TORSIONAL OSCILLATIONS OF A PLANE IN A VISCO-ELASTIC MAXWELL FLUID

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The flow induced in a visco-elastic Maxwell fluid due to torsional oscillations of a plane is investigated in the present paper. The amplitude of the oscillations is assumed to be small. The effect of relaxation time parameter $\bar{\lambda}$ is shown graphically on the transverse and radial components of velocity. It is observed that for small value of $\bar{\lambda}$ the viscous effects predominate, but for large $\bar{\lambda}$, the elastic effects permeate into the entire flow.

1. INTRODUCTION

The problem of small torsional oscillations of an infinite disc in an incompressible viscous fluid has been discussed by Rosenblat (1959). The velocity components are expressed in power series of ϵ, the amplitude of oscillation. The main characteristic of the solution arises in second approximation, which reveals that a secondary radial-axial flow consists of a steady term and a fluctuating term of frequency twice that of the frequency of oscillation of the disc. The former component persists outside the boundary layer and is confined within a secondary layer whose thickness is of order ϵ^{-1} times that of the shear layer. Outside this secondary layer there is merely a constant axial inflow. The same problem has also been studied recently by Benny (1964) by introducing two length scales and an additional independent variable to obtain a single expression valid both near and at large distances from the disc. A similar problem for a second order fluid has been solved by Srivastava (1963) by expanding the velocity components and the pressure in powers of amplitude of oscillations of the plate.

The present paper is devoted to the study of torsional oscillations of a plate in visco-elastic Maxwell fluid. The constitutive equation of the fluid is

$$\left(1+\lambda\frac{\partial}{\partial t}\right)\tau_{ij} = 2\mu e_{ij}, \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

where τ_{ij} is the deviatoric stress tensor, e_{ij} is strain rate tensor, μ is viscosity and λ is the relaxation time parameter. The amplitude of oscillation ϵ is assumed to be small. Therefore the velocity components and pressure are expressed in the form of a power series in ϵ. Non-Newtonian effects are exhibited through the dimensionless relaxation time parameter $\bar{\lambda}\ (=\lambda n)$.

The first approximation of transverse velocity is compared for various values of $\bar{\lambda}$. Comparison of first and third approximation has been depicted graphically for small values of $\bar{\lambda}$ ($= 0{\cdot}5$). The non-Newtonian effect on fluctuating component of radial velocity u_f has also been discussed. It has been found that for small values of relaxation time parameter the viscous effects predominate, but for large values of $\bar{\lambda}$, the disturbances are transmitted through a larger distance and the fluctuations in u_f are more rapid in the immediate neighbourhood of the plate than the Newtonian case.

2. Equations of Motions

Let (r, θ, z) be the polar cylindrical coordinates, and u, v, w be the radial, transverse and axial components of velocity. Let the plane $z = 0$ represent a lamina of infinite extent and the space $z > 0$ be occupied by incompressible visco-elastic Maxwell fluid. The lamina performs torsional oscillations about the axis $r = 0$ with frequency n. The boundary conditions of the problem are

$$\left.\begin{array}{ll} u = 0,\ v = r\omega \exp(int), & w = 0 \text{ at } z = 0, \\ \text{and} & \\ u \to 0,\ v \to 0 & \text{as } z \to \infty. \end{array}\right\} \quad \ldots \quad \ldots \quad (2)$$

We assume

$$\left.\begin{array}{l} u = r\omega \dfrac{\partial}{\partial y} F(y, T), \quad v = r\omega G(y, T), \quad w = -2\omega\sqrt{\dfrac{2\nu}{n}}\, F(y, T), \quad p = p(z, t), \\ z = \sqrt{\dfrac{2\nu}{n}}\, y,\ t = (T/n), \quad \lambda = (\bar{\lambda}/n) \text{ and } \omega = n\epsilon, \end{array}\right\} \quad (3)$$

where ν is kinematic viscosity.

Equation of continuity

$$\frac{\partial u}{\partial r} + \frac{u}{r} + \frac{\partial w}{\partial z} = 0$$

is satisfied identically with the velocity components defined in (3).

We assume a series solution for F and G in the form

$$\left.\begin{array}{l} F(y, T) = F_0(y, T) + \epsilon F_1(y, T) + \epsilon^2 F_2(y, T) + \ldots, \\ G(y, T) = G_0(y, T) + \epsilon G_1(y, T) + \epsilon^2 G_2(y, T) + \ldots \end{array}\right\} \quad \ldots \quad \ldots \quad (4)$$

