FLOW OF VISCO-ELASTIC MAXWELL FLUID BETWEEN TORSIONALLY OSCILLATING DISCS

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The flow of an incompressible visco-elastic Maxwell fluid between two parallel infinite discs executing small torsional oscillations in their own plane is discussed. Two cases, (i) one disc oscillating and the other at rest, and (ii) both discs oscillating with same frequency and speed but with a phase difference of 180°, have been studied. In both cases the effect of relaxation time parameter has been shown graphically on: (*a*) the amplitude of transverse velocity, (*b*) the steady part of radial velocity and (*c*) the typical stream-lines of the steady part of radial-axial flow. It is found that with the increase of relaxation time parameter the elastic effects dominate in the region away from the oscillating disc and for small relaxation time, viscous effects permeate the entire flow.

§ 1. INTRODUCTION

In a recent paper Rosenblat (1960) has investigated the flow induced in a viscous fluid from small torsional oscillations of two infinite discs. The following two cases have been studied:

(i) One disc is oscillating and the other is at rest;

(ii) Both discs oscillate with same frequency and speed but with phase difference of 180°. The theoretical analysis has been extended by Rajeswari (1961) for Reiner-Rivlin fluid. She found that the radial-axial flow has a mean steady component and a fluctuating component of frequency twice that of the oscillating disc, a result similar to that for the Newtonian case obtained by Rosenblat. Bhatnagar and Rajeswari (1962) and Srivastava (1963) have studied the same problem for a special case of the Rivlin-Ericksen 'second order' fluid. Frater (1964) has discussed only the first case for Oldroyd fluid. Bhatnagar and Rajeswari have found that a reversal of the direction of the steady secondary flow is a characteristic feature of the Rivlin-Ericksen fluid and pointed out that it is always possible to find a value of the Reynolds number above which the flow is reversed in direction. This reversal of flow phenomenon is also predicted by Frater for Oldroyd fluid with elastic parameter $\sigma < (1/3)$ and a critical range of fluid parameter S.

The present note is devoted to the study of the above-mentioned problem of Rosenblat for Maxwell fluid. The constitutive equation of the fluid is

$$\left(1+\lambda\frac{\partial}{\partial t}\right)\tau_{ij} = 2\mu e_{ij}, \quad \cdots \quad (1)$$

where τ_{ij} is the deviatoric stress tensor, e_{ij} is the strain rate tensor, μ is the coefficient of viscosity and the parameter λ has the dimensions of time. If the motion is stopped the stress will relax as exp $(-t/\lambda)$, consequently λ is known as the relaxation time.

The method adopted for analysis is similar to that of Rosenblat (1960). The amplitude of oscillation has been assumed to be very small and, therefore, terms of the order of $(\Omega/n)^2$ have been neglected in the equations of motion. It is observed that the radial-axial flow has a steady component and a fluctuating component of frequency twice that of the oscillating disc. For small relaxation time, viscous effects permeate the entire flow. The pattern of typical streamlines is similar to that of Newtonian fluid and there is no reversal of flow contrary to the reversal of flow as predicted by Frater for Oldroyd fluid. The effect of the relaxation time parameter on (i) the amplitude of transverse velocity, (ii) steady component of radial velocity and (iii) typical streamlines of the steady part of the radial-axial flow has been shown diagrammatically.

§ 2. Equations of Motion

Let u, v, w be the velocity components in the directions r, θ, z respectively. Let $z = 0$ and $z = d$ represent two infinite parallel plane discs bounding the Maxwell fluid. The discs perform torsional oscillations about the axis $r = 0$. If the disc $z = 0$ performs torsional oscillations of frequency n and angular speed Ω while the disc $z = d$ remains at rest, then the boundary conditions are

$$\left.\begin{aligned} &u = w = 0, \quad v = \mathrm{Re}\,[r\Omega \exp(\mathrm{int})], && \text{at } z = 0,\\ &u = v = w = 0, && \text{at } z = d. \end{aligned}\right\} \quad \cdots \quad (2)$$

