. In this paper, we consider a mixed boundary value problem when two equal strips on the boundary are maintained at arbitrary temperatures, the remainder being insulated. A normal pressure is applied on the surface only on the strips. In such a situation, the thermal problem leads to triple integral equations while the usual mixed boundary conditions lead to dual integral equations. These triple integral equations can be reduced (Tranter 1960) to dual trigonometric series which were solved by Tranter (1959).

The above procedure gives the temperature as the sum of two terms involving a series of Bessel functions. The displacement components and the stress are then obtained in the form of Fourier sine and cosine transforms. However, it is found that some of the integral transforms are divergent. Finite results are obtained by taking the finite parts of these integrals, the final results in almost all the cases being expressed in terms of elementary functions. It has been verified that in the particular case, when temperature effects are absent, the results obtained by taking finite parts of divergent integrals agree with those deduced by the complex variable method. It is believed that this solution has not been obtained previously.

Statement of the Problem

The x-axis is taken along the boundary of the half-plane and the y-axis is taken perpendicular to the boundary and directed into the medium. The boundary conditions on $y = 0$ are thus

$$\frac{\partial T}{\partial y} = 0, \quad 0 < |x| < a \text{ or } b < |x| < \infty,$$

$$T = \theta(x) = f(x) + g(x), \quad a < |x| < b,$$

where

$$f(x) = \frac{\theta(x) + \theta(-x)}{2},$$

$$g(x) = \frac{\theta(x) - \theta(-x)}{2};$$

$$\left.\begin{aligned} \tau_{yx} &= 0, \\ \tau_{yy} &= -P, \quad a < |x| < b \\ &= 0, \quad \text{elsewhere.} \end{aligned}\right\} \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \qquad (1)$$

Solution of the Thermal Problem

The temperature change T can be expressed as the sum of T_1 and T_2, both satisfying the heat conduction equation, with T_1 and T_2 satisfying on $y = 0$ the boundary conditions

$$\frac{\partial T_i}{\partial y} = 0, \quad 0 < |x| < a \text{ or } b < |x| < \infty, \quad i = 1, 2,$$

$$T_1 = f(x), \quad a < |x| < b,$$

$$T_2 = g(x), \quad a < |x| < b.$$

Let

$$T_2 = \int_0^\infty \xi_2(\eta) e^{-y\eta} \sin x\eta \, d\eta.$$

The boundary conditions give

$$\int_0^\infty \eta \xi_2(\eta) \sin x\eta \, d\eta = 0, \qquad 0 < x < a,$$

$$\int_0^\infty \xi_2(\eta) \sin x\eta \, d\eta = g(x), \quad a < x < b,$$

$$\int_0^\infty \eta \xi_2(\eta) \sin x\eta \, d\eta = 0, \qquad b < x < \infty.$$

The solution of these triple integral equations has been reduced by Tranter (1960) to that of the dual trigonometric series

$$\sum_{n=1}^{\infty} (-1)^{n-1} c_n \cos \frac{2n-1}{2} \theta = 0, \quad c < \theta < \pi,$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1} c_n \cos \frac{2n-1}{2} \theta = g\left(b \cos \frac{\theta}{2}\right), \quad 0 < \theta < c$$

by the substitutions

$$\xi_2(\eta) = \frac{1}{\eta} \sum_{n=1}^{\infty} c_n J_{2n-1}(b\eta) \quad \ldots \quad (2)$$

$$x = b \cos \frac{\theta}{2}, \quad a = b \cos \frac{c}{2} \quad \ldots \quad (3)$$

Tranter (1959) has also given for c_n the formulae

$$\frac{(-1)^{n-1} c_n \operatorname{cosec} \frac{c}{2}}{2} = \Phi_n\left(\sin \frac{c}{2}\right) + DP_{n-1}(\cos c), \quad \ldots \quad (4)$$

where

$$\Phi_n\left(\sin \frac{c}{2}\right) = \xi(1) F\left(n, 1-n;\ 1;\ \sin^2 \frac{c}{2}\right) - \int_0^1 \xi'(s) F\left(n, 1-n;\ 1;\ s^2 \sin^2 \frac{c}{2}\right) ds,$$

$$\xi(s) = \frac{4}{\pi} \int_0^s \frac{\rho g'\left[b\sqrt{1-\rho^2 \sin^2 \frac{c}{2}}\right] d\rho}{\sqrt{s^2-\rho^2}}$$

and D is a constant.

