GRAVITATIONAL INSTABILITY OF AN INFINITE ISOTHERMAL STRATIFIED MEDIUM UNDER UNIFORM ROTATION USING THE PRINCIPLE OF EXCHANGE OF STABILITIES

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In this paper we consider the gravitational instability of an infinitely extending isothermal and stratified medium under uniform rotation using the principle of exchange of stabilities. The method adopted here closely follows the one used by Bhatnagar in solving a similar problem with $\gamma = 2$. We find that rotation has a stabilizing effect on the system and that for each value of rotation there exist two critical values of wave-number $\bar{k}_1$ and $\bar{k}_2$ such that the system is stable for $\bar{k} < \bar{k}_1$ and $\bar{k} > \bar{k}_2$. We also find that when $\frac{\pi G \rho_c}{2\Omega^2} < 2 \cdot 1$, where Ω is the magnitude of rotation, the system is stable for all wave-numbers. The present investigation corroborates the general conclusions arrived at by Goldreich and Lynden-Bell who have dealt with the same problem by a different method but differs in a neighbourhood of $\bar{k} = 0$ from the result of an earlier investigation by us using series in Chebyshev polynomials.

1. INTRODUCTION

In this note we consider the gravitational instability of an infinitely extending isothermal medium which is stratified in plane parallel layers and is under uniform rotation. In a previous paper (Uberoi *et al.*, *communicated*) we have considered the same problem making use of Clenshaw's method (1957) of solving ordinary linear differential equations employing series in Chebyshev polynomials. We have shown in that paper that rotation has a stabilizing effect on the system in that the critical wave-length of the disturbance beyond which the system becomes unstable increases when rotational frequency increases. In fact, there exists a critical value for rotation above which the system is stable for disturbances of all wave-lengths. We have also shown that there is only one critical wave-length corresponding to a given value of rotation which is in conformity with the results obtained by Ledoux (1951) and Chandrasekhar (1961) for a stratified medium without rotation and a uniform medium under uniform rotation respectively. Here we may point out that our result is different in this respect from that of Goldreich and Lynden-Bell (1965), who find that there are two critical wave-lengths for a given value of rotation. However, we have, in that paper, demonstrated beyond doubt the efficacy of the Clenshaw's method in dealing with gravitational instability problems. In fact, the critical value of rotation above which the system becomes stable for disturbances of all wave-lengths, as predicted by us using this method, does not differ very much from that obtained by Goldreich and Lynden-Bell (1965).

In the present paper we deal with the problem directly, following closely the method given by Bhatnagar (1967) who has solved the problem of gravitational instability of a rotating polytropic sheet with $\gamma = 2$. The present investigation is motivated by the fact that the above-mentioned work has yielded completely new results which could not be obtained by the method employed by Goldreich and Lynden-Bell, indicating that the method used here is more general and comprehensive. The method does not involve any approximation in the analysis and is based on the fact that the parameter corresponding to the rotational frequency is still retained in the boundary conditions though it gets out the main equations, when ω, the frequency of disturbance, is set to be zero for the purpose of investigating the neutral stability.

The present investigation corroborates the general conclusions arrived at by Goldreich and Lynden-Bell and differs from our earlier result in the neighbourhood of $\bar{k} = 0$. We attribute this to the fact that the truncated series solution remains continuous and bounded even in the neighbourhood of $\bar{k} = 0$, while the present investigation shows that $\frac{1}{R}$, i.e. $\frac{\pi G \rho_c}{2\Omega^2} \to \infty$ as $\bar{k} \to 0$.

2. Equilibrium State

We consider an infinitely extending self-gravitating medium rotating with angular velocity $\vec{\Omega}(0, 0, \Omega)$ about the z-axis. The medium is assumed to be uniform in the planes parallel to $z = 0$. Let ρ_0 and ψ_0 be the equilibrium density and the gravitational potential respectively. If we assume an applied gravitational field $-\Omega^2(x, y, 0)$ towards the axis of rotation following Goldreich and Lynden-Bell (1965), then ρ_0 and ψ_0 are given by

$$\rho_0 = \rho_c(1-\mu^2), \qquad (2.1)$$

$$\psi_0 = c^2 \ln (1-\mu^2) + \tfrac{1}{2}\frac{A_0}{\alpha} \ln \frac{1+\mu}{1-\mu} + B_0, \qquad (2.2)$$

where

$$\mu = \tanh \alpha z, \quad \alpha = \sqrt{\frac{2\pi G \rho_c}{c^2}}, \qquad (2.3)$$

ρ_c is the density at the central plane $z = 0$ and c is the velocity of sound.

