CERTAIN THEOREMS IN BILATERAL OPERATIONAL CALCULUS WITH THEIR APPLICATIONS

by R. K. GUPTA, *Department of Applied Mathematics, Applied Sciences and Humanities, M.N.R. Engineering College, Allahabad*

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The paper deals with properties of the bilateral Laplace transformation of the two variables. In the beginning the author has started with the operational relation

$$f(x_1, x_2)U(x_1, x_2) \doteqdot F(p_1, p_2)$$

and later on x_1 and x_2 were replaced by the general functions

$$a_1^{\phi_1(x_1)} \text{ and } a_1^{\phi_2(x_2)}$$

respectively, which gives rise to a more general bilateral operational relation in two variables. As an application of the same large number of operational relations have been investigated.

§ 1. In this paper we shall take $f(x, y)$ to be a function of two real variables, x and y, defined for all the pairs (x, y) in $-\infty < x < \infty$ and $-\infty < y < \infty$, and, integrable in any finite rectangle:

$$R_{xy};\ -X \leqslant x \leqslant X,\ -Y \leqslant y \leqslant Y.$$

If for a pair of complex quantities p and q the limit of the following four integrals:

$$\lim_{\substack{X \to \infty \\ Y \to \infty}} \int_0^X \int_0^Y e^{-px-qy} f(x, y)\, dx\, dy, \qquad \ldots \quad \ldots \quad \text{(A)}$$

$$\lim_{\substack{X \to \infty \\ Y \to \infty}} \int_0^X \int_{-Y}^0 e^{-px-qy} f(x, y)\, dx\, dy, \qquad \ldots \quad \ldots \quad \text{(B)}$$

$$\lim_{\substack{X \to \infty \\ Y \to \infty}} \int_{-X}^0 \int_0^Y e^{-px-qy} f(x, y)\, dx\, dy, \qquad \ldots \quad \ldots \quad \text{(C)}$$

$$\lim_{\substack{X \to \infty \\ Y \to \infty}} \int_{-X}^0 \int_{-Y}^0 e^{-px-qy} f(x, y)\, dx\, dy, \qquad \ldots \quad \ldots \quad \text{(D)}$$

exist, then evidently the double integral

$$L_\pi^2\{f\} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-px-qy} f(x, y)\, dx\, dy \qquad \ldots \quad \ldots \quad (1)$$

exists, and we shall call it the two-dimensional bilateral Laplace integral of $f(x, y)$ for the pair of values of p and q and when $L^2_\pi\{f\}$ exists not only for a particular pair of values of (p, q) but for every pair of values in a certain associated region of complex p and q planes, then we shall call it the two-dimensional bilateral Laplace transform of $f(x, y)$. So we shall take

$$F(p, q) = pq \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-px-qy} f(x, y)\, dx\, dy \qquad \ldots \qquad \ldots \qquad (2)$$

and denote it by $F(p, q) \;\overset{\cdot\cdot}{\underset{\cdot\cdot}{=\!=}}\; f(x, y)$ or $F(p, q) \longrightarrow_2 f(x, y)$
and in one variable by $F(p) \;\overset{\cdot}{\underset{\cdot}{=\!=}}\; f(x)$ or $F(p) \longrightarrow f(x)$.

In analogy with Heaviside's unit function $U(x, y) \equiv U(x)U(y)$,

$$U(x, y) = 0 \quad \text{for } x \text{ or } y < 0$$

and

$$U(x, y) = 1 \quad \text{for } x \geqslant 0,\ y \geqslant 0.$$

Obviously $U(x, y)$ is valid for Re $(p) > 0$, Re $(q) > 0$
$U(-x, -y)$ is valid for Re $(p) < 0$, Re $(q) < 0$ $\;\overset{\cdot\cdot}{\underset{\cdot\cdot}{=\!=}}\; 1.$

THEOREM 1. Let

(i) $F(p_1, p_2) \;\overset{\cdot\cdot}{\underset{\cdot\cdot}{=\!=}}\; G(s_1, s_2)$, $\dfrac{1}{p_r} \;\overset{\cdot}{\underset{\cdot}{=\!=}}\; s_r$, $r = 1, 2$.

where $L^2_\pi(G)$ is absolutely convergent and $G(s_1, s_2)$ is integrable L in $(-\infty, \infty)$ in a pair of associated convergence domains S_{p_1} and S_{p_2}, which may be the complex p_1 and p_2 planes or the above halves of these planes, viz. $-\infty < R(p_r) < \infty$.