Substituting in the second of the dual series and putting $\theta = 0$, we obtain D from the resulting equation

$$DK\left(\cos \frac{c}{2}\right) = g(b) \operatorname{cosec} \frac{c}{2} - 2 \sum_{n=1}^{\infty} \frac{\Phi_n\left(\sin \frac{c}{2}\right)}{2n-1}, \quad \ldots \quad (5)$$

using the result

$$\sum_{n=1}^{\infty} \frac{P_{n-1}(\cos c)}{2n-1} = \tfrac{1}{2} K\left(\cos \frac{c}{2}\right)$$

in the usual notation for Legendre polynomials and complete elliptic integrals. Equations (4) and (5) determine $\xi_2(\eta)$ and thus T_2.

We now proceed to determine $\xi_1(\eta)$, where

$$T_1 = \int_0^{\infty} \xi_1(\eta) e^{-y\eta} \cos x\eta \, d\eta.$$

The boundary conditions yield the triple integral equations

$$\int_0^\infty \eta\xi_1(\eta)\cos x\eta\, d\eta = 0, \qquad 0 < x < a,$$

$$\int_0^\infty \xi_1(\eta)\cos x\eta\, d\eta = f(x), \qquad a < x < b,$$

$$\int_0^\infty \eta\xi_1(\eta)\cos x\eta\, d\eta = 0, \qquad b < x < \infty.$$

The substitutions (3) and the expression

$$\xi_1(\eta) = \frac{1}{\eta}\sum_{n=1}^{\infty} d_n J_{2n}(b\eta) \quad \ldots \quad (6)$$

enable us, following Tranter's procedure, to reduce the triple integral equations to the dual series

$$\sum_{n=1}^{\infty} (-1)^n d_n \cos n\theta = 0, \qquad c < \theta < \pi,$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n d_n \cos n\theta}{n} = 2f\left(b\cos\frac{\theta}{2}\right), \qquad 0 < \theta < c.$$

On integrating, with respect to θ, the first equation between θ and π and the second between 0 and θ, we obtain

$$\sum_{n=1}^{\infty} (-1)^n \frac{d_n \sin n\theta}{n} = 0, \qquad c < \theta < \pi,$$

$$\sum_{n=1}^{\infty} (-1)^n \frac{d_n \sin n\theta}{n^2} = f_1(\theta), \qquad 0 < \theta < c,$$

where

$$f_1(\theta) = 2\int_0^\theta f\left(b\cos\frac{\theta}{2}\right) d\theta.$$

The constants d_n are given by (Tranter 1960)

$$\frac{(-1)^n d_n \pi \operatorname{cosec}\frac{c}{2}}{4n^2} = \int_0^1 F\left(n, -n;\ 1;\ s^2\sin^2\frac{c}{2}\right)\left[\frac{d}{ds}\int_0^s \frac{\rho f_1\left\{2\sin^{-1}\left(\rho\sin\frac{c}{2}\right)\right\} d\rho}{\sqrt{(s^2-\rho^2)\left(1-\rho^2\sin^2\frac{c}{2}\right)}}\right] ds. \quad \ldots \quad (7)$$

Thus the temperature change T is given by

$$T = \int_0^\infty e^{-y\eta}[\xi_1(\eta)\cos x\eta + \xi_2(\eta)\sin x\eta]\, d\eta, \quad \ldots \quad (8)$$

where $\xi_1(\eta)$ and $\xi_2(\eta)$ are given by the equations (2) to (7).