3. Linearized Equations

Let ρ_1, p, ψ_1 and $\vec{v}$ represent the small perturbations in density, pressure, gravitational potential and velocity respectively. We assume that these small perturbations vary as $[\exp (i\omega t + ikx)]$ with time t and spatial coordinate

x in order to study the stability of the system against disturbances propagating perpendicular to the axis of rotation. Then the linearized equations governing the small perturbations are

$$i\omega v_z = \frac{d\chi_1}{dz}, \quad \dots \quad (3.1)$$

$$\chi_1 = \psi_1 - c^2\frac{\rho_1}{\rho_0},$$
$$= \psi_1 - c^2\Theta,$$

where

$$\Theta = \frac{\rho_1}{\rho_0} \quad \dots \quad (3.2)$$

$$\frac{d^2\psi_1}{dz^2} - k^2\psi_1 = -4\pi G\rho_c \operatorname{sech}^2 \alpha z\, \Theta \quad \dots \quad (3.3)$$

and

$$i\omega\rho_1 - i\omega\frac{k^2\rho_0\chi_1}{4\Omega^2-\omega^2} + \frac{d}{dz}(\rho_0 v_z) = 0. \quad \dots \quad (3.4)$$

4. Dispersion Relation for Neutral Stability

From (3.1) and (3.2) we have

$$\rho_0 v_z = \frac{\rho_c\alpha}{i\omega}(1-\mu^2)^2\frac{d}{d\mu}(\psi_1 - c^2\Theta). \quad \dots \quad (4.1)$$

Substituting for v_z from (4.1) in (3.4), we obtain

$$-\omega^2\rho_1 + \omega^2\frac{k^2\rho_0}{4\Omega^2-\omega^2}(\psi_1 - c^2\Theta) + \rho_c\alpha^2(1-\mu^2)\frac{d}{d\mu}\left[(1-\mu^2)^2\frac{d}{d\mu}(\psi_1-c^2\Theta)\right] = 0. \quad \dots \quad (4.2)$$

Since it has been shown that overstability cannot occur in the present case Goldreich and Lynden-Bell (1965), we set $\omega = 0$ to discuss neutral stability. Equation (4.2) then reduces to

$$\frac{d}{d\mu}[\psi_1 - c^2\Theta] = \frac{A}{(1-\mu^2)^2},$$

from which we obtain

$$\psi_1 = c^2\Theta + B + A\left[\tfrac{1}{2}\frac{\mu}{1-\mu^2} + \tfrac{1}{4}\ln\frac{1+\mu}{1-\mu}\right].$$

Since ψ_1 and Θ have to be bounded as $\mu \to \pm 1$, we must have $A = 0$. So we have

$$\psi_1 = c^2\Theta + B. \quad \dots \quad (4.3)$$

Substituting in eqn. (3.3), we get

$$\frac{d^2}{dz^2}(c^2\Theta + B) - k^2(c^2\Theta + B) = -4\pi G\rho_c(1-\mu^2)\Theta,$$

which yields

$$(1-\mu^2)\frac{d^2\Theta}{d\mu^2}-2\mu\frac{d\Theta}{d\mu}+\left[2-\frac{\bar{k}^2}{1-\mu^2}\right]\Theta=\frac{\bar{B}\bar{k}^2}{1-\mu^2}, \quad \ldots \quad \ldots \quad (4.4)$$

where

$$\bar{k}=\frac{k}{\alpha},\ \bar{B}=\frac{B}{\alpha}.$$

To solve eqn. (4.4) we first consider the homogeneous equation

$$(1-\mu^2)\frac{d^2\Theta}{d\mu^2}-2\mu\frac{d\Theta}{d\mu}+\left[2-\frac{\bar{k}^2}{1-\mu^2}\right]\Theta=0. \quad \ldots \quad \ldots \quad (4.5)$$