Substituting (1), (3) and (4) in the equations of motion in r- and θ-directions and equating like powers of ϵ we get

$$\left.\begin{array}{l} \left(1 + \bar{\lambda}\dfrac{\partial}{\partial T}\right)\left(\dfrac{\partial F_0'}{\partial T}\right) = \tfrac{1}{2} F_0''', \\ \left(1 + \bar{\lambda}\dfrac{\partial}{\partial T}\right)\left(\dfrac{\partial F_1'}{\partial T} + F_0'^2 - G_0^2 - 2F_0 F_0''\right) = \tfrac{1}{2} F_1''', \\ \left(1 + \bar{\lambda}\dfrac{\partial}{\partial T}\right)\left(\dfrac{\partial F_2'}{\partial T} + 2F_0' F_1' - 2F_0 F_1'' - 2F_0'' F_1 - 2G_0 G_1\right) = \tfrac{1}{2} F_2''', \end{array}\right\} \quad \ldots \quad (5)$$

etc., and

$$\left.\begin{aligned}\left(1+\bar{\lambda}\frac{\partial}{\partial T}\right)\left(\frac{\partial G_0}{\partial T}\right) &= \tfrac{1}{2}G_0'',\\ \left(1+\bar{\lambda}\frac{\partial}{\partial T}\right)\left(\frac{\partial G_1}{\partial T}+2F_0'G_0-2F_0G_0'\right) &= \tfrac{1}{2}G_1'',\\ \left(1+\bar{\lambda}\frac{\partial}{\partial T}\right)\left(\frac{\partial G_2}{\partial T}+2F_0'G_1-2F_1'G_0-2F_0G_1'-2F_1G_0'\right) &= \tfrac{1}{2}G_2'',\end{aligned}\right\} \quad .. \quad (6)$$

etc.

The boundary conditions (2) are

$$\left.\begin{aligned}&F_j = F_j' = 0,\ G_0 = \exp(iT),\ G_{j+1} = 0, &&\text{at } y = 0,\\ &F_j' \to 0,\ G_j \to 0, &&\text{as } y \to \infty,\end{aligned}\right\} \quad .. \quad (7)$$

for $j = 0, 1, 2, 3, \ldots$

3. First Approximation to Velocity Components

First equation of (5) with boundary conditions (7) gives

$$F_0(y, T) = 0. \quad .. \quad .. \quad .. \quad .. \quad (8)$$

To obtain a unique solution for G_0 from (6) and (7) we assume

$$G_0(y, T) = \phi_0(y)\exp(iT). \quad .. \quad .. \quad .. \quad (9)$$

First equations of (6), (7) and (9) give

$$\phi_0 = \exp\left\{-\left(c+\frac{i}{c}\right)y\right\}, \quad .. \quad .. \quad .. \quad (10)$$

where

$$c^2 = \sqrt{\bar{\lambda}^2+1}-2\bar{\lambda}.$$

In real notation

$$G_0 = \exp(-cy)\cos\left(T-\frac{y}{c}\right). \quad .. \quad .. \quad .. \quad (11)$$

In Fig. 1 ($G_0 - \cos T$) at various time intervals $T = 0$, $(\pi/3)$, $(\pi/2)$ and $(2\pi/3)$ has been plotted against y for $\bar{\lambda} = 0$ (Newtonian case), 0·5, 2·0. We note that for small values of $\bar{\lambda}$, the behaviour of transverse velocity components for Maxwell fluid is similar to that of Newtonian fluid. That is, the viscous effects predominate over the elastic effects. With increase of $\bar{\lambda}$, the magnitude of transverse velocity components increases in the vicinity of the disc. Disturbances, set in the fluid, will die out at larger distances from the plane than in the case of Newtonian fluids. The velocity components to the first approximation are

$$\left.\begin{aligned}u &= 0,\\ v &= r\omega\exp\left(-\frac{1}{\sqrt{2}}cmz\right)\cos\left(nt-\frac{m}{\sqrt{2}c}z\right),\\ w &= 0,\end{aligned}\right\} \quad .. \quad .. \quad (12)$$

where $n = \nu m^2$.

The transverse shearing stress τ on the disc to the first approximation is given by

$$\left(1+\lambda\frac{\partial}{\partial t}\right)\tau = -\rho r\omega\sqrt{\frac{n\nu}{2}}\left(c+\frac{i}{c}\right)\exp\,(int). \qquad (13)$$