If $y > 0$, it can be shown (Watson 1958, p. 386) that

$$T = \sum_{n=1}^{\infty} \frac{c_n}{(2n-1)b^{2n-1}} \operatorname{Im}\left[\sqrt{(y-ix)^2+b^2}-(y-ix)\right]^{2n-1}$$

$$+\sum_{n=1}^{\infty} \frac{d_n}{2nb^{2n}} \operatorname{Re}\left[\sqrt{(y-ix)^2+b^2}-(y-ix)\right]^{2n}. \qquad (9)$$

If $y = 0$, using known integral transforms (Erdelyi 1954, pp. 43, 98) we obtain

$$T = \sum_{n=1}^{\infty} \frac{c_n}{2n-1} \sin\left[(2n-1)\sin^{-1}\frac{x}{b}\right] + \sum_{n=1}^{\infty} \frac{d_n}{2n} \cos\left[2n \sin^{-1}\frac{x}{b}\right], \quad 0 < x < b,$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n-1} c_n b^{2n-1}}{[x+\sqrt{x^2-b^2}]^{2n-1}} + \sum_{n=1}^{\infty} \frac{(-1)^n d_n b^{2n}}{[x+\sqrt{x^2-b^2}]^{2n}}, \qquad b < x < \infty. \qquad (9a)$$

Determination of Stresses and Displacements

Using the usual stress-strain relations for the plane-strain problem and substituting

$$\left.\begin{aligned} u &= \int_0^\infty [f_1(y,\eta)\cos x\eta + f_2(y,\eta)\sin x\eta]\,d\eta \\ v &= \int_0^\infty [f_3(y,\eta)\cos x\eta + f_4(y,\eta)\sin x\eta]\,d\eta \end{aligned}\right\} \qquad (10)$$

in the equations of equilibrium (Timoshenko and Goodier 1951, pp. 421, 433), it is found that they are satisfied if

$$\left.\begin{aligned} [(1-2\nu)D^2-2(1-\nu)\eta^2]f_1(y,\eta)+\eta Df_4(y,\eta) &= 2(1+\nu)\alpha\eta\xi_2(\eta)e^{-y\eta} \\ [2(1-\nu)D^2-(1-2\nu)\eta^2]f_4(y,\eta)-\eta Df_1(y,\eta) &= -2(1+\nu)\alpha\eta\xi_2(\eta)e^{-y\eta} \end{aligned}\right\} \qquad (11)$$

and

$$\left.\begin{aligned} [(1-2\nu)D^2-2(1-\nu)\eta^2]f_2(y,\eta)-\eta Df_3(y,\eta) &= -2(1+\nu)\alpha\eta\xi_1(\eta)e^{-y\eta} \\ [2(1-\nu)D^2-(1-2\nu)\eta^2]f_3(y,\eta)+\eta Df_2(y,\eta) &= -2(1+\nu)\alpha\eta\xi_1(\eta)e^{-y\eta} \end{aligned}\right\} \qquad (12)$$

where

$$D \equiv \frac{d}{dy}.$$

Solving (11) and (12), we obtain

$$u = \int_0^\infty [(e_1+e_2y)\cos x\eta + (e_3+e_4y)\sin x\eta]e^{-y\eta}\,d\eta \qquad (13)$$

$$v = \int_0^\infty \frac{1}{\eta}[\{\eta e_3+(3-4\nu)e_4+\eta e_4y-2(1+\nu)\alpha\xi_1(\eta)\}\cos x\eta$$

$$-\{\eta e_1+(3-4\nu)e_2+\eta e_2y+2(1+\nu)\alpha\xi_2(\eta)\}\sin x\eta]e^{-y\eta}\,d\eta \qquad (14)$$

where the constants e_1, e_2, e_3 and e_4, determined from the boundary conditions (1), are given by

$$\left.\begin{aligned} e_1 &= -\frac{(1+\nu)\alpha\xi_2(\eta)}{\eta}, \\ e_2 &= 0, \\ e_3 &= \frac{P(1-2\nu)}{\pi G}\left(\frac{\sin a\eta-\sin b\eta}{\eta^2}\right)+\frac{(1+\nu)}{\eta}\alpha\xi_1(\eta), \\ e_4 &= -\frac{P}{\pi G}\times\frac{\sin a\eta-\sin b\eta}{\eta}. \end{aligned}\right\} \quad \ldots \quad \ldots \quad (15)$$