If both discs are oscillating with the same frequency and angular speed, but in opposite directions, the boundary conditions are

$$\left.\begin{aligned} &u = w = 0,\ v = \mathrm{Re}\,[r\Omega \exp(\mathrm{int})], && \text{at } z = 0,\\ &u = w = 0,\ v = -\mathrm{Re}\,[r\Omega \exp(\mathrm{int})], && \text{at } z = d. \end{aligned}\right\} \quad \cdots \quad (3)$$

In both cases we assume the velocity components and pressure in the following form:

$$\left.\begin{aligned} &u = \frac{r\Omega^2}{n}\frac{\partial}{\partial y}F(y, T),\ v = r\Omega \exp(iT)G(y),\ w = -(2d\Omega^2/n)F(y, T),\\ &(p/\rho) = (1/2)\,\Omega^2 r^2 P(y, T) + 2\Omega^2 d^2 K(y, T),\ y = (z/d), \text{ and } T = nt, \end{aligned}\right\} \quad (4)$$

where p and ρ are the pressure and density respectively.

The velocity components satisfy the equation of continuity

$$\frac{\partial u}{\partial r}+\frac{u}{r}+\frac{\partial w}{\partial z} = 0.$$

Substituting (1) and (4) in equations of motion and neglecting $(\Omega/n)^2$, as the amplitude (Ω/n) is small and equating the coefficients of equal powers of r we get

$$\left.\begin{aligned}
&\left(1+\bar{\lambda}\frac{\partial}{\partial T}\right)\left[\frac{\partial^2 F}{\partial y \partial T}-\{G\exp(iT)\}^2\right] = -\left(1+\bar{\lambda}\frac{\partial}{\partial T}\right)P+\frac{1}{R}\frac{\partial^3 F}{\partial y^3},\\
&\left(1+\bar{\lambda}\frac{\partial}{\partial T}\right)[iG\exp(iT)] = \frac{1}{R}\exp(iT)\frac{\partial^2 G}{\partial y^2},\\
&\left(1+\bar{\lambda}\frac{\partial}{\partial T}\right)\frac{\partial P}{\partial y} = 0,\\
&\text{and}\\
&\left(1+\bar{\lambda}\frac{\partial}{\partial T}\right)\left(\frac{\partial F}{\partial T}\right) = \left(1+\bar{\lambda}\frac{\partial}{\partial T}\right)\left(\frac{\partial K}{\partial y}\right)+\frac{1}{R}\frac{\partial^2 F}{\partial y^2},
\end{aligned}\right\} \quad .. \quad (5)$$

where $R = (nd^2\rho/\mu)$ and $\bar{\lambda} = n\lambda$.

From (2), (3) and (4) the boundary conditions are

$$\left.\begin{aligned}
&F = \frac{\partial F}{\partial y} = 0, \quad G = 1, \quad \text{at } y = 0,\\
&F = \frac{\partial F}{\partial y} = G = 0, \qquad \text{at } y = 1,
\end{aligned}\right\} \quad .. \quad .. \quad (6)$$

and

$$\left.\begin{aligned}
&F = \frac{\partial F}{\partial y} = 0, \quad G = 1, \qquad \text{at } y = 0,\\
&F = \frac{\partial F}{\partial y} = 0, \quad G = -1, \quad \text{at } y = 1.
\end{aligned}\right\} \quad .. \quad .. \quad (7)$$

§ 3. One Disc Oscillating

The second equation of (5) with boundary conditions (6) gives

$$G(y) = \frac{\sinh \frac{1}{2} s\alpha y_1}{\sinh \frac{1}{2} sx}, \quad .. \quad .. \quad .. \quad .. \quad (8)$$

where

$$s = \sqrt{2R}, \quad c^2 = \sqrt{1+\bar{\lambda}^2}-\bar{\lambda}, \quad \alpha = c+\frac{i}{c}, \quad y_1 = 1-y.$$