The form of eqn. (13) suggests

$$\tau = \phi_1 \exp\,(int). \qquad (14)$$

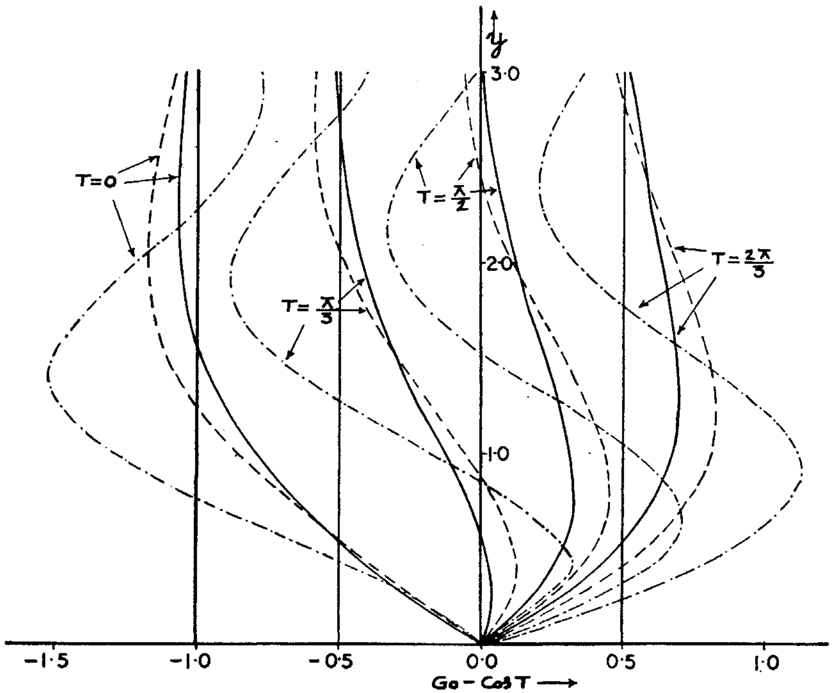

FIG. 1. Variation of $G_0-\cos T$ with y at times $T = 0, \frac{\pi}{3}, \frac{\pi}{2}$ and $\frac{2\pi}{3}$.

——— $\bar{\lambda} = 0$, - - - - $\bar{\lambda} = 0{\cdot}5$, — . — . — . — . $\bar{\lambda} = 2{\cdot}0$.

From (13) and (14) we have

$$\phi_1 = -\frac{\rho r\omega}{1+in\lambda}\sqrt{\frac{n\nu}{2}}\left(c+\frac{i}{c}\right). \qquad (15)$$

Therefore, shearing stress in real notation is given by

$$\tau = -\sqrt{\frac{n\nu}{2}}\,\frac{\rho r\omega A}{1+n^2\lambda^2}\cos\,(nt+B), \qquad (16)$$

where

$$A^2 = (1+c^4)(1+n^2\lambda^2)/c^2,$$

$$B = \tan^{-1}\left(\frac{1-n\lambda c^2}{n\lambda+c^2}\right).$$

Thus the shearing stress has a phase lead of B over the oscillations of the plate. The frictional torque (for the two sides of the disc of radius a) is

$$M = -4\pi\int_0^a r^2(\tau)_{z=0}\,dr. \quad .. \quad .. \quad .. \quad (17)$$

Hence to the first approximation

$$M = \sqrt{\frac{n\nu}{2}}\,\frac{\pi\rho\omega A a^4}{1+n^2\lambda^2}\cos(nt+B), \quad .. \quad .. \quad .. \quad (18)$$

provided the edge effects are neglected. These results reduce to the Newtonian case when $\lambda = 0$.

4. Second Approximation to Velocity Components

Equations (7) to (9) and second equation of (6) give

$$G_1 = 0. \quad .. \quad .. \quad .. \quad .. \quad .. \quad (19)$$

The structure of equation (5) indicates that the solution of $F_1(y, T)$ is of the form

$$F_1(y, T) = f(y)+h(y)\exp(2iT). \quad .. \quad .. \quad .. \quad (20)$$

Therefore radial and axial velocities consist of a steady term and a fluctuating term with frequency twice that of the disc-like Newtonian fluid. The boundary conditions (7) are now modified to

$$\left.\begin{array}{ll} f = f' = h = h' = 0, & \text{at } y = 0, \\ f' \to 0,\ h' \to 0, & \text{as } y \to \infty. \end{array}\right\} \quad .. \quad .. \quad .. \quad (21)$$

Equations (19) to (21) and second equation of (5) give

$$f = \{-1+2cy+\exp(-2cy)\}/8c^3,$$

and

$$\left.\begin{array}{l} h = \dfrac{1+2i\bar{\lambda}}{16}\left[\dfrac{\sqrt{2}(1+iE^2)}{E\sqrt{4\bar{\lambda}^2+1}}\left\{\exp\left(-\sqrt{2}\left[E+\dfrac{i}{E}\right]y\right)-1\right\}\right. \\ \qquad \left. -\dfrac{1+ic^2}{c\sqrt{\bar{\lambda}^2+1}}\left\{\exp\left(-2\left[c+\dfrac{i}{c}\right]y\right)-1\right\}\right], \end{array}\right\} \quad .. \quad (22)$$

where

$$E^2 = \sqrt{4\bar{\lambda}^2+1}-2\bar{\lambda}.$$

The first equation of (22), however, does not satisfy the condition $f' \to 0$ as $y \to \infty$. This is again discussed in § 6. The steady and fluctuating components of radial and axial velocities are