(ii) $\phi_r(p_r, s_r) \;\overset{\cdot}{\underset{\cdot}{=\!=}}\; \exp\left[-s_r\theta_r(x_r^k)\right]\psi_r(x_r)$, $\dfrac{1}{p_r} \;\overset{\cdot}{\underset{\cdot}{=\!=}}\; x_r$,

where s_r denotes a real parameter and the definition integral implied in the operational relation is absolutely convergent in a pair of associated convergence domains, say D_{p_r}, which may also be the complex p_1 and p_2 planes.

(iii) $\phi_r(p_r, s_r)$ is bounded and absolutely integrable in s_r in $(-\infty, \infty)$.

(iv) $\theta_1(p_1^k) \in S_{p_1}$ and $\theta_2(p_2^k) \in S_{p_2}$. Then

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \phi_1(p_1, s_2)\phi_2(p_2, s_2) G(s_1, s_2)\, ds_1\, ds_2 \;\overset{\cdot\cdot}{\underset{\cdot\cdot}{=\!=}}\; \frac{F\{\theta_1(x_1^k), \theta_2(x_2^k)\}}{\theta_1(x_1^k)\theta_2(x_2^k)} \psi_1(x_1)\psi_2(x_2),$$

provided that $L^2_\pi\{f\}$ is absolutely convergent in a pair of associated domains Ω_{p_r}, where Ω_{p_r} are the common region of S_{p_r} and D_{p_r}. Some particular cases of the above theorem are worth noticing.

Corollary 1: Let $k = 1$, $\psi_r(x_r) = 1$ and $\theta_r(x_r) = \cosh\,(x_r + a_r x_r)$.
Then (Watson 1944, p. 256)

$$4p_1p_2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} K_{p_1+a_1s_1}(s_1)K_{p_2+a_2s_2}(s_2)G(s_1,\,s_2)\,ds_1\,ds_2$$

$$\doteqdot \frac{F(\cosh x_1+a_1x_1,\,\cosh x_2+a_2x_2)}{(\cosh x_1+a_1x_1)(\cosh x_2+a_2x_2)}.$$

Corollary 2: Let $k = 2$, $\psi_r(x_r) = 1$ and $\theta_r(x_r) = x_r$, $r = 1, 2$.
Then (Watson 1944, p. 386)

$$\pi p_1p_2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{e^{\frac{p_1^2}{4s_1}+\frac{p_2^2}{4s_2}}}{(s_1s_2)^{\frac{1}{2}}}G(s_1,\,s_2)\,ds_1\,ds_2 \doteqdot \frac{F(x_1^2,\,x_2^2)}{(x_1x_2)^2}.$$

Corollary 3: Let $k = 1$, $\psi_r(x_r) = e^{-e^{-x_r}}$, $\theta_r(x_r) = x_r$, $r = 1, 2$.
Then (Watson 1944, p. 47)

$$p_1p_2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Gamma(p_1+s_1)\Gamma(p_2+s_2)G(s_1,\,s_2)\,ds_1\,ds_2 \doteqdot \frac{F(x_1,\,x_2)}{x_1x_2}e^{-e^{-x_1}}e^{-e^{-x_2}}.$$

Corollary 4: Let $k = 1$, $\psi_r(x_r) = 1$, $\theta_r(x_r) = \cosh x_r$.
Then (Watson 1944, p. 393)

$$4p_1p_2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} K_{p_1}(s_1)K_{p_2}(s_2)G(s_1,\,s_2)\,ds_1\,ds_2 \doteqdot \frac{F(\cosh x_1,\,\cosh x_2)}{(\cosh x_1)(\cosh x_2)}.$$

Corollary 5: Let $k = -1$, $\psi_r(x_r) = \dfrac{U(x_r)}{\nu}$, $\theta_r(x_r) = x_r$, $r = 1, 2$.

Then (Watson 1944, p. 393)

$$4(p_1p_2)^{\frac{1}{2}(\nu+1)}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (s_1s_2)^{\frac{1}{2}(1-\nu)}K_{\nu-1}(2\sqrt{s_1p_1})K_{\nu-1}(2\sqrt{s_2p_2})G(s_1,\,s_2)\,ds_1\,ds_2$$

$$\doteqdot \frac{F\left(\dfrac{1}{x_1},\,\dfrac{1}{x_2}\right)}{(x_1x_2)^{\nu-1}}U(x_1)U(x_2).$$

Corollary 6: Let $k = 1$, $\psi_r(x_r) = \dfrac{(1-e^{-x_r})^{\nu-1}}{\Gamma(\nu)}U(x_r)$, $\theta_r(x_r) = x_r$.