Two linearly independent solutions of this equation, when

$\bar{k}\neq 0, \pm 1, \pm 2, \ldots, -1<\mu<1$, are

$$P_1^{\bar{k}}(\mu)=\frac{1}{\Gamma(1-\bar{k})}\left(\frac{1+\mu}{1-\mu}\right)^{\frac{\bar{k}}{2}}{}_2F_1\left(-1,2;\ 1-\bar{k};\ \frac{1-\mu}{2}\right)$$

$$=\frac{1}{\Gamma(2-\bar{k})}\left(\frac{1+\mu}{1-\mu}\right)^{\frac{\bar{k}}{2}}(\mu-\bar{k}) \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (4.6)$$

and

$$P_1^{-\bar{k}}(\mu)=\frac{1}{\Gamma(1+\bar{k})}\left(\frac{1-\mu}{1+\mu}\right)^{\frac{\bar{k}}{2}}{}_2F_1\left(-1,2;\ 1+\bar{k};\ \frac{1-\mu}{2}\right)$$

$$=\frac{1}{\Gamma(2+\bar{k})}\left(\frac{1-\mu}{1+\mu}\right)^{\frac{\bar{k}}{2}}(\mu+\bar{k}). \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (4.7)$$

Now the solution of the inhomogeneous eqn. (4.4) can be written, using variation of parameters, as

$$\Theta(\mu)=C\cdot\frac{1}{\Gamma(2-\bar{k})}\left(\frac{1+\mu}{1-\mu}\right)^{\frac{\bar{k}}{2}}(\mu-\bar{k})$$

$$+D\cdot\frac{1}{\Gamma(2+\bar{k})}\left(\frac{1-\mu}{1+\mu}\right)^{\frac{\bar{k}}{2}}(\mu+\bar{k})$$

$$+\frac{\bar{B}\bar{k}}{2(1-\bar{k}^2)}\left[(\bar{k}+\mu)\left(\frac{1-\mu}{1+\mu}\right)^{\frac{\bar{k}}{2}}I_1-(\bar{k}-\mu)\left(\frac{1+\mu}{1-\mu}\right)^{\frac{\bar{k}}{2}}I_2\right], \quad \ldots \quad (4.8)$$

where

$$I_1=\int^{\mu}\frac{\bar{k}-\xi}{1-\xi^2}\left(\frac{1+\xi}{1-\xi}\right)^{\frac{\bar{k}}{2}}d\xi, \quad \ldots \quad \ldots \quad \ldots \quad (4.9)$$

$$I_2=\int^{\mu}\frac{\bar{k}+\xi}{1-\xi^2}\left(\frac{1-\xi}{1+\xi}\right)^{\frac{\bar{k}}{2}}d\xi. \quad \ldots \quad \ldots \quad \ldots \quad (4.10)$$

Again, since Θ must be bounded as $\mu \to \pm 1$, we must have $C = D = 0$.

Let $\eta(z) \exp [i\omega t + ikx]$ represent the displacement in the boundary so that we have

$$i\omega\eta(z) = v_z(z).$$

Substituting in (4.4) we obtain

$$\frac{d}{d\mu}\left[(1-\mu^2)\bar{\eta}\right] = \frac{\bar{k}^2}{8\bar{\Omega}^2}\bar{B} - \Theta, \qquad \ldots \quad \ldots \quad \ldots (4.11)$$

where $\bar{\eta} = \alpha\eta$ and $\bar{\Omega}^2 = \dfrac{\Omega^2}{4\pi G\rho_c}$.

Integrating eqn. (4.11) from -1 to $+1$ and making use of the fact that $\bar{\eta}$ must be bounded as $\mu \to \pm 1$,
we get

$$\int_{-1}^{1} \Theta(\mu)\, d\mu = \frac{Bk^2}{4\Omega^2}, \quad \ldots \quad \ldots \quad \ldots \quad \ldots (4.12)$$

where we have dropped the bar over $\bar{B}$ and $\bar{k}$ for convenience.

Substituting the value of Θ from (4.8) we find that

$$\frac{k(1-k^2)}{2\Omega^2} = J_2 - J_1,$$

where

$$J_1 = \int_{-1}^{1} (k-\mu)\left(\frac{1+\mu}{1-\mu}\right)^{\frac{k}{2}} d\mu \int_{1}^{\mu} \frac{d\xi}{1-\xi^2}(k+\xi)\left(\frac{1-\xi}{1+\xi}\right)^{\frac{k}{2}}, \qquad \ldots (4.13)$$

$$J_2 = \int_{-1}^{1} (k+\mu)\left(\frac{1-\mu}{1+\mu}\right)^{\frac{k}{2}} d\mu \int_{-1}^{\mu} \frac{d\xi}{1-\xi^2}(k-\xi)\left(\frac{1+\xi}{1-\xi}\right)^{\frac{k}{2}}. \qquad \ldots (4.14)$$

However, it is easy to show that $J_2 = -J_1$ and hence the final form of the dispersion relation is

$$\frac{k(1-k^2)}{4\Omega^2} + J_1 = 0.$$

5. Evaluation of J_1

We have

$$J_1 = \int_{-1}^{1} d\mu(k-\mu)\left(\frac{1+\mu}{1-\mu}\right)^{\frac{k}{2}} I_2,$$

where

$$I_2 = \int_{1}^{\mu} \frac{d\xi}{1-\xi^2}(k+\xi)\left(\frac{1-\xi}{1+\xi}\right)^{\frac{k}{2}}.$$

We shall first evaluate I_2.