$$
\left.
\begin{aligned}
u_s &= \frac{r\omega\epsilon}{4c^2}\{1-\exp(-\sqrt{2}cmz)\},\\
u_f &= \frac{r\omega\epsilon}{4}\Big[\exp(-Emz)\Big\{2\bar{\lambda}\cos\Big(2nt-\frac{m}{E}z\Big)+\sin\Big(2nt-\frac{m}{E}z\Big)\Big\}\\
&\quad -\exp(-\sqrt{2}cmz)\Big\{2\bar{\lambda}\cos\Big(2nt-\frac{\sqrt{2}m}{c}z\Big)+\sin\Big(2nt-\frac{\sqrt{2}m}{c}z\Big)\Big\}\Big],\\
w_s &= \frac{\omega\epsilon}{2\sqrt{2}mc^3}\{1-\sqrt{2}cmz-\exp(-\sqrt{2}cmz)\},\\
w_f &= \frac{\nu m\epsilon}{4\sqrt{2}}\Big[(\cos 2nt-2\bar{\lambda}\sin 2nt)\Big\{\gamma\Big(1-\exp[-Emz]\cos\frac{m}{E}z\\
&\quad -E^2\exp[-Emz]\sin\frac{m}{E}z\Big)+\delta\Big(-1+\exp[-\sqrt{2}cmz]\cos\frac{\sqrt{2}m}{c}z\\
&\quad +c^2\exp[-\sqrt{2}cmz]\sin\frac{\sqrt{2}m}{c}z\Big)\Big\}\\
&\quad -(2\bar{\lambda}\cos 2nt+\sin 2nt)\Big\{\gamma\Big(E^2-E^2\exp[-Emz]\cos\frac{m}{E}z\\
&\quad +\exp[-Emz]\sin\frac{m}{E}z\Big)+\delta\Big(-c^2+c^2\exp[-\sqrt{2}cmz]\cos\frac{\sqrt{2}m}{c}z\\
&\quad -\exp[-\sqrt{2}cmz]\sin\frac{\sqrt{2}m}{c}z\Big)\Big\}\Big],
\end{aligned}
\right\} \quad (23)
$$

where

$$\gamma = \frac{\sqrt{2}}{E\sqrt{4\bar{\lambda}^2+1}} \quad \text{and} \quad \delta = \frac{1}{c\sqrt{\bar{\lambda}^2+1}},$$

and subscripts s and f denote the steady and the fluctuating parts respectively.

We observe that as z increases the components decrease exponentially. The rate of diminution depends on the parameters ν, n and $\bar{\lambda}$. Fig. 2 illustrates the variation of $(4u_f/r\omega\epsilon)$ against y at times $T = 0$, $(\pi/3)$, $(2\pi/3)$ for $\bar{\lambda} = 0$ and 0·5. The elastic effects of the fluid increase the magnitude of u_f. The disturbances will also be carried over a larger distance along the axis of oscillation than ordinary viscous fluids. Due to elastic effects of the fluid, the fluctuations take longer time to relax than viscous fluids and do not die out quickly. The effect of large relaxation time parameter ($\bar{\lambda} = 10$) is shown in Fig. 3. We note that fluctuations in u_f are more sharp, and also more frequent than for small values of $\bar{\lambda}$. From the first two equations of (23) we have

$$u_s \to (r\omega\epsilon/4c^2) \text{ and } u_f \to 0 \text{ as } z \to \infty. \quad \ldots \quad \ldots \quad (24)$$

Hence, at large distances only the steady radial flow persists, the magnitude of which does not depend on viscosity of the fluid, but does depend on the

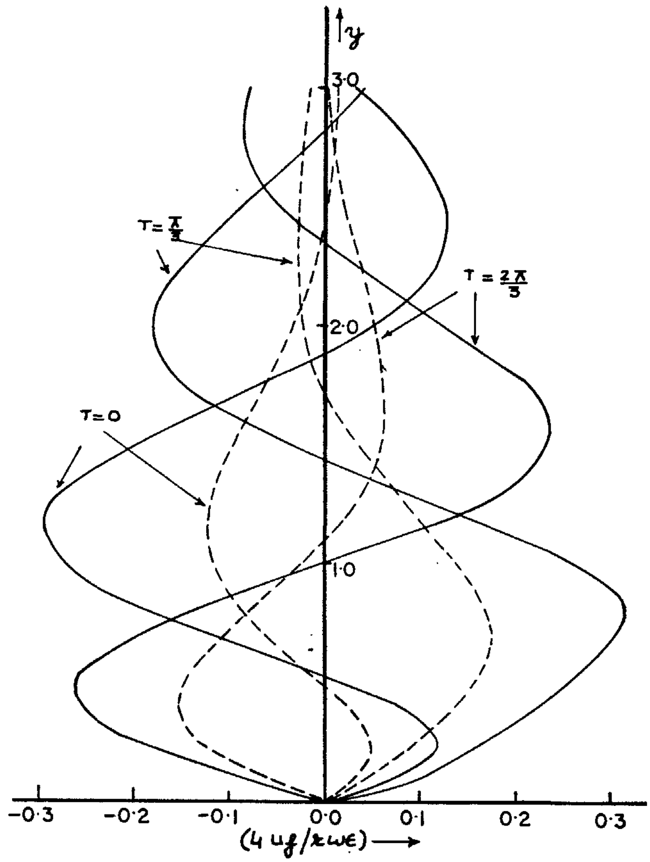

FIG. 2. Variation of $(4u_f/r\omega\epsilon)$ with y at times $T = 0, \frac{\pi}{3}, \frac{2\pi}{3}$.