Then (Watson 1944, p. 400)

$$p_1p_2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\Gamma(p_1+s_1)\Gamma(p_2+s_2)}{\Gamma(p_1+s_1+\nu)\Gamma(p_2+s_2+\nu)}G(s_1,\,s_2)\,ds_1\,ds_2$$

$$\doteqdot \frac{1}{x_1x_2\{\Gamma(\nu)\}^2}(1-e^{-x_1})^{\nu-1}(1-e^{-x_2})^{\nu-1}F(x_1,\,x_2)\,U(x_1)U(x_2).$$

Corollary 7: Let $k = -1$, $\psi_r(x_r) = (2x_r)^{\nu-1}U(x_r)$, $\theta_r(x_r) = x_r$.
Then (Watson 1944, p. 393)

$$4(p_1p_2)^{1-\frac{1}{2}\nu}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(s_1s_2)^{\frac{1}{2}\nu}K_\nu(2\sqrt{p_1s_1})K_\nu(2\sqrt{p_2s_2})G(s_1, s_2)\,ds_1\,ds_2$$
$$\doteqdot F\left(\frac{1}{x_1}, \frac{1}{x_2}\right)(x_1x_2)^\nu U(x_1)U(x_2).$$

As applications of the above corollaries we find the following correspondences in one variable only.

1. Let $G(s) = \dfrac{e^{-\frac{1}{4s}}}{2\sqrt{\pi}s^{\frac{3}{2}}}U(s) \doteqdot pe^{-\sqrt{p}} = F(p)$ (Watson 1944, p. 386).

Then by Cor. 6, we get

$$\frac{p}{2\sqrt{\pi}}\int_0^\infty \frac{\Gamma(p+s)}{\Gamma(p+s+\nu)}\,\frac{e^{-\frac{1}{4s}}}{(s)^{\frac{3}{2}}}\,ds \doteqdot \frac{e^{-\sqrt{x}}}{\Gamma(\nu)}\,(1-e^{-x})^{\nu-1}U(x),$$
$$\operatorname{Re}(\nu) > 0,\ \infty > \operatorname{Re}(p) >$$

2. Consider $G(s) = \dfrac{e^{-\frac{1}{4s}}}{\sqrt{\pi s}}U(s) \doteqdot \sqrt{p}e^{-\sqrt{p}} = F(p)$ (Watson 1944, p. 386)

$$\frac{2p^{1-\frac{1}{2}\nu}}{\sqrt{\pi}}\int_0^\infty s^{\frac{1}{2}(\nu-1)}K_\nu(2\sqrt{ps})e^{-\frac{1}{4s}}\,ds \doteqdot x^{\nu-\frac{1}{2}}e^{-\frac{1}{\sqrt{x}}}U(x),\quad \operatorname{Re}(p) > 0.$$

3. Let $G(s) = \operatorname{erfc}\left(\dfrac{1}{2\sqrt{s}}\right)U(s) \doteqdot e^{-\sqrt{p}} = F(p)$ (Watson 1944, p. 388).

By Cor. 7, we get

$$2p^{1-\frac{1}{2}\nu}\int_0^\infty s^{\frac{1}{2}\nu}\operatorname{erfc}\left(\frac{1}{2\sqrt{s}}\right)K_\nu(2\sqrt{ps})\,ds \doteqdot x^\nu e^{-\frac{1}{\sqrt{x}}}U(x).$$

4. Let $G(s) = \dfrac{e^{-\frac{a+b}{s}}}{s}K_\nu\left(\dfrac{2\sqrt{ab}}{s}\right)U(s) \doteqdot 2pK_\nu(2\sqrt{ap})K_\nu(2\sqrt{bp}) = F(p)$.