Partially integrating once, we get

$$I_2 = -\frac{1}{k}\left(\frac{1-\mu}{1+\mu}\right)^{\frac{k}{2}}(k+\mu) - \frac{2}{k}\Phi\left(\frac{1-\mu}{1+\mu}\right),$$

where

$$\Phi(x) = \int_0^x \frac{t^{\frac{k}{2}}}{(1+t)^2}\, dt.$$

Hence,

$$J_1 = \int_{-1}^{1} d\mu(k-\mu)\left(\frac{1+\mu}{1-\mu}\right)^{\frac{k}{2}}\left[-\frac{1}{k}\left(\frac{1-\mu}{1+\mu}\right)^{\frac{k}{2}}(k+\mu)-\frac{2}{k}\Phi\left(\frac{1-\mu}{1+\mu}\right)\right],$$

$$= \frac{2}{k}\left(\frac{1}{3}-k^2\right)-\frac{2}{k}K, \qquad \text{(5.1)}$$

where

$$K = \int_{-1}^{1} d\mu(k-\mu)\left(\frac{1+\mu}{1-\mu}\right)^{\frac{k}{2}}\Phi\left(\frac{1-\mu}{1+\mu}\right),$$

$$= 2\int_0^\infty \frac{x^{-\frac{k}{2}}}{(1+x)^3}[(k-1)+(k+1)x]\, dx \int_{t=0}^{t=x}\frac{t^{\frac{k}{2}}}{(1+t)^2}\, dt,$$

where $x = \dfrac{1-\mu}{1+\mu}$.

Partially integrating the integral inside once, we obtain

$$K = K_1 + kK_2 \qquad \text{(5.2)}$$

where

$$K_1 = -2\int_0^\infty \frac{(k-1)+(k+1)x}{(1+x)^4}\, dx, \qquad \text{(5.3)}$$

$$K_2 = \int_0^\infty dx \frac{x^{-\frac{k}{2}}}{(1+x)^3}[(k-1)+(k+1)x]\int_0^x \frac{t^{\frac{k}{2}-1}}{1+t}\, dt. \qquad \text{(5.4)}$$

Now K_1 can be integrated easily to give

$$K_1 = \tfrac{1}{3}-k.$$

Hence

$$J_1 = 2(1-k)-2K_2. \qquad \text{(5.5)}$$

To evaluate K_2 we first change the order of integration, obtaining

$$K_2 = \int_{t=0}^{t=\infty}\int_{x=t}^{x=\infty}\frac{x^{-\frac{k}{2}}\, t^{\frac{k}{2}-1}}{(1+x)^3(1+t)}[(k-1)+(k+1)x]\, dx\, dt. \qquad \text{(5.6)}$$

Changing to polar coordinates through $x = r\sin\theta$, $t = r\cos\theta$, we have

$$K_2 = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} d\theta \frac{(\cot\theta)^{\frac{k}{2}}}{\cos\theta}\int_0^\infty \frac{(k-1)+(k+1)r\sin\theta}{(1+r\sin\theta)^3(1+r\cos\theta)}\, dr,$$

$$= (k+1)\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} d\theta \frac{(\cot\theta)^{\frac{k}{2}}}{\cos^2\theta\sin^2\theta}K_3(\theta), \qquad \text{(5.7)}$$

where

$$K_3(\theta) = \int_0^\infty \frac{r+\frac{k-1}{k+1}\operatorname{cosec}\theta}{(r+\operatorname{cosec}\theta)^3(r+\sec\theta)}\,dr,$$

$$= \int_0^\infty \frac{r+\gamma}{(r+\alpha)^3(r+\beta)}\,dr,$$

where

$$\alpha = \operatorname{cosec}\theta\,;\ \beta = \sec\theta\,;\ \gamma = \frac{k-1}{k+1}\operatorname{cosec}\theta.$$