From (8), the amplitude of transverse velocity is

$$\frac{|v|}{r\Omega} = \left(\frac{\cosh scy_1-\cos\frac{s}{c}y_1}{\cosh sc-\cos\frac{s}{c}}\right)^{\frac{1}{2}}, \quad .. \quad .. \quad .. \quad (9)$$

and the phase angle is

$$\tan^{-1}\left[\frac{\sinh\frac{sc}{2}(1+y_1)\sin\frac{s}{2c}(1-y_1)-\sinh\frac{sc}{2}(1-y_1)\sin\frac{s}{2c}(1+y_1)}{\cosh\frac{sc}{2}(1+y_1)\cos\frac{s}{2c}(1-y_1)-\cosh\frac{sc}{2}(1-y_1)\cos\frac{s}{2c}(1+y_1)}\right]. \quad .. \quad (10)$$

The skin friction on the disc $z = 0$ is

$$(\tau_{\theta z})_{z=0} = -\frac{\mu r\Omega s}{2cd\left(\cosh sc-\cos\frac{s}{c}\right)(1+n^2\lambda^2)}\left[\left\{(n\lambda+c^2)\sin\frac{s}{c}\right.\right.$$

$$\left.-(1-n\lambda c^2)\sinh sc\right\}\sin nt+\left\{(1-n\lambda c^2)\sin\frac{s}{c}+(n\lambda+c^2)\sinh sc\right\}\cos nt\bigg] \quad .. \quad (11)$$

and on the disc $z = d$ is

$$(\tau_{\theta z})_{z=d} = -\frac{\mu r\Omega s}{cd\left(\cosh sc-\cos\frac{s}{c}\right)(1+n^2\lambda^2)}\left[\left\{(n\lambda+c^2)\cosh\frac{sc}{2}\sin\frac{s}{2c}\right.\right.$$

$$\left.-(1-n\lambda c^2)\sinh\frac{sc}{2}\cos\frac{s}{2c}\right\}\sin nt+\left\{(1-n\lambda c^2)\cosh\frac{sc}{2}\sin\frac{s}{2c}\right.$$

$$\left.\left.+(n\lambda+c^2)\sinh\frac{sc}{2}\cos\frac{s}{2c}\right\}\cos nt\right]. \quad .. \quad ... \quad .. \quad .. \quad (12)$$

The third equation of (5) on integration gives

$$\left(1+\bar{\lambda}\frac{\partial}{\partial T}\right)P = L(T). \quad .. \quad .. \quad .. \quad .. \quad (13)$$

The structure of the equations (5) and (13) demands that the solution should be in the form

$$\left.\begin{aligned} F(y, T) &= f(y)+h(y)\exp(2iT), \\ K(y, T) &= K_0(y)+K_1(y)\exp(2iT), \\ \text{and} \qquad L(T) &= L_0+L_1\exp(2iT). \end{aligned}\right\} \quad .. \quad .. \quad (14)$$

Therefore, we note that the radial-axial flow has a mean steady component and a fluctuating component of frequency twice that of the oscillating plates. The boundary conditions (6) and (7) are

$$\left.\begin{aligned} f = f' &= 0 \quad \text{at} \quad y=0 \quad \text{and} \quad y=1, \\ h = h' &= 0 \quad \text{at} \quad y=0 \quad \text{and} \quad y=1. \end{aligned}\right\} \quad .. \quad .. \quad (15)$$

From (5), (8), (13), (14) and (15) the mean steady component is

$$f(y) = \frac{1}{4sc^3\left(\cosh sc - \cos\frac{s}{c}\right)}\left[scy(1-2y+y^2)\left(\cosh sc + c^4 \cos\frac{s}{c}\right)\right.$$

$$-(1-3y^2+2y^3)\left(\sinh sc + c^6 \sin\frac{s}{c}\right) - scy^2(1-y)(1+c^4)$$

$$\left.+\sinh sc(1-y) + c^6 \sin\frac{s}{c}(1-y)\right], \quad \ldots \quad (16)$$

and the time-dependent component of radial-axial flow is

$$\frac{4}{2\mu - i}\, s\alpha\beta(\cosh s\alpha - 1)(2 - 2\cosh s\beta + s\beta \sinh s\beta)h(y)$$