By Cor. 2, we get

$$\frac{\sqrt{\pi}p\Gamma(\frac{1}{2}-\nu)}{\cos(\nu\pi)\{(a+b-\frac{1}{4}p^2)^2-4ab\}^{\frac{1}{4}}}Q^\nu_{-\frac{1}{2}}\left[\frac{a+b-\frac{1}{4}p^2}{\{(a+b-\frac{1}{4}p^2)^2-4ab\}^{\frac{1}{2}}}\right]$$
$$\doteqdot 2K_\nu(2\sqrt{ax})K_\nu(2\sqrt{bx}),\ \operatorname{Re}(\tfrac{1}{2}\pm\nu) > 0,\ (\sqrt{a}\pm\sqrt{b}) > \tfrac{1}{2}p > 0.$$

5. Let $G(s) = \dfrac{e^{-\frac{1}{4s}}}{2\sqrt{\pi}s^{\frac{3}{2}}}U(s) \doteqdot pe^{-\sqrt{p}} = F(p)$.

By Cor. 4, we get

$$\frac{p}{\sqrt{2}}\,\frac{1}{\pi}S_4\left(\frac{1}{4}, -\frac{1}{4}, \frac{p-1}{2}, -\frac{p+1}{2}; \frac{1}{16}\right) \doteqdot e^{-\sqrt{\cosh x}}.$$

6. Let $G(s) = s^\mu J_\mu(s)U(s) \doteqdot \dfrac{1}{\sqrt{\pi}}2^\mu\Gamma(\mu+\tfrac{1}{2})p(p^2+1)^{-\mu-\frac{1}{2}} = F(p)$.

By Cor. 7, we get

$$p^{\frac{\nu+1}{2}}\int_0^\infty K_{\nu-1}(2\sqrt{sp})J_\mu(s)s^{\mu-\frac{1}{2}\nu-\frac{1}{2}}\,ds \doteqdot \frac{\Gamma(\mu+\frac{1}{2})2^{\mu-1}x^{2\mu-\nu+1}}{\sqrt{\pi}(x^2+1)^{\mu+\frac{1}{2}}}\,U(x),$$

$\operatorname{Re}(\mu) > -\frac{1}{2}$ and $\operatorname{Re}(2\mu-\nu) > -2$.

THEOREM 2. Let

(i) $F(p_1, p_2) \doteqdot G(s_1, s_2),\ \frac{1}{p_r} \doteqdot s_r,\ (r = 1, 2),$

where $L^2_\pi\{G\}$ is absolutely convergent in a pair of associated convergence domains S_{p_1} and S_{p_2} which may be the half-planes $-\infty < \operatorname{Re}(p_i) < \infty$, $i = 1, 2$.

(ii) $\theta_r(p_r, s_r) \doteqdot \exp\left[-s_r\phi_r^k(x_r)\right]\psi_r(x_r),$

valid in a pair of associated convergence domains D_{p_1} and D_{p_2} which also may be half-planes $-\infty < \operatorname{Re}(p) < \infty$, and s_1, s_2 are real parameters.

(iii) $p_1+\phi_1^k(p_1) \in S_{p_1},\ \ p_2+\phi_2^k(p_2) \in S_{p_2}.$

Then

$$\int_{-\infty}^\infty\int_{-\infty}^\infty \frac{F\left\{x_1+\phi_1^k(x_1),\, x_2+\phi_2^k(x_2)\right\}}{\left\{x_1+\phi_1^k(x_1)\right\}\left\{x_2+\phi_2^k(x_2)\right\}}\,\psi_1(x_1)\psi_2(x_2)\,dx_1\,dx_2$$

$$= \int_{-\infty}^\infty\int_{-\infty}^\infty \theta_1(s_1, s_1)\theta_2(s_2, s_2)G(s_1, s_2)(s_1s_2)^{-1}\,ds_1\,ds_2,$$

provided that both the double integrals are absolutely convergent in $(-\infty, \infty)$.

Corollary 1: Let $k = 1,\ \psi_r(x_r) = \frac{U(x_r)}{\sqrt{x_r}},\ \phi_r(x_r) = \frac{1}{x_r},\ r = 1, 2,$

then we get

$$\int_0^\infty\int_0^\infty \frac{F\left(x_1+\frac{1}{x_1},\, x_2+\frac{1}{x_2}\right)}{(1+x_1^2)(1+x_2^2)}(x_1x_2)^{\frac{1}{2}}\,dx_1\,dx_2 = \pi\int_{-\infty}^\infty\int_{-\infty}^\infty \frac{e^{-2(s_1+s_2)}}{\sqrt{s_1s_2}}\,G(s_1, s_2)\,ds_1\,ds_2.$$