Hence from (13), (14) and (15), we obtain

$$\begin{aligned} u &= \int_0^\infty e^{-y\eta}\left[\frac{(1+\nu)\alpha}{\eta}\{\xi_1(\eta)\sin x\eta-\xi_2(\eta)\cos x\eta\}\right. \\ &\quad \left.+\frac{P}{\pi G}\left(\frac{\sin a\eta-\sin b\eta}{\eta}\right)\left(\frac{1-2\nu}{\eta}-y\right)\sin x\eta\right]d\eta \quad \ldots \quad \ldots \quad \ldots \quad (16) \\ &= u_1+u_2+u_3, \end{aligned}$$

$$\begin{aligned} v &= -\int_0^\infty \frac{e^{-y\eta}}{\eta}\left[\left\{\frac{P}{\pi G}(\sin a\eta-\sin b\eta)\left(y+\frac{2-2\nu}{\eta}\right)+(1+\nu)\alpha\xi_1(\eta)\right\}\cos x\eta\right. \\ &\quad \left.+(1+\nu)\alpha\xi_2(\eta)\sin x\eta\right]d\eta \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (17) \\ &= v_1+v_2+v_3, \end{aligned}$$

where (u_1, v_1) are the displacement components due to surface traction, (u_2, v_2) due to the surface temperature $f(x)$ and (u_3, v_3) due to $g(x)$.

The associated stress components are obtained in the forms

$$\left.\begin{aligned} \tau_{xx} &= \frac{2P}{\pi}\int_0^\infty e^{-y\eta}\left(\frac{1}{\eta}-y\right)(\sin a\eta-\sin b\eta)\cos x\eta\,d\eta, \\ \tau_{yy} &= \frac{2P}{\pi}\int_0^\infty e^{-y\eta}\left(\frac{1}{\eta}+y\right)(\sin a\eta-\sin b\eta)\cos x\eta\,d\eta, \\ \tau_{xy} &= \frac{2Py}{\pi}\int_0^\infty e^{-y\eta}(\sin a\eta-\sin b\eta)\sin x\eta\,d\eta. \end{aligned}\right\} \quad \ldots \quad \ldots \quad (18)$$

$$\tau_{zz} = -2G(1+\nu)\alpha T+\frac{2P\nu}{\pi}\left[\tan^{-1}\frac{2ay}{x^2+y^2-a^2}-\tan^{-1}\frac{2by}{x^2+y^2-b^2}\right].$$

This completes the solution of the problem. We note that the expressions for the displacement components u, v involve divergent integrals.

Evaluation of the Stresses and Displacements

The contributions to the stress and displacement components arising from the surface traction and from the prescribed temperature on the surface will now be evaluated in terms of series involving elementary functions. It is clearly sufficient to evaluate the integrals for positive values of x.

Contributions from the surface traction:

The stress components and the displacement component u_1 are easily evaluated by integration in the following forms:

$$\tau_{xx_1}+\tau_{yy_1} = \frac{2P}{\pi}[\theta_3-\theta_4-\theta_1+\theta_2],$$

$$\tau_{xx_1}-\tau_{yy_1} = \frac{4Py}{\pi}\left[\frac{a(x^2-y^2-a^2)}{(x^2-y^2-a^2)^2+4a^2y^2}-\frac{b(x^2-y^2-b^2)}{(x^2-y^2-b^2)^2+4b^2y^2}\right],$$

$$\tau_{xy_1} = \frac{4Pxy^2}{\pi}\left[\frac{a}{(x^2+y^2-a^2)^2+4a^2y^2}-\frac{b}{(x^2+y^2-a^2)^2+4b^2y^2}\right],$$

$$u_1 = -\frac{P}{2\pi G}\left[(1-2\nu)\{\theta_2\rho_2\cos\theta_2-\theta_1\rho_1\cos\theta_1+\theta_3\rho_3\cos\theta_3-\theta_4\rho_4\cos\theta_4\}\right.$$
$$\left.+2(1-\nu)y\log\frac{\rho_1\rho_4}{\rho_2\rho_3}\right],$$

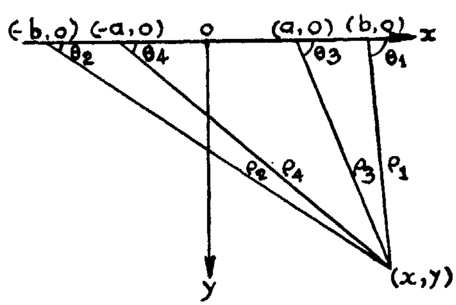

FIG. 1.