Integrating $K_3(\theta)$ by expressing the integrand in partial fractions, we get

$$K_3(\theta) = \sin^2\theta\cos\theta\left[-\frac{1}{k+1}\frac{1}{\sin\theta-\cos\theta}+\frac{\sin\theta-\frac{k-1}{k+1}\cos\theta}{(\sin\theta-\cos\theta)^2}\right.$$

$$\left.+\frac{\sin\theta-\frac{k-1}{k+1}\cos\theta}{(\sin\theta-\cos\theta)^3}\cos\theta.\ \ln\cot\theta\right].$$

Substituting this in K_2 and making the transformation $t = \tan\theta$ we obtain

$$K_2 = \int_1^\infty dt\, t^{-\frac{k}{2}}\left[-\frac{1}{t-1}+\frac{(k+1)t+(1-k)}{(t-1)^2}-\frac{(k+1)t+(1-k)}{(t-1)^3}\ln t\right],$$

$$= \int_0^\infty \frac{du}{(1+u)^{\frac{k}{2}}}\left[-\frac{1}{u}+\frac{(k+1)u+2}{u^2}-\frac{(k+1)u+2}{u^3}\ln(1+u)\right], \quad \ldots \quad (5.8)$$

where $u = t-1$.

Equation (5.5) now reads

$$J_1 = 2(1-k)+2K_4, \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (5.9)$$

where

$$K_4 = \int_0^\infty\left[-\frac{k}{u}-\frac{2}{u^2}+\left\{\frac{k+1}{u^2}+\frac{2}{u^3}\right\}\ln(1+u)\right]\frac{du}{(1+u)^{\frac{k}{2}}},$$

$$= \left[\frac{1}{(1+u)^{\frac{k}{2}}}\left\{\frac{1}{u}-\ln(1+u)\left(k+\frac{k+1}{u}+\frac{1}{u^2}\right)\right\}\right]_0^\infty$$

$$+\frac{k}{2}\int_0^\infty\left[\frac{1}{u}-\ln(1+u)\left(k+\frac{k+1}{u}+\frac{1}{u^2}\right)\right]\frac{1}{(1+u)^{\frac{k}{2}+1}}\,du.$$

The integrated part vanishes as $u\to\infty$ and tends to $-(k+\frac{1}{2})$ as $u\to 0$.

Hence,

$$K_4 = k+\tfrac{1}{2}+\frac{k}{2}\int_0^\infty \left(\frac{1}{u}-\frac{1}{u^2}\ln\,(1+u)\right)\frac{1}{(1+u)^{\frac{k}{2}+1}}\,du$$

$$-\frac{k}{2}\int_0^\infty \frac{\left(k+\dfrac{k+1}{u}\right)\ln\,(1+u)}{(1+u)^{\frac{k}{2}+1}}\,du.$$

The first integral on the right can be shown to be equal to

$$\frac{k}{2}\left[-1+\left(\frac{k}{2}+1\right)\int_0^\infty \frac{\ln\,(1+u)}{u(1+u)^{\frac{k}{2}+1}}\,du\right].$$

Hence,

$$K_4 = \frac{k}{2}+\tfrac{1}{2}-\frac{k^2}{4}\int_0^\infty \frac{\ln\,(1+u)}{u(1+u)^{\frac{k}{2}+1}}\,du-\frac{k^2}{2}\int_0^\infty \frac{\ln\,(1+u)}{(1+u)^{\frac{k}{2}+1}}\,du. \qquad \text{.. (5.10)}$$

But,

$$\int_0^\infty \frac{\ln\,(1+u)}{(1+u)^{\frac{k}{2}+1}}\,du = \frac{4}{k^2}.$$

Therefore,

$$K_4 = \frac{k}{2}-\tfrac{3}{2}-\frac{k^2}{4}\int_0^\infty \frac{\ln\,(1+u)}{u(1+u)^{\frac{k}{2}+1}}\,du. \qquad \text{.. .. .. (5.11)}$$

Thus,

$$J_1 = -(1+k)-\frac{k^2}{2}\int_0^\infty \frac{\ln\,(1+u)}{u(1+u)^{\frac{k}{2}+1}}\,du. \qquad \text{.. .. (5.12)}$$