——— $\lambda = 0{\cdot}5$, - - - - $\bar{\lambda} = 0$.

relaxation time parameter. However, the steady radial flow is discussed in more details in § 6. The axial flow for large values of z is given by the equations

$$\left.\begin{aligned} w_s &= \omega\epsilon(1-\sqrt{2}cmz)/2\sqrt{2}mc^3, \\ w_f &= \frac{\nu m\epsilon}{4\sqrt{2}}[(\gamma-\delta-2\bar{\lambda}\gamma E^2+2\bar{\lambda}\delta c^2)\cos 2nt \\ &\quad +(-\gamma E^2+\delta c^2-2\bar{\lambda}\gamma+2\bar{\lambda}\delta)\sin 2nt]. \end{aligned}\right\} \quad \ldots \quad \ldots \quad (25)$$

The stream function for the steady radial-axial flow is obtained from (23) in the form

$$\psi_s = r^2\omega\epsilon\{-1+\sqrt{2}cmz+\exp(-\sqrt{2}cmz)\}/4\sqrt{2}mc^3. \quad .. \quad (26)$$

For large values of z,

$$\psi_s = r^2\omega\epsilon(-1+\sqrt{2}cmz)/4\sqrt{2}mc^3. \quad .. \quad .. \quad .. \quad (27)$$

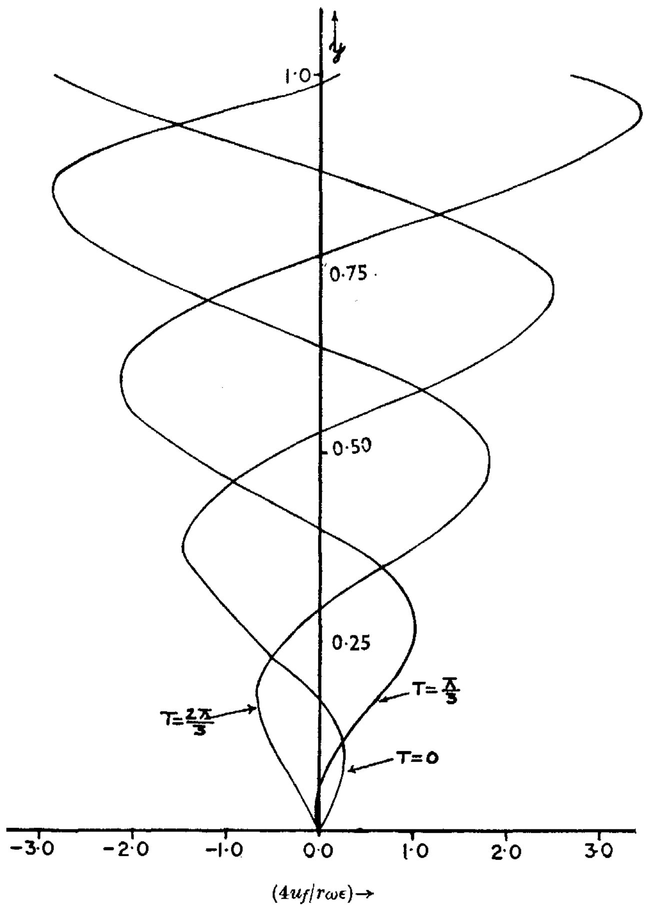

FIG. 3. Variation of $(4u_f/r\omega\epsilon)$ with at times $T = 0, \frac{\pi}{3}, \frac{2\pi}{3}$. $\bar{\lambda} = 10{\cdot}0$.

Putting $\bar{\lambda} = 0$ in (27) we obtain the stream function of an axially symmetric stagnation flow for viscous fluids against an imaginary wall distant $z = (1/\sqrt{2m})$

from the plate. The pressure gradient normal to the plate to the first approximation in ϵ is given by

$$
\begin{aligned}
-\frac{2}{\rho\nu\omega\epsilon}\frac{\partial p}{\partial y} = \frac{2}{c}\exp(-2cy) - \frac{\sqrt{2}}{E}\exp(-\sqrt{2}Ey)\left\{E^2\sin\left(2T-\frac{\sqrt{2}}{E}y\right)\right. \\
\left.+\cos\left(2T-\frac{\sqrt{2}}{E}y\right)\right\} + \frac{2}{c}\exp(-2cy)\left\{c^2\sin\left(2T-\frac{2y}{c}\right)+\cos\left(2T-\frac{2y}{c}\right)\right\} \\
+(2\lambda\cos 2T-\sin 2T)\left[\gamma\left\{1-\exp(-\sqrt{2}Ey)\cos\frac{\sqrt{2}y}{E}-E^2\exp(-\sqrt{2}Ey)\sin\frac{\sqrt{2}y}{E}\right\}\right. \\
\left.+\delta\left\{-1+\exp(-2cy)\cos\frac{2y}{c}+c^2\exp(-2cy)\sin\frac{2y}{c}\right\}\right] \\
+(2\lambda\sin 2T-\cos 2T)\left[\gamma\left\{E^2-E^2\exp(-\sqrt{2}Ey)\cos\frac{\sqrt{2}y}{E}+\exp(-\sqrt{2}Ey)\sin\frac{\sqrt{2}y}{E}\right\}\right. \\
\left.+\delta\left\{-c^2+c^2\exp(-2cy)\cos\frac{2y}{c}-\exp(-2cy)\sin\frac{2y}{c}\right\}\right]. \quad \ldots \quad \ldots \quad (28)
\end{aligned}
$$