$$= \cosh s\beta y[\alpha \sinh s\beta(\cosh s\alpha - 1) + s\alpha\beta(1 - \cosh s\alpha \cosh s\beta)$$

$$+\beta \sinh s\alpha(\cosh s\beta - 1)] + \sinh s\beta y[\alpha(1 - \cosh s\beta)(\cosh s\alpha - 1)$$

$$+\beta \sinh s\beta(s\alpha \cosh s\alpha - \sinh s\alpha)] + \alpha \sinh s\beta(1 - \cosh s\alpha)$$

$$+s\alpha\beta(\cosh s\alpha \cosh s\beta - 1) - \beta \sinh s\alpha(1 - \cosh s\beta + s\beta \sinh s\beta)$$

$$+ys\beta[\beta \sinh s\alpha \sinh s\beta + \alpha(1 - \cosh s\beta)(1 + \cosh s\alpha)]$$

$$+\beta \sinh s\alpha(1-y)(2 - 2\cosh s\beta + s\beta \sinh s\beta), \quad \ldots \quad (17)$$

where

$$\beta = \frac{1}{\sqrt{2}}\left(E + \frac{i}{E}\right) \text{ and } E^2 = \sqrt{4\bar{\lambda}^2 + 1} - 2\bar{\lambda}.$$

The results with $\bar{\lambda} = 0$ corresponding to the Newtonian case are in agreement with those of Rosenblat.

§ 4 Discussion (on § 3)

The effect of relaxation time parameter on amplitude of transverse velocity is shown in Fig. 1 and it has been compared with the Newtonian case. The amplitude of the transverse velocity ($|v|/r\Omega$) increases with the relaxation time parameter and this increase is less for small y than for values of y near the plane of symmetry. Figure 2 shows the dimensionless radial velocity f' for various values of $\bar{\lambda} = 0$, 0·5 and 1·0. The fluid is thrown radially outwards near the oscillating disc. To maintain the continuity of flow a radial pressure gradient is developed which causes a radial inward flow near the stationary disc. The profile near the oscillating disc is of boundary layer type and the flow near the stationary disc is similar to Poiseuille flow with radial pressure gradient. Thus the effect of relaxation time parameter is to increase the magnitude of the radial velocity while the character of the flow is similar to that of Newtonian fluid.

The stream function for steady radial-axial flow is given by

$$\psi = \frac{d\Omega^2}{n} r^2 f(y). \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (18)$$