where the meanings of θ_1, ρ_1, etc., are clear from Fig. 1. The integral determining v_1 is, however, divergent so that we must take its finite part. For this purpose, we use the result (A4) proved in the appendix and obtain

$$v_1 = \frac{P}{2\pi G}[(1-2\nu)\{\theta_1-\theta_2-\theta_3+\theta_4\}+4(1-\nu)(1-\gamma)(b-a)$$
$$+2(1-\nu)\{\rho_1\cos\theta_1\log\rho_1-\rho_2\cos\theta_2\log\rho_2-\rho_3\cos\theta_3\log\rho_3+\rho_4\cos\theta_4\log\rho_4\}]$$

where γ is Euler's constant. As a check, these results were compared, after putting $a = 0$, with the corresponding results obtained by the complex variable method (Muskhelishvili 1953, p. 388). It is found that the two sets of results agree apart from the fact that u_1, as obtained by the complex variable method, does not tend to zero at infinity on the real axis. It is worth noting in this connection that for comparison we have used the additional condition that u_1 is an odd function of x while v_1 is an even function, in order to evaluate the arbitrary constants of integration obtained by the complex variable method.

Contributions from the symmetric surface temperature distribution $f(x)$:

These are

$$u_2 = (1+\nu)\alpha \sum_{n=1}^{\infty} dn \int_0^{\infty} \frac{e^{-y\eta}J_{2n}(b\eta)}{\eta^2} \sin x\eta \, d\eta,$$

$$v_2 = (1+\nu)\alpha \sum_{n=1}^{\infty} dn \int_0^{\infty} \frac{e^{-y\eta}J_{2n}(b\eta)}{\eta^2} \cos x\eta \, d\eta.$$

v_2 and u_2 are the real and imaginary parts of

$$I = (1+\nu)\alpha \sum_{n=1}^{\infty} d_n \int_0^{\infty} \frac{e^{-(y-ix)\eta}J_{2n}(b\eta)d\eta}{\eta^2}$$

$$= (1+\nu)\alpha \sum_{n=1}^{\infty} \frac{d_n b}{2n(2n-1) \cdot 2^{2n}} \left[\frac{b^{2n-1}F(n-\frac{1}{2}, n+1;\ 2n+1;\ z)}{\{(y-ix)^2+b^2\}^{n-\frac{1}{2}}}\right]$$

where the substitution

$$z = \frac{b^2}{(y-ix)^2+b^2} \quad \dots \quad \dots \quad \dots \quad \dots \quad (19)$$

and the value of the integral $\int_0^{\infty} e^{-at}J_\nu(bt)t^{\mu-1}\,dt$ (Watson 1958, p. 385) for positive values of Re a have been used.

The following results are particular cases of known formulae (Magnus and Oberhettinger 1954, pp. 8, 9):

$$F(n-\tfrac{1}{2}, n+1;\ 2n+1;\ z) = (1-z)^{-n+\frac{1}{2}}F\left(n-\tfrac{1}{2}, n;\ 2n+1;\ \frac{z}{z-1}\right),$$

$$F\left(n-\tfrac{1}{2}, n;\ 2n+1;\ \frac{4\zeta}{(1+\zeta)^2}\right) = (1+\zeta)^{2n-1}F(2n-1, -1;\ 2n+1;\ \zeta).$$

Using these results and substituting

$$\zeta = \frac{\sqrt{1-z}-1}{\sqrt{1-z}+1}, \quad \dots \quad \dots \quad \dots \quad \dots \quad (20)$$

it is easily seen that

$$F(n-\tfrac{1}{2}, n+1;\ 2n+1;\ z) = (1-\zeta)^{2n-1}\left[1-\frac{2n-1}{2n+1}\zeta\right].$$