5. Third Approximation to Velocity Components

From eqns. (5) to (7), we have

$$
\left.\begin{aligned} F_{2j}(y, T) &= 0, \\ G_{2j+1}(y, T) &= 0, \end{aligned}\right\} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (29)
$$

for $j = 0, 1, 2, \ldots$

The form of eqn. (6) suggests

$$
G_2(y, T) = \chi(y)\exp(iT)+\zeta(y)\exp(3iT). \quad \ldots \quad \ldots \quad (30)
$$

In real notation

$$
G_2(y, T) = \mathrm{Re}\,(\chi)\cos T-\mathrm{Im}\,(\chi)\sin T+\mathrm{Re}\,(\zeta)\cos 3T-\mathrm{Im}\,(\zeta)\sin 3T. \quad (31)
$$

From (7) the boundary conditions for $\chi(y)$ and $\zeta(y)$ are

$$
\left.\begin{aligned} x = \zeta = 0, \quad &\text{at} \quad y = 0, \\ x \to 0,\ \zeta \to 0, \quad &\text{as} \quad y \to \infty. \end{aligned}\right\} \quad \ldots \quad \ldots \quad \ldots \quad (32)
$$

Equations (6), (30) and (32) give

$$
\left.
\begin{aligned}
\operatorname{Re}(\chi) &= \exp(-cy)\Big[-(\alpha_1+\alpha_3)\cos\frac{y}{c}-(\beta_1+2\beta_2+\beta_3)\sin\frac{y}{c} \\
&\quad +\frac{y}{4c^3(1+c^4)}\Big\{(1-3\bar{\lambda}c^2-2c^4)\cos\frac{y}{c}+(\bar{\lambda}+3c^2-2\bar{\lambda}c^4)\sin\frac{y}{c}\Big\} \\
&\quad -\frac{y^2}{4c^2}\Big(\cos\frac{y}{c}+\bar{\lambda}\sin\frac{y}{c}\Big)\Big]+\exp(-3cy)\Big(\alpha_1\cos\frac{y}{c}+\beta_1\sin\frac{y}{c}\Big) \\
&\quad +\exp\{-(c+\sqrt{2}E)y\}\Big(\alpha_3\cos\frac{\sqrt{2}c-E}{Ec}y+\beta_3\sin\frac{\sqrt{2}c-E}{Ec}y\Big), \\
\operatorname{Im}(x) &= \exp(-cy)\Big[(\alpha_1+2\alpha_2+\alpha_3)\sin\frac{y}{c}-(\beta_1+\beta_3)\cos\frac{y}{c} \\
&\quad +\frac{y}{4c^3(1+c^4)}\Big\{(\bar{\lambda}+3c^2-2\bar{\lambda}c^4)\cos\frac{y}{c}-(1-3\bar{\lambda}c^2-2c^4)\sin\frac{y}{c}\Big\} \\
&\quad -\frac{y^2}{4c^2}\Big(\bar{\lambda}\cos\frac{y}{c}-\sin\frac{y}{c}\Big)\Big]+\exp(-3cy)\Big(\beta_1\cos\frac{y}{c}-\alpha_1\sin\frac{y}{c}\Big) \\
&\quad +\exp\{-(c+\sqrt{2}E)y\}\Big(\beta_3\cos\frac{\sqrt{2}c-E}{Ec}y-\alpha_3\sin\frac{\sqrt{2}c-E}{Ec}y\Big), \\
\operatorname{Re}(\zeta) &= -\exp(-\sqrt{3}ky)\Big\{(\alpha_4+\alpha_5+\alpha_6)\cos\frac{\sqrt{3}}{k}y+(\beta_4+\beta_5+\beta_6)\sin\frac{\sqrt{3}}{k}y\Big\} \\
&\quad +\exp(-cy)\Big(\alpha_4\cos\frac{y}{c}+\beta_4\sin\frac{y}{c}\Big)+\exp(-3cy)\Big(\alpha_5\cos\frac{3y}{c}+\beta_5\sin\frac{3y}{c}\Big) \\
&\quad +\exp\{-(c+\sqrt{2}E)y\}\Big(\alpha_6\cos\frac{\sqrt{2}c+E}{Ec}y+\beta_6\sin\frac{\sqrt{2}c+E}{Ec}y\Big), \\
\operatorname{Im}(\zeta) &= \exp(-\sqrt{3}ky)\Big\{-(\beta_4+\beta_5+\beta_6)\cos\frac{\sqrt{3}}{k}y+(\alpha_4+\alpha_5+\alpha_6)\sin\frac{\sqrt{3}}{k}y\Big\} \\
&\quad +\exp(-cy)\Big(\beta_4\cos\frac{y}{c}-\alpha_4\sin\frac{y}{c}\Big)+\exp(-3cy)\Big(\beta_5\cos\frac{3y}{c}-\alpha_5\sin\frac{3y}{c}\Big) \\
&\quad +\exp\{-(c+\sqrt{2}E)y\}\Big(\beta_6\cos\frac{\sqrt{2}c+E}{Ec}y-\alpha_6\sin\frac{\sqrt{2}c+E}{Ec}y\Big),
\end{aligned}
\right\}\quad (33)
$$