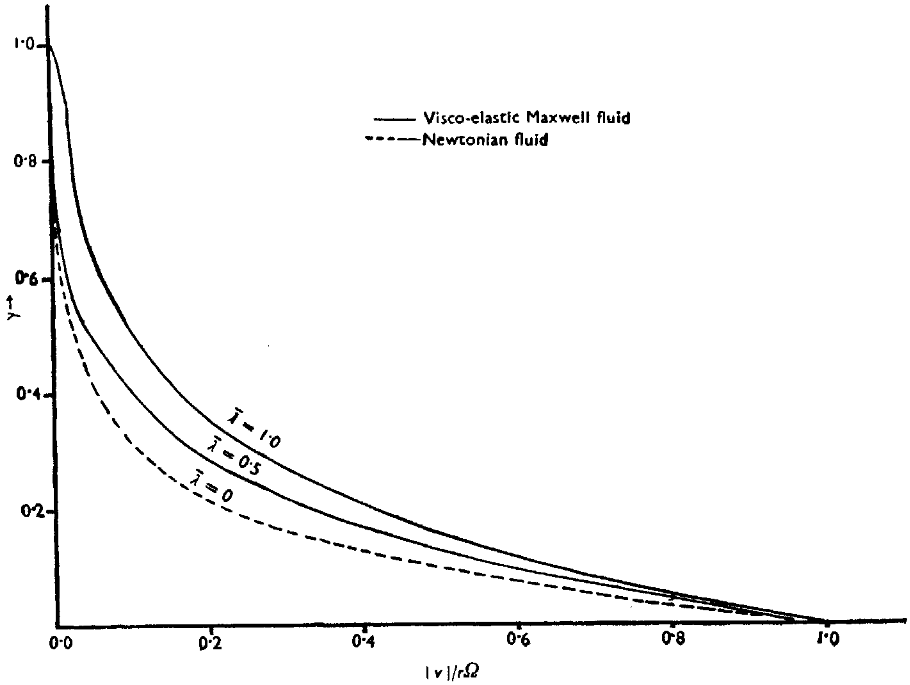

FIG. 1. One disc oscillating. Amplitude of transverse velocity $|v|/r\Omega$ for $R = 100$; $\bar{\lambda} = 0$, 0·5 and 1·0.

Figure 3 depicts schematically, typical streamlines of steady radial-axial flow for the visco-elastic case ($\bar{\lambda} = 1$ and 5) and the Newtonian case ($\bar{\lambda} = 0$). The effect of increase in relaxation time parameter is to increase the radial velocity. There is a lateral shift in the streamlines which shows that the fluid particles are taken nearer to the common axis of the discs. For small relaxation time, viscous effects permeate the entire flow.

§ 5. BOTH DISCS OSCILLATING

We now consider the second case when both the discs are oscillating with same frequency and angular speed with phase difference of 180°. Following Rosenblat (1960) we anticipate that the results will be symmetrical (or anti-symmetrical) about the plane $y = (1/2)$. The second equation of (5) with boundary conditions (7) gives

$$G(y) = \frac{\sinh s\alpha(1-y) - \sinh s\alpha y}{\sinh s\alpha}. \qquad \ldots \quad \ldots \quad \ldots \quad (19)$$

Hence, the amplitude of transverse velocity is

$$\frac{|v|}{r\Omega} = \left[\frac{\cosh\frac{sc}{2}(1-2y)-\cos\frac{s}{2c}(1-2y)}{\cosh\frac{sc}{2}-\cos\frac{s}{2c}}\right]^{\frac{1}{2}}, \quad \ldots \quad \ldots \quad (20)$$

and the phase angle is

$$\tan^{-1}\left[\frac{\sinh\frac{sc}{2}(1-2y)\,\sin\frac{s}{2c}y-\sinh\frac{sc}{2}y\,\sin\frac{s}{2c}(1-2y)}{\cosh\frac{sc}{2}(1-2y)\,\cos\frac{s}{2c}y-\cosh\frac{sc}{2}y\,\cos\frac{s}{2c}(1-2y)}\right]. \quad \ldots \quad (21)$$