Therefore,

$$I = (1+\nu)\alpha b \sum_{n=1}^{\infty} \frac{d_n}{2n(4n^2-1)} \left[\frac{z^{n-\frac{1}{2}}(2n+\sqrt{1-z})}{(1+\sqrt{1-z})^{2n}}\right]$$

z being given by (19) and v_2, u_2 being the real and imaginary parts of this series. The above result is true provided $y > 0$. If $y = 0$, the results are obtained as follows from the tables of sine and cosine transforms (Erdelyi 1954, pp. 44, 99):

$$\{u_2, v_2\} = (1+\nu)\alpha \sum_{n=1}^{\infty} d_n\{u_{2n}(x), V_{2n}(x)\}$$

where

$$u_{2n}(x) = \begin{cases} \dfrac{\sqrt{b^2-x^2}\sin\left[2n\sin^{-1}\dfrac{x}{b}\right]}{4n^2-1} - \dfrac{x\cos\left[2n\sin^{-1}\dfrac{x}{b}\right]}{2n(4n^2-1)}, & 0 < x < b, \\ (-1)^{2n+1}\dfrac{b^{2n}[x+2n\sqrt{x^2-b^2}]}{2n(4n^2-1)[x+\sqrt{x^2-b^2}]^{2n}}, & b < x < \infty, \end{cases}$$

$$V_{2n}(x) = \begin{cases} \dfrac{b\cos\left[(2n-1)\sin^{-1}\dfrac{x}{b}\right]}{4n(2n-1)} + \dfrac{b\cos\left[(2n+1)\sin^{-1}\dfrac{x}{b}\right]}{4n(2n+1)}, & 0 < x < b, \\ 0 & , b < x < \infty. \end{cases}$$

The component of the displacement normal to the boundary is thus seen to vanish outside the region $-b < x < b$.

Contributions from the anti-symmetric surface temperature distribution $g(x)$:

These are

$$(u_3, v_3) = -(1+\nu)\alpha \sum_{n=1}^{\infty} c_n \int_0^{\infty} \frac{e^{-y\eta}J_{2n-1}(b\eta)}{\eta^2}\{\cos x\eta, \sin x\eta\}\, d\eta.$$

As before, v_3 is obtained (for $y > 0$) to be

$$v_3 = -(1+\nu)\alpha \sum_{n=1}^{\infty} \frac{c_n b}{2^{2n-1}(2n-1)(2n-2)} \operatorname{Im}[z^{n-1}F(n-1, n+\tfrac{1}{2};\ 2n;\ z)].$$

Now (Magnus and Oberhettinger 1954, pp. 8, 9)

$$F(n-1, n+\tfrac{1}{2};\ 2n;\ z) = (1-z)^{-n-\frac{1}{2}}F\left(n+\tfrac{1}{2}, n+1;\ 2n;\ \frac{z}{z-1}\right)$$
$$= (1-\zeta)^{2n+1}F(2n+1, 1;\ 2n;\ \zeta)$$

by substitution (20).
Since (Magnus and Oberhettinger 1954, p. 9)

$$F(2n+1, 1;\ 2n;\ \zeta) - F(2n+1, 1;\ 2n+1;\ \zeta) = \frac{\zeta}{2n}F(2n+2, 2;\ 2n+2;\ \zeta),$$

$$F(2n+1, 1;\ 2n+1;\ \zeta) = \frac{1}{1-\zeta} \text{ and } F(2n+2, 2;\ 2n+2;\ \zeta) = \frac{1}{(1-\zeta)^2},$$

it is seen that

$$F(n-1, n+\tfrac{1}{2};\ 2n;\ z) = (1-\zeta)^{2n+1}\cdot\frac{2n-(2n-1)\zeta}{2n(1-\zeta)^2},$$

and

$$v_3 = -(1+\nu)\alpha \sum_{n=1}^{\infty} \frac{c_n b}{2n(2n-1)(2n-2)} \operatorname{Im}\left[\frac{z^{n-1}(4n-1+\sqrt{1-z})}{(1+\sqrt{1-z})^{2n}}\right].$$

The expression for u_3 ($y > 0$) is the real part of the expression of which the above is the imaginary part, except for the first term which is a divergent integral; its finite part has been calculated in the Appendix (A5).