Corollary 2: Let $k = \frac{1}{2},\ \psi_r(x_r) = (2x_r)^{\nu-1}U(x_r),\ \phi_r(x_r) = x_r,$
then we get

$$\frac{2^{2(\nu-1)}}{\{\Gamma(2\nu)\}^2}\int_0^\infty\int_0^\infty \frac{F(x_1+\sqrt{x_1},\, x_2+\sqrt{x_2})}{(x_1+\sqrt{x_1})(x_2+\sqrt{x_2})}(x_1x_2)^{\nu-1}\,dx_1\,dx_2$$

$$= \int_0^\infty\int_0^\infty \frac{e^{\frac{s_1+s_2}{8}}}{(s_1s_2)^\nu}\,D_{-2\nu}\left(\sqrt{\frac{s_1}{2}}\right)D_{-2\nu}\left(\sqrt{\frac{s_2}{2}}\right)G(s_1, s_2)\,ds_1\,ds_2.$$

Examples

1. Consider $G(s) = \frac{1}{\sqrt{\pi}}\sin(2\sqrt{s})U(s) \doteqdot p^{-\frac{1}{2}}e^{-\frac{1}{p}} = F(p).$

By Cor. 1, we get

$$\int_0^\infty x(x^2+1)^{-\frac{3}{2}} e^{-\frac{x}{x^2+1}}\,dx = -i\sqrt{\frac{\pi}{2}}\,e^{-\frac{1}{2}}\,\mathrm{erfc}\left(\frac{1}{\sqrt{2}}\right).$$

2. Consider $G(s) = \mathrm{erfc}\left(\frac{1}{2}\sqrt{\frac{a}{s}}\right)U(s) \doteqdot e^{-\sqrt{ap}} = F(p)$.

By Cor. 1, we get

$$\int_0^\infty \frac{e^{-\sqrt{a\left(x+\frac{1}{x}\right)}}}{(1+x^2)}\sqrt{x}\,dx = \sqrt{\pi}\int_0^\infty \mathrm{erfc}\left(\frac{1}{2}\sqrt{\frac{a}{s}}\right)\frac{e^{-2s}}{\sqrt{s}}\,ds.$$

§ 2. We have already proved (Gupta 1959, p. 197) the theorem:

(i) Let $F(p_1, p_2) \doteqdot f(x_1, x_2)U(x_1, x_2)$,

where $L_\pi^2\{f\}$ is absolutely convergent in the half-planes H_{p_i} (say), $i = 1, 2$.

(ii) $e^{-\lambda_i\psi_i(p_i)} \doteqdot h_i(\lambda_i, x_i)$, $\lambda_i > 0$, $h_i(\lambda_i, x_i)$ is a continuous function of x_i in $(-\infty, \infty)$ and $L_\pi\{h\}$ is absolutely convergent in the half-planes D_{p_i}, defined by $R(p_i) \geqslant 0$, and $\psi_i(p_i)$ is bounded and integrable in $(-\infty, \infty)$. Then

$$K(p_1, p_2) = \frac{p_1p_2}{(\log a)^2}\int_{-\infty}^\infty\int_{-\infty}^\infty h_1(p_1, x_1)h_2(p_2, x_2)\frac{F(x_1, x_2)}{x_1x_2}\,dx_1\,dx_2$$
$$\doteqdot f\{a^{\phi_1(x_1)}, a^{\phi_2(x_2)}\}\phi_1'(x_1)\phi_2'(x_2),\quad a \neq 1.$$

It is sufficient if $h_1(p_1, x_1)h_2(p_2, x_2)\dfrac{F(x_1, x_2)}{x_1x_2}$ is absolutely convergent in $(-\infty, \infty)$ in each of the variables and $f\left\{a^{\phi_1(x_1)}, a^{\phi_2(x_2)}\right\}\phi_1'(x_1)\phi_2'(x_2)$ is bounded and integrable in $(-\infty, \infty)$.

(iii) $\phi_i(x_i)$ is a single-valued function of x_i and differentiable in x_i, $a^{\phi_i(x_i)}$ tends to zero as x tends to $-\infty$, and to $+\infty$ as x tends to $+\infty$ or vice versa.

(iv) $x_i = \phi_i^{-1}\left(\dfrac{\log s_i}{\log a}\right) = \psi_i(s_i)$,

where ϕ_i^{-1} denotes the function inverse to ϕ_i.

Some particular cases of the above theorem are given below.