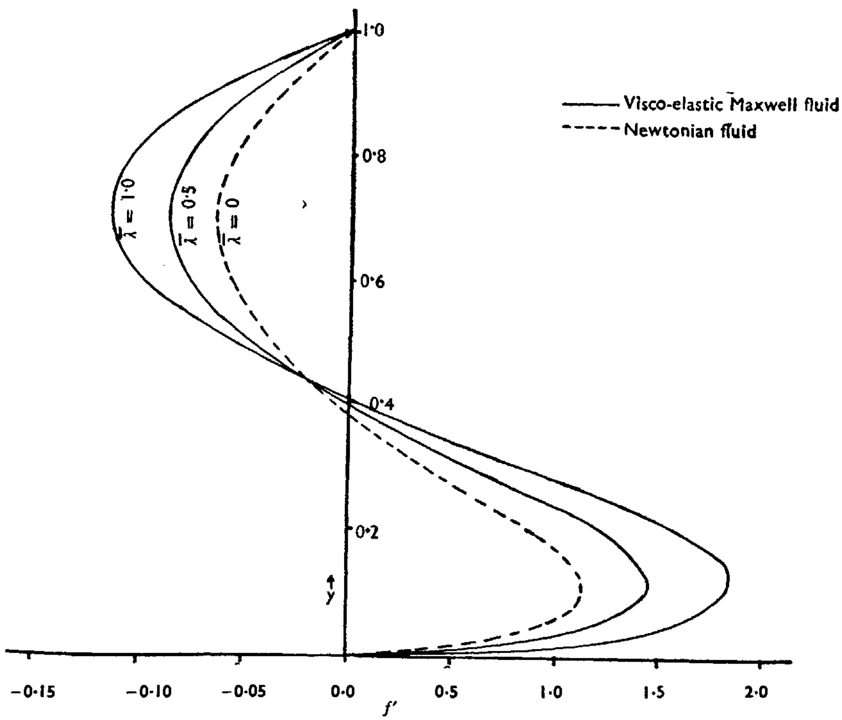

FIG. 2. One disc oscillating; steady radial velocity f' for $R = 100$; $\lambda = 0$, 0·5 and 1.

The skin friction on the disc $z = 0$ is

$$(\tau_{\theta z})_{z=0} = -\frac{\mu r\Omega s}{2cd\left(\cosh\frac{sc}{2}-\cos\frac{s}{2c}\right)(1+n^2\lambda^2)}\left[\left\{(n\lambda+c^2)\sin\frac{s}{2c}-(1-n\lambda c^2)\sinh\frac{sc}{2}\right\}\right.$$

$$\left.\times\sin nt+\left\{(1-n\lambda c^2)\sin\frac{s}{2c}+(n\lambda+c^2)\sinh\frac{sc}{2}\right\}\cos nt\right]. \quad \ldots \quad \ldots \quad (22)$$

From (5), (13), (14), (15) and (19) the mean steady component is

$$\frac{1}{R}f(y)=\frac{\cosh\frac{sc}{2}+\cos\frac{s}{2c}}{\cosh sc-\cos\frac{s}{c}}\left[-(1-6y^2+4y^3)\left(\frac{1}{s^3c^3}\sinh\frac{sc}{2}+\frac{c^3}{s^3}\sin\frac{s}{2c}\right)\right.$$

$$+(y-3y^2+2y^3)\left(\frac{1}{s^2c^2}\cosh\frac{sc}{2}+\frac{c^2}{s^2}\cos\frac{s}{2c}\right)$$

$$\left.+\left\{\frac{1}{s^3c^3}\sinh\frac{sc}{2}(1-2y)+\frac{c^3}{s^3}\sin\frac{s}{2c}(1-2y)\right\}\right], \qquad \ldots \quad \ldots \quad \ldots \quad (23)$$

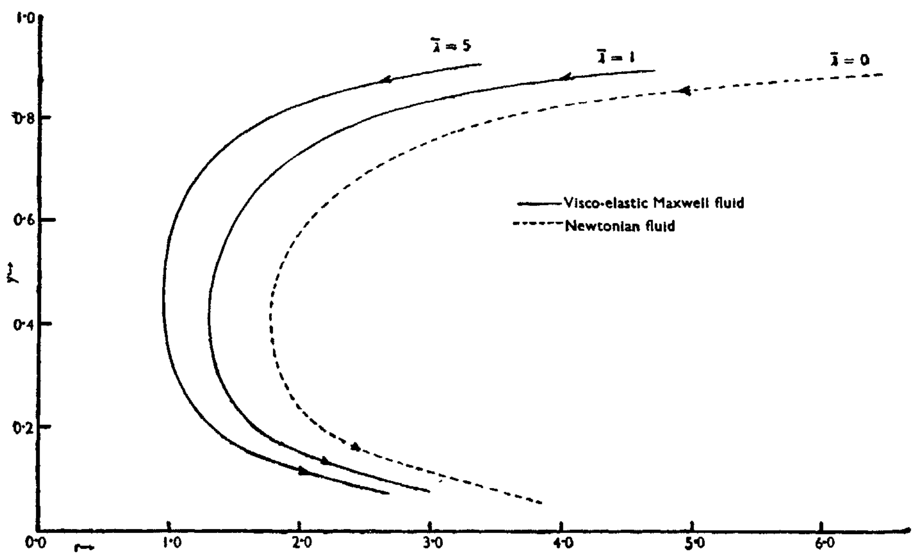

FIG. 3. One disc oscillating. Typical streamlines of steady radial-axial for $R=100$; $\bar{\lambda}=0$, 1 and 5.