Thus the dispersion relation is

$$\frac{\bar{k}(1-\bar{k}^2)}{4\bar{\Omega}^2} = 1+\bar{k}+\frac{\bar{k}^2}{2}J,$$

or

$$\frac{1}{R} = \frac{1+\bar{k}+\dfrac{\bar{k}^2}{2}J}{2\bar{k}(1-\bar{k}^2)}, \qquad \text{.. .. .. (5.13)}$$

where

$$R = \frac{2\Omega^2}{\pi G\rho_c} = 8\bar{\Omega}^2, \qquad \text{.. .. .. (5.14)}$$

and

$$J = \int_0^\infty \frac{\ln\,(1+u)}{u(1+u)^{\frac{\bar{k}}{2}+1}}\,du$$

$$= \int_0^\infty \frac{x\exp\left[-\left(\dfrac{\bar{k}}{2}+1\right)x\right]}{1-e^{-x}}\,dx$$

$$= \zeta\left(2,\,\frac{\bar{k}}{2}+1\right), \qquad \text{.. .. .. (5.15)}$$

where

$$\zeta(s, \nu) = \sum_{n=0}^{\infty} (\nu+n)^{-s}$$

is the generalized zeta function (Erdélyi 1953).

The expression for J differs from that of Goldreich and Lynden-Bell in as much as $\frac{\bar{k}}{2}$ occurs in the place of $\frac{\bar{k}}{2}+1$.

6. Conclusions

In the figure, the continuous curve represents the plot of $\frac{1}{R}$ against the wave-number $\bar{k}$ based on the present investigation, while the dotted line represents the same curve on the basis of our investigation in the earlier paper,

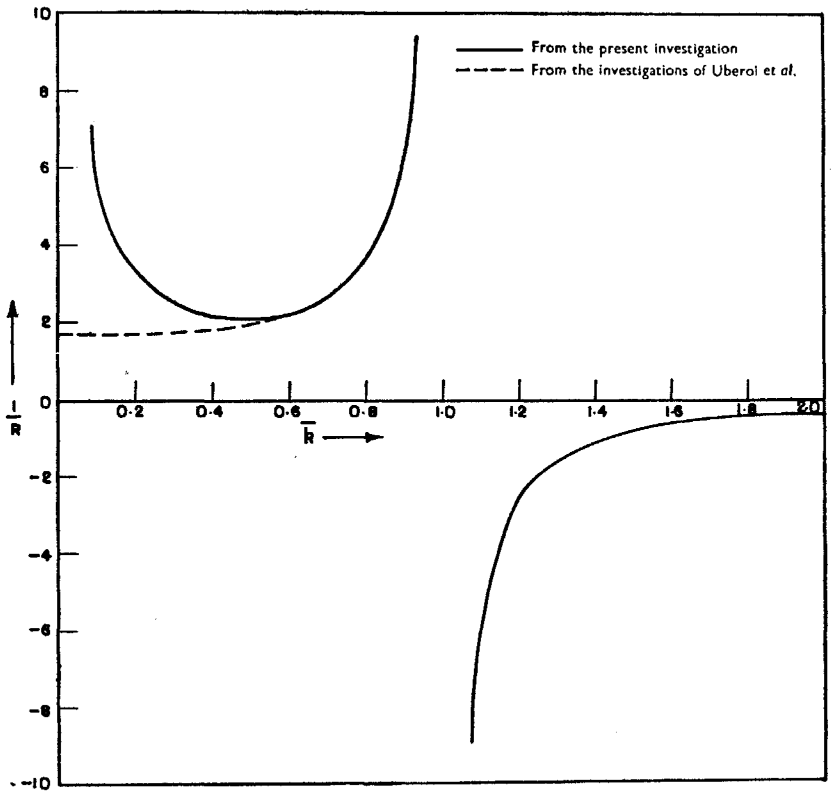

Fig. 1. Neutral stability curves.

where, in the series solution, only 48 terms were retained. From the graph it is clear that except in a close neighbourhood of $\bar{k} = 0$, the two investigations predict identical results. We also mention that the continuous curves are identical with those obtained by Goldreich and Lynden-Bell by a different method. When $\frac{\pi G \rho_c}{2\Omega^2} < 2{\cdot}1$ the system is stable for all $\bar{k}$ but for $\frac{\pi G \rho_c}{2\Omega^2} > 2{\cdot}1$ there are two critical wave-numbers $\bar{k}_1$ and $\bar{k}_2$, such that the system is stable for $\bar{k} < \bar{k}_1$ and $\bar{k} > \bar{k}_2$. Moreover, as $\Omega \to 0$, $\bar{k}_1 \to 0$ so that for small rotations, large wave-length disturbances are unstable. When $\bar{k} > 1$, i.e. $k^2c^2 > 2\pi G \rho_c$, the system is stable for all rotations.

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