where

$$\alpha_1 = [8Q(1-2\bar{\lambda}^2c^4-3\bar{\lambda}c^2+6\bar{\lambda}c^6-c^4)-c^4(\bar{\lambda}+c^2)]/64Qc^4(1+4c^4),$$

$$\alpha_2 = [-\bar{\lambda}+32Q(S+\bar{\lambda}R)]/64Q,$$

$$\alpha_3 = Ec[(4R-4S\bar{\lambda}+3\bar{\lambda})(1-\sqrt{2}\bar{\lambda}Ec+E^2c^2)-(1-4S-4\bar{\lambda}R-2\bar{\lambda}^2)(c^2-E^2)]/2H,$$

$$\alpha_4 = [c^2(\beta+\bar{\lambda}\alpha+12\bar{\lambda}^2\beta)-Q(1+14\bar{\lambda}^2)]/2c^2(16\bar{\lambda}^2+1),$$

$$\alpha_5 = (c^2-4Q)(1-6\bar{\lambda}^2)/24c^2,$$

$$\alpha_6 = Ec[(4\alpha+12\bar{\lambda}\beta-5\bar{\lambda})(1-\sqrt{2}\bar{\lambda}Ec-E^2c^2)-(1-4\beta+12\bar{\lambda}\alpha-6\bar{\lambda}^2)(c^2+E^2)]/2H,$$

$$\beta_1 = [16Q(\bar{\lambda}+3c^2+2c^6+\bar{\lambda}c^4-4\bar{\lambda}^2c^6)+c^4(1-4\bar{\lambda}c^2)]/128Qc^4(1+4c^4),$$

$$\beta_2 = [1+64Q(S\bar{\lambda}-R)]/128Q,$$

$$\beta_3 = Ec[(-1+4S+2\bar{\lambda}^2+4\bar{\lambda}R)(1-\sqrt{2}\bar{\lambda}Ec+E^2c^2) + (4S\bar{\lambda}-4R-3\bar{\lambda})(c^2-E^2)]/2H,$$

$$\beta_4 = [c^2(\alpha-\bar{\lambda}\beta+12\bar{\lambda}^2\alpha)-Q\bar{\lambda}(1+24\bar{\lambda}^2)]/2c^2(16\bar{\lambda}^2+1),$$

$$\beta_5 = 5\bar{\lambda}(c^2-4Q)/24c^2,$$

$$\beta_6 = Ec[(1-4\beta+12\bar{\lambda}\alpha-6\bar{\lambda}^2)(1-\sqrt{2}\bar{\lambda}Ec-E^2c^2)+(4\alpha+12\bar{\lambda}\beta-5\bar{\lambda})(c^2+E^2)]/2H,$$

$$\alpha = (E^2+2\bar{\lambda}-c^2+2\bar{\lambda}E^2c^2)/8Ec\sqrt{2(4\bar{\lambda}^2+1)},$$

$$\beta = (1+2\bar{\lambda}c^2-2\bar{\lambda}E^2+E^2c^2)/8Ec\sqrt{2(4\bar{\lambda}^2+1)},$$

$$P = (1+2\bar{\lambda}^2)/8\sqrt{\bar{\lambda}^2+1},$$

$$Q = 1/8\sqrt{\bar{\lambda}^2+1},$$

$$R = (c^2+E^2+2\bar{\lambda}-2\bar{\lambda}E^2c^2)/8Ec\sqrt{2(4\bar{\lambda}^2+1)},$$

$$S = (E^2c^2+2\bar{\lambda}c^2+2\bar{\lambda}E^2-1)/8Ec\sqrt{2(4\bar{\lambda}^2+1)},$$

$$K^2 = \sqrt{9\bar{\lambda}^2+1}-3\bar{\lambda},$$

$$H = 2\sqrt{2}[(1-\sqrt{2}\bar{\lambda}Ec-E^2c^2)^2+(c^2+E^2)^2].$$

The transverse velocity component to third approximation is

$$v = r\omega(G_0+\epsilon^2G_2). \qquad \text{(34)}$$

The effects of relaxation time parameter on first and third approximation to v are depicted in Fig. 4 at time intervals $T = 0$, $(\pi/3)$, $(\pi/2)$ and $(2\pi/3)$ with $\bar{\lambda} = 0{\cdot}5$ and $\epsilon = 0{\cdot}5$. For small relaxation time parameter $\bar{\lambda}$, the two approximations are very close in the neighbourhood of the plate and the pattern of the graphs is similar to that of Newtonian case. Therefore eqn. (12) is taken as a good approximation to transverse velocity. The root mean square of transverse velocity is

$$\bar{v} = (r\omega/\sqrt{2}) \exp(-cmz/\sqrt{2}). \qquad \text{(35)}$$

6. The Steady Radial-axial Flow

We have noted from eqn. (22) that the steady part of radial velocity does not tend to zero as $y \to \infty$, which is in contradiction to the boundary conditions. Hence we consider the equations for the steady velocity components again. The governing equations are

$$\left.\begin{aligned} u_s\frac{\partial u_s}{\partial r}+w_s\frac{\partial u_s}{\partial z}-\frac{\bar{v}^2}{r} &= \nu\frac{\partial^2 u_s}{\partial z^2}, \\ \frac{\partial u_s}{\partial r}+\frac{u_s}{r}+\frac{\partial w_s}{\partial z} &= 0. \end{aligned}\right\} \qquad \text{(36)}$$