and the fluctuating component is

$$\frac{2s\alpha(\cosh s\alpha-1)}{(2\mu-i)\left(1+\cosh\frac{s\alpha}{2}\right)}(2-2\cosh s\beta+s\beta\sinh s\beta)h(y)$$

$$=\left(s\alpha\cosh\frac{s\alpha}{2}-2\sinh\frac{s\alpha}{2}\right)[\cosh s\beta y-\cosh s\beta(1-y)]$$

$$+\sinh\frac{s\alpha}{2}(1-2y)(2-2\cosh s\beta+s\beta\sinh s\beta)$$

$$+(1-2y)\left[s\alpha\cosh\frac{s\alpha}{2}(\cosh s\beta-1)-s\beta\sinh s\beta\sinh\frac{s\alpha}{2}\right]. \qquad \ldots \quad (24)$$

§ 6. Discussion (on § 5)

Figure 4 depicts the effect of relaxation time parameter on $(|v|/r\Omega)$. We observe that with the increase of $\bar{\lambda}$, the elastic effects predominate in the central region and viscous effects are limited in the neighbourhood of the oscillating plates. In Fig. 5 we note that with the increase of $\bar{\lambda}$ the radial velocity increases. The fluid is thrown out radially near the oscillating plates and it is sucked radially from infinity towards the common axis in the central

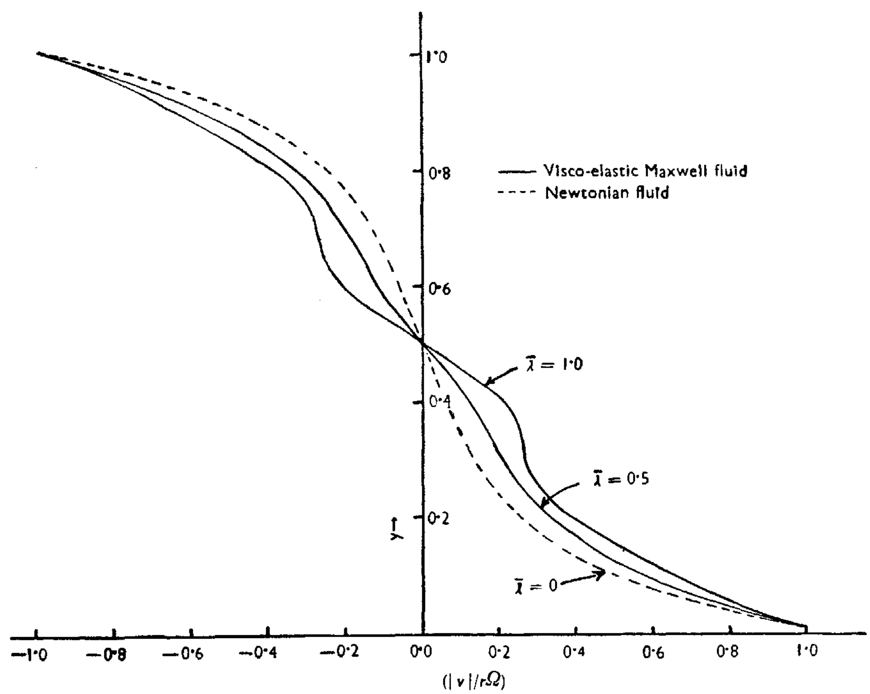

Fig. 4. Two discs oscillating. Amplitude of transverse velocity $\frac{|v|}{r\Omega}$ for $R = 100$; $\bar{\lambda} = 0$, 0·5 and 1·0.

region developing a radial pressure gradient which causes a Poiseuille flow in this region. In Fig. 6 the effect of relaxation time parameter has been shown on typical streamlines of steady radial-axial flow. The pattern of the streamlines also predicts that the liquid is sucked in the central region and thrown out near the two plates. As already noted, with the increase of $\bar{\lambda}$, the radial velocity increases; hence the typical streamlines will shift towards the common axis of the two plates.