Corollary 1: Let $\phi_i(x_i) = x_i$, then we obtain

$$\frac{1}{\Gamma(p_1')\Gamma(p_2')}\int_0^\infty\int_0^\infty x_1^{p_1'-1}x_2^{p_2'-1}F(x_1, x_2)\,dx_1\,dx_2 \doteqdot f(a^{x_1}, a^{x_2}),$$

where $p' = p/\log a$, provided the integral on the left is convergent and the definition integral exists.

Corollary 2: If we assume the image integral to be

$$F(a^{-x_1}, a^{-x_2}) \doteqdot p_1p_2\int_{-\infty}^\infty\int_{-\infty}^\infty e^{-p_1x_1-p_2x_2}F(a^{-x_1}, a^{-x_2})\,dx_1\,dx_2$$

and let $a^{-x_i} = s_i$, we obtain

$$F(a^{-x_1}, a^{-x_2}) \doteq \Gamma(p_1'+1)\Gamma(p_2'+1)K(p_1', p_2') = \int_0^\infty \int_0^\infty s_1^{p_1'-1} s_2^{p_2'-1} F(s_1, s_2)\, ds_1\, ds_2.$$

As applications of the above corollaries we find the following operational relations in one variable.

1. Consider

$$x^\lambda J_\lambda^\mu(bx^\mu)U(x) \doteq p^{-\lambda}e^{-b/p\mu}, \quad b > 0,\ 1 > \mu > 0,\ \mathrm{Re}\,(p) > 0,\ \mathrm{Re}\,(\lambda) > 1.$$

By Cor. 1, we get

$$a^{\lambda x} J_\lambda^\mu(ba^{\mu x}) \doteq \frac{\Gamma\left(\frac{\lambda-p'}{\mu}\right)}{\mu\Gamma(p')b^{\frac{\lambda-p'}{\mu}}}, \quad \mathrm{Re}\,(\lambda-p') > 0,\ 1 > \mu > 0.$$

When $\mu = 1$, the result is true if $\frac{1}{2}(\lambda-\frac{3}{2}) < \mathrm{Re}\,(p') < \mathrm{Re}\,(\lambda)$.

By Cor. 2, we get

$$a^{\lambda x} e^{-ba^{\mu x}} \doteq \frac{p'\Gamma\left(\frac{\lambda-p'}{\mu}\right)}{\mu b^{\frac{\lambda-p'}{\mu}}},$$

if $\mu = 1$, $\lambda = \frac{1}{2}$, we get

$$\sin\,(2\sqrt{ba^x}) \doteq \frac{\sqrt{\pi}b^{p'}\Gamma(\frac{1}{2}-p')}{\Gamma(p')}, \quad \tfrac{1}{2} > \mathrm{Re}\,(p') > -\tfrac{1}{2},\ b > 0.$$

2. Consider $x^{-\frac{1}{2}(n+1)}e^{-\frac{b^2}{4x}}\,\mathrm{He}_n\left(\frac{b}{\sqrt{2x}}\right)U(x) \doteq 2^{\frac{1}{2}n}\sqrt{\pi}p^{\frac{1}{2}(n+1)}e^{-b\sqrt{p}}$, $\mathrm{Re}\,(p, a) > 0$ (Erdelyi 1954, I, p. 246, No. 8).

By Cor. 1, we get

$$(a^x)^{-\frac{1}{2}(n+1)}e^{-\frac{b^2}{4a^x}}\,\mathrm{He}_n\left(\frac{b}{\sqrt{2}a^{\frac{1}{2}x}}\right) \doteq \frac{2^{1+\frac{1}{2}n}\sqrt{\pi}\Gamma(2p'+n+1)}{\Gamma(p')b^{2p'+n+1}}, \quad \mathrm{Re}\,(2p'+n+1) > 0.$$

3. Let $f(x)U(x) = \sqrt{\frac{2}{\pi}}\,\frac{e^{-\frac{b^2}{8x}}}{(2x)^n}D_{2n-1}\left(\frac{b}{\sqrt{2x}}\right)U(x) \doteq p^n e^{-b\sqrt{p}} = F(p)$, $\mathrm{Re}\,(p, a) > 0$ (Erdelyi 1954, p. 52).