The boundary conditions are

$$u_s = w_s = 0 \text{ at } z = 0 \text{ and } u_s \to 0 \text{ as } z \to \infty. \quad \ldots \quad (37)$$

We assume

$$u_s = r\omega\epsilon f'(y),\ v_s = \bar{v},\ w_s = -2\omega\epsilon\sqrt{(2\nu/n)}f(y),\ z = \sqrt{(2\nu/n)}y. \quad \ldots \quad (38)$$

First equations of (36) and (38) give

$$\epsilon^2(f'^2-2ff'')-(1/2)\exp(-2cy) = (1/2)f''', \quad \ldots \quad (39)$$

which agrees with Rosenblat (1959) for $\bar{\lambda} = 0$. Following Rosenblat we assume that the steady radial flow will be confined to a secondary layer.

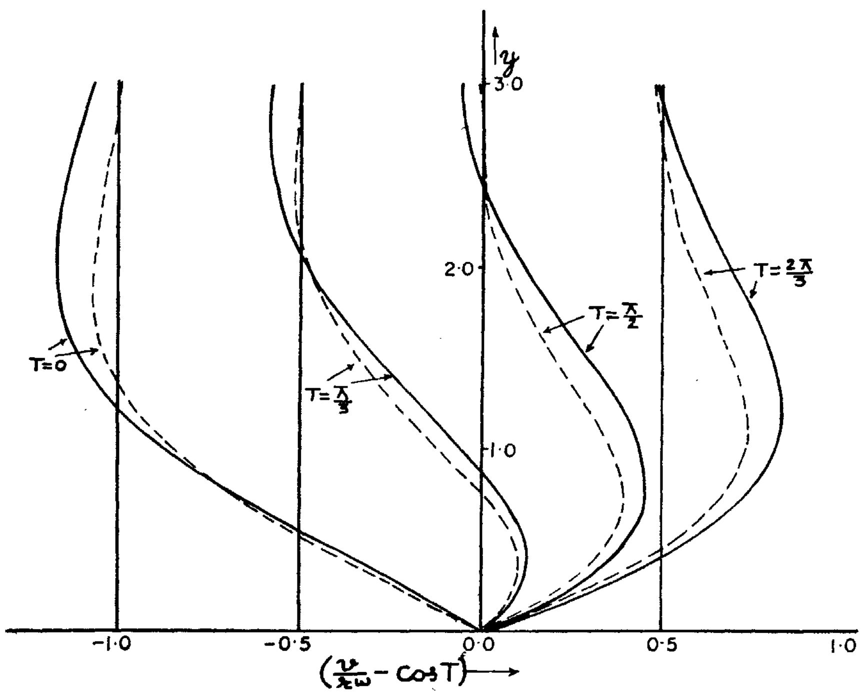

FIG. 4. Variation of $\left(\frac{v}{r\omega} - \cos T\right)$ with y at times $T = 0, \frac{\pi}{2}, \frac{2\pi}{3}$.
——— first approximation, - - - - second approximation.

The thickness of this layer is estimated approximately by Pohlhausen type method. Hence it is assumed that the flow takes place within a distance d of the oscillating plane and the conditions at infinity are satisfied at $y = d$. The transformed boundary conditions are

$$\left.\begin{array}{ll} f = 0, f' = 0, f''' = -1, & \text{at } y = 0, \\ f' = f'' = f''' = \ldots = 0, & \text{at } y = d. \end{array}\right\} \quad \ldots \quad (40)$$

Integrating (39) and using (40) we have

$$3\epsilon^2 \int_0^d (f'^2)dy - (1/4c) = -(1/2)f''(O). \quad (41)$$

Let

$$f' = \{\exp(-cy/d) - \exp(-2cy)\}/4c^2, \quad (42)$$

which nearly satisfies all boundary conditions. From (41) and (42) we have

$$\frac{3\epsilon^2}{2c^4}\left(\frac{1}{4} - \frac{2d}{2d+1} + \frac{d}{2}\right) = \frac{1}{d}, \quad (43)$$

which agrees with the result of Rosenblat (1959) for $\bar{\lambda} = 0$.

Equation (43) determines the thickness of this secondary layer for the case of a Maxwell fluid. We have

$$d \approx (1{\cdot}155c^2/\epsilon) + 0{\cdot}75, \quad (44)$$

for the order of thickness of secondary layer. For Newtonian fluids $\bar{\lambda} = 0$, eqn. (44) transforms to

$$d \approx (1{\cdot}155/\epsilon) + 0{\cdot}75, \quad (45)$$

which differs from the equation obtained by Rosenblat (1959, eqn. 61), in the constant term.

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