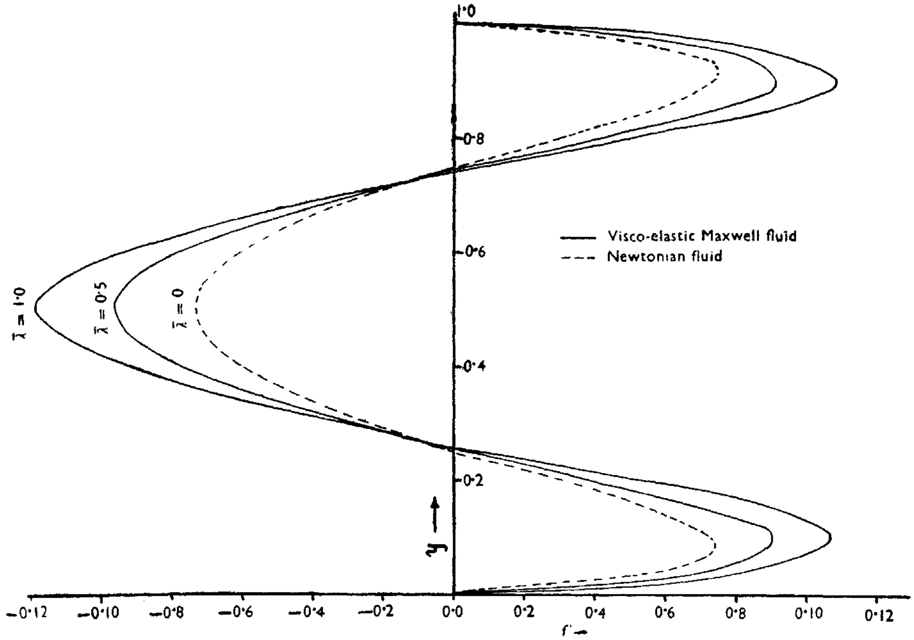

FIG. 5. Two discs oscillating; steady radial velocity f', for $R = 100$; $\bar{\lambda} = 0$, 0·5 and 1·0.

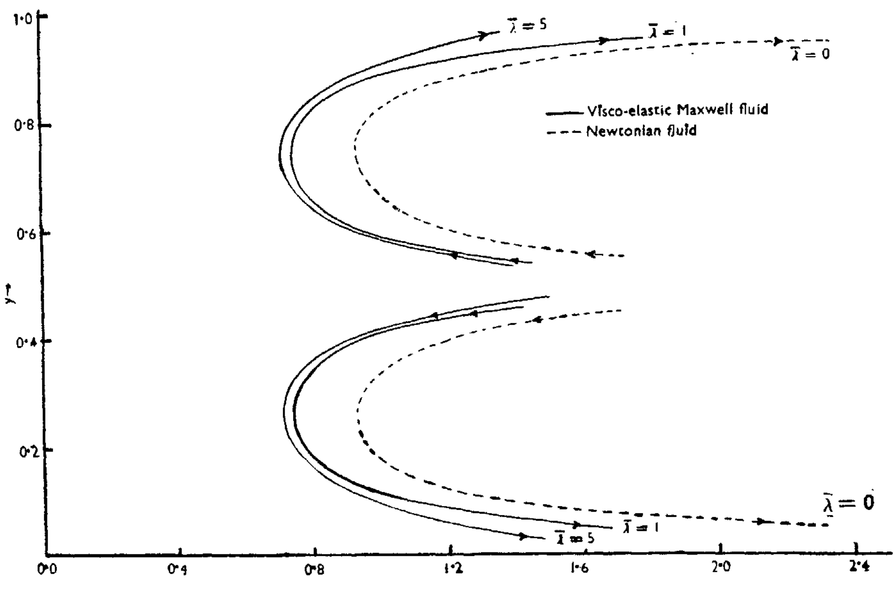

FIG. 6. Two discs oscillating. Typical streamlines of steady radial-axial flow for $R = 100$; $\bar{\lambda} = 0$, 1 and 5.

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