By Cor. 1, we get

$$(2a^x)^{-n}e^{-\frac{a^{-x}b^2}{8}}D_{2n-1}\left(a^{-\frac{1}{2}x}\frac{b}{\sqrt{2}}\right) \doteq \frac{\sqrt{2\pi}\Gamma(2p'+2n)}{\Gamma(p')b^{2p'+2n}}, \quad \mathrm{Re}\,(n+p') > 0.$$

4. Let $f(x)U(x) = (4x)^{\frac{1}{2}n}i^n\,\mathrm{erfc}\left(\frac{b}{2\sqrt{x}}\right)U(x) \doteq p^{-\frac{1}{2}n}e^{-b\sqrt{p}} = F(p)$, $\mathrm{Re}\,(p, b) < 0$ and $n = 0, 1, 2, \ldots$

By Cor. 1, we get

$$(4a^x)^{\frac{1}{2}n}i^n\,\mathrm{erfc}\,(\tfrac{1}{2}ba^{-\frac{1}{2}x}) \doteq \frac{2\Gamma(2p'-n)}{\Gamma(p')b^{2p'-n}}, \quad \mathrm{Re}\,(p') > \tfrac{1}{2}n.$$

5. Let $f(x)U(x) = \left\{2\sqrt{\frac{x}{\pi}}\, e^{-\frac{b}{4x}} - \sqrt{b}\,\operatorname{erfc}\left(\frac{1}{2}\sqrt{\frac{b}{x}}\right)\right\} U(x) \doteq p^{-\frac{1}{2}} e^{-\sqrt{bp}} = F(p)$,

Re $(p, a) > 0$ (Erdelyi 1954, I, p. 38).

By Cor. 1, we get

$$\frac{2a^{\frac{1}{2}x}}{\sqrt{\pi}} e^{-\frac{1}{4}ba^{-x}} - \sqrt{b}\,\operatorname{erfc}\left(\frac{1}{2}\sqrt{\frac{b}{a^x}}\right) \doteq \frac{2\Gamma(2p'-1)}{\Gamma(p')b^{p'-\frac{1}{2}}},\quad \text{Re }(p') > \tfrac{1}{2},\ b > 0.$$

6. Let $f(x)U(x) = \left\{(x+\tfrac{1}{2}b)\,\operatorname{erfc}\left(\frac{1}{2}\sqrt{\frac{b}{x}}\right) - \sqrt{\frac{bx}{\pi}}\, e^{-\frac{b}{4x}}\right\} U(x) = \frac{e^{-(bp)^{\frac{1}{2}}}}{p} = F(p)$,

Re $(p, b) > 0$ (McLachlan *et al.* 1950, p. 52).

By Cor. 1, we get

$$(a^x+\tfrac{1}{2}b)\,\operatorname{erfc}\left(\frac{1}{2}\sqrt{\frac{b}{a^x}}\right) - \sqrt{\frac{b}{\pi}}\, a^{\frac{1}{2}x} e^{-\frac{1}{4}ba^{-x}} \doteq \frac{2\Gamma(2p'-2)}{\Gamma(p')b^{p'-1}},\quad \text{Re }(p') > 1,\ b > 0.$$

7. Let $f(x)U(x) = \dfrac{(x+\sqrt{x^2-1})^{\nu}+(x+\sqrt{x^2-1})^{-\nu}}{\sqrt{x^2-1}}\, U(x-1) \doteq 2pK_{\nu}(p) = F(p)$,

Re $(p) > 0$ (McLachlan *et al.* 1950, p. 23).

By Cor. 1, we get

$$\frac{(a^x+\sqrt{a^{2x}-1})^{\nu}+(a^x+\sqrt{a^{2x}-1})^{-\nu}}{\sqrt{a^{2x}-1}}\, U(a^x-1) \doteq \frac{2^{p'}\Gamma\left(\frac{p'+\nu+1}{2}\right)\Gamma\left(\frac{p'-\nu+1}{2}\right)}{\Gamma(p')}$$

Re $(p') > |\text{Re }(\nu)| - 1$.

8. Let $f(x)U(x) = \dfrac{(-1)^m b^n m!}{2^{n+1}x^m}\, e^{-\frac{1}{4}\frac{b^2}{x}} L_m^n\left(\frac{1}{4}\frac{b^2}{x}\right) U(x) \doteq p^{\frac{1}{2}n+m+1} K_n(b\sqrt{p}) = F(p)$,

Re $(p, b) > 0$, and m, n are positive integers (Van der Pol and Bremmer 1955, p. 283, No. 42).

By Cor. 1, we get

$$(-1)^m a^{-mx} m!\, