NOTE ON THE HAHN-BANACH THEOREM IN NORMED SPACES

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Simple geometric conditions are given under which the extension of a linear functional is unique.

Shukla (1966) has stated and proved a necessary and sufficient condition under which the extension of a linear functional in a normed space is unique. His proof is directly deduced from the proof of the existence theorem given by Banach (1932). However, the Hahn-Banach theorem can be proved now in a much simpler manner as it has been shown by Bourbaki (1953). It follows that a much simpler proof, as well as a much simpler enunciation, of a uniqueness theorem is available. It is the purpose of this paper to show it.

Bourbaki's proof is based on what this author calls the 'geometric' form of the Hahn-Banach theorem. It runs as follows:

Let E be a locally convex topological vector space over the real numbers; let A be an open, non-empty, convex set in E and M a linear manifold whose intersection with A is empty. Then there exists a closed hyperplane (*i.e. of codimension one*) *containing M and whose intersection with A is empty.*

This proposition leads directly to the proof of what Bourbaki calls the 'analytic' form of the Hahn-Banach theorem. With the same notation as above, let $f \neq 0$ be a linear functional defined on M and let us assume that M contains the origin of E. In the case of a normed space, the only case considered here, we assume that $|f(x)| \leq \| x \|$, $x \in M$, $\| x \|$ standing for the norm of x in E. Let A be the open set defined by $\| x \| < 1$, and N the hyperplane of M defined by $f(x) = 1$. Then, there exists a linear functional $\bar{f}$ defined in E, an extension of f, such that $\bar{f}(x) = 1$ is the equation of a hyperplane of E passing through f and whose intersection with A is empty. This implies clearly that $|\bar{f}(x)| \leq \| x \|$, $x \in E$.

Now, it is evident that the extension $\bar{f}$ is unique if and only if there exists a unique hyperplane of E passing through N and whose intersection with A is empty. Let us call, with Bourbaki, hyperplane of support (un hyperplan d'appui) a hyperplane having at least one point in common with the closure of A but whose intersection with A itself is empty. We obtain at once a necessary and sufficient condition for the extension $\bar{f}$ to be unique, namely, that there should exist only one hyperplane of support of A through N. This is a geometric condition; but, in practical cases, it is one which is much easier to verify than the complicated analytic conditions given by Shukla.

At any rate there is an analytic condition simpler than the one of Shukla (1966) which is easily derived from the geometric one. Let x_0 be a point of $\bar{N} \cap \bar{A}$ ($\bar{A}$: the closure of A) and λ be a negative real number. Let $\bar{f}$ be an extension of f and $x_1 \epsilon E$ such that $\bar{f}(x_1) = 1$. Let $X = \lambda\, x_0$ and consider the open, convex, set B defined by $\{x \mid \|x - X\| < 1 - \lambda\}$. For sufficiently large values of λ the straight line joining X to x_1 intersects the boundary of B at a point x_2 and one has $\|X - x_2\| = 1 - \lambda$. If the hyperplane defined by $\bar{f}(x) = 1$ is the unique hyperplane of support at x_0 through N then

$$\lim_{\lambda \to -\infty} \{\|X - x_1\| - (1 - \lambda)\} = 0.$$

If, on the contrary, there exists an extension $\bar{f}$ for which this limit is not zero, this extension is not unique and at x_0 through N there exists a bundle of hyperplanes of support.

As it has been said this condition is not easily verifiable except in a few particular cases.

As an application of the geometric method, let us consider a Hilbert space, H. If $x,\ y \epsilon H$, $\|x\|$ and $< x,\ y >$ denote respectively the norm and the scalar product. Let A be the ball: $\|x\| < 1$ and let x and y be such that $\|x\| = \|y\| = 1$ and $< x,\ y > = 0$. Then

$$\|x + \lambda y\|^2 = \|x\|^2 + \lambda^2 \|y\|^2 = 1 + \lambda^2.$$

So, if $\lambda \neq 0$, $\|x + \lambda y\| > 1$. This shows that the hyperplane defined by $\{y \mid x + \lambda\, y\}$, for all y such that $\|y\| = 1$ and $< x,\ y > = 0$ is a hyperplane of support for A at x. If this hyperplane of support were not unique, there should exist y' such that $\|y'\| = 1$ and that $\|x + \lambda\, y'\| \geq 1$, with $y' \neq y$, on at least one straight line through 0. Hence, $< x,\ y' > = a \neq 0$. The condition $\|x + \lambda\, y'\| \geq 1$ implies that $\lambda\ (\lambda + 2a) \geq 0$ for all real values of λ and this is clearly impossible.

An example of a normed space where the ball $\|x\| < 1$ has at certain points of its boundary a bundle of hyperplanes of support provided by a space E which is the Cartesian sum of two normed spaces E' and E''. If $x' \epsilon E'$ and $x'' \epsilon E''$, if the norms of x' in E' and of x'' in E'' are denoted $\|x'\|$ and $\|x''\|$, and if $x = x' + x''$ has the norm $\|x\| = \|x'\| + \|x''\|$ in E, then it is not difficult to show that at a point $(x',\ 0)$, where x' is of norm 1 in E', there exist an infinity of hyperplanes of support.

Some more examples will be studied in another paper.

References

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