Therefore,

$$u_3 = -(1+\nu)\alpha b\left[-\gamma-\tfrac{1}{2}\log(x^2+y^2)+\frac{1}{b}\int_0^\infty \frac{e^{-y\eta}\cos x\eta}{\eta^2}\{J_1(b\eta)-b\eta\}\,d\eta\right.$$
$$\left.+\sum_{n=2}^{\infty}\frac{c_n}{2n(2n-1)(2n-2)}\operatorname{Re}\left\{\frac{z^{n-1}(4n-1+\sqrt{1-z})}{(1+\sqrt{1-z})^{2n}}\right\}\right].$$

If $y = 0$, we obtain from the tables of integral transforms (Erdelyi 1954, pp. 44, 99), the expressions

$$u_3 = -(1+\nu)\alpha\left[\sum_{n=2}^{\infty} c_n U_{3n}(x)-\gamma b-\frac{b}{2}\log(x^2+y^2)+\int_0^\infty \frac{\cos x\eta\,\{J_1(b\eta)-b\eta\}}{\eta^2}\,d\eta\right],$$

$$v_3 = -(1+\nu)\alpha\sum_{n=1}^{\infty} c_n V_{3n}(x),$$

where

$$U_{3n}(x) = \begin{cases} \dfrac{b\cos\left[(2n-2)\sin^{-1}\dfrac{x}{b}\right]}{2(2n-1)(2n-2)}+\dfrac{b\cos\left[2n\sin^{-1}\dfrac{x}{b}\right]}{2(2n-1)2n}, & 0<x<b,\\[2ex] \dfrac{(-1)^{n-1}b^{2n-1}}{2(2n-1)(2n-2)[x+\sqrt{x^2-b^2}]^{2n-2}}-\dfrac{(-1)^{n-1}b^{2n+1}}{2(2n-1)2n[x+\sqrt{x^2-b^2}]^{2n}}, & b<x<\infty,\end{cases}$$

$$V_{3n}(x) = \begin{cases} \dfrac{\sqrt{b^2-x^2}\sin\left[(2n-1)\sin^{-1}\dfrac{x}{b}\right]}{2n(2n-2)}-\dfrac{x\cos\left[(2n-1)\sin^{-1}\dfrac{x}{b}\right]}{2n(2n-1)(2n-2)}, & 0<x<b,\\[2ex] 0, & b<x<\infty.\end{cases}$$

Thus, we find again that v_3 also vanishes on the boundary outside the interval $-b<x<b$.

The solutions obtained above are purely formal since, in general, it is very difficult to examine the convergence of the series obtained. However, in particular cases when c_n and d_n have simple forms the question of convergence may be studied. Thus $\theta(x) = \sin x$ (*see* Tranter 1960), we have

$$(-1)^{n-1}c_n = \frac{2P_{n-1}(\cos c)}{K\left(\cos\dfrac{c}{2}\right)},$$
$$d_n = 0,$$

and the series may be easily shown to be convergent since

$$|P_{n-1}(\cos c)|<1$$

and

$$\left|\sqrt{1+\left(\frac{y-ix}{b}\right)^2}+\frac{y-ix}{b}\right|>1 \quad \text{if } y>0 \text{ or } |x|>b.$$

If $y = 0$ and $|x| < b$, the convergence is obvious from the forms of the relevant series. Lastly, we note that the stresses, apart from τ_{zz} which contains a term $-2G(1+\nu)\alpha T$, are independent of the temperature.

Appendix

Since

$$\int_{\epsilon}^{\infty} \frac{\cos \lambda\eta}{\eta}\, d\eta = \int_{\epsilon}^{1} \frac{\cos \lambda\eta - 1}{\eta}\, d\eta + \int_{1}^{\infty} \frac{\cos \lambda\eta}{\eta}\, d\eta - \log \epsilon,$$

$$FP\int_{0}^{\infty} \frac{\cos \lambda\eta}{\eta}\, d\eta = \int_{\lambda}^{\infty} \frac{\cos t}{t}\, dt - \int_{0}^{\lambda} \frac{1-\cos t}{t}\, dt$$

$$= \gamma - \log \lambda, \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \text{(A1)}$$

by a well-known formula (Magnus and Oberhettinger 1954, p. 97); γ is Euler's constant.

Integrating by parts, we have

$$\int_{\epsilon}^{\infty} \frac{e^{-y\eta} \sin \tau\eta}{\eta^2}\, d\eta = \frac{e^{-y\epsilon} \sin \tau\epsilon}{\epsilon} + \tau \int_{\epsilon}^{\infty} \frac{e^{-y\eta} \cos \tau\eta}{\eta}\, d\eta - y \int_{\epsilon}^{\infty} \frac{e^{-y\eta} \sin \tau\eta}{\eta}\, d\eta$$

so that

$$FP\int_{0}^{\infty} \frac{e^{-y\eta} \sin \tau\eta}{\eta^2}\, d\eta = \tau - y \tan^{-1}\frac{\tau}{y} + \tau FP\int_{0}^{\infty} \frac{e^{-y\eta} \cos \tau\eta}{\eta}\, d\eta. \qquad \ldots \quad \ldots \quad \text{(A2)}$$

On integrating the relation

$$\int_{0}^{\infty} e^{-y\eta} \cos \tau\eta\, d\eta = \frac{y}{\tau^2 + y^2}$$

and using the result

$$\lim_{y \to 0} \int_{0}^{\infty} \frac{1 - e^{-y\eta}}{\eta} \cos \tau\eta\, d\eta = 0$$

we obtain

$$\int_{0}^{\infty} \frac{1 - e^{-y\eta}}{\eta} \cos \tau\eta\, d\eta = \tfrac{1}{2} \log \frac{\tau^2 + y^2}{\tau^2}.$$

Therefore,

$$FP\int_{0}^{\infty} \frac{e^{-y\eta} \cos \tau\eta}{\eta}\, d\eta = FP\int_{0}^{\infty} \frac{\cos \tau\eta}{\eta}\, d\eta - \tfrac{1}{2} \log \left(\frac{\tau^2 + y^2}{\tau^2}\right)$$

$$= -\gamma - \tfrac{1}{2} \log (\tau^2 + y^2), \qquad \ldots \quad \ldots \quad \text{(A3)}$$

using (A1). Hence

$$FP\int_{0}^{\infty} \frac{e^{-y\eta} \sin \tau\eta}{\eta^2}\, d\eta = (1-\gamma)\,\tau - y \tan^{-1}\frac{\tau}{y} - \frac{\tau}{2} \log (\tau^2 + y^2). \quad \ldots \quad \text{(A4)}$$

Finally, we have

$$FP\int_0^\infty \frac{e^{-y\eta}J_1(b\eta)\cos x\eta}{\eta^2}\,d\eta = \int_0^\infty \frac{e^{-y\eta}\cos x\eta\{J_1(b\eta)-b\eta\}}{\eta^2}\,d\eta + FPb\int_0^\infty \frac{e^{-y\eta}\cos x\eta}{\eta}\,d\eta$$

$$= -\gamma b - \frac{b}{2}\log(y^2+x^2) + \int_0^\infty \frac{e^{-y\eta}\cos x\eta\{J_1(b\eta)-b\eta\}}{\eta^2}\,d\eta. \quad \ldots \quad \ldots \quad (A5)$$

References

Erdelyi, A. (Ed.) (1954). Tables of Integral Transforms. Vol. 1. McGraw-Hill Book Co., Inc., New York.

Magnus, W., and Oberhettinger, F. (1954). Formulas and Theorems for the Functions of Mathematical Physics. Chelsea Publishing Co., New York.

Melan, E., and Parkus, H. (1953). Warmespannungen infolge stationarer Temperaturfelder. Springer, Wien.

Muskhelishvili, N. I. (1953). Some Basic Problems of the Mathematical Theory of Elasticity. P. Noordhoff, Groningen.

Sen, B. (1951). Note on the stresses produced by nuclei of thermoelastic strain in a semi-infinite elastic solid. *Q. appl. Math.*, 8, 365.

Timoshenko, S., and Goodier, J. N. (1951). Theory of Elasticity. McGraw-Hill Book Co., Inc., New York.

Tranter, C. J. (1959). Dual trigonometric series. *Proc. Glasg. math. Ass.*, **4**, 49.

———— (1960). Some triple integral equations. *Proc. Glasg. math. Ass.*, **4**, 200.

Watson, G. N. (1958). A Treatise on the Theory of Bessel Functions. Cambridge University Press, p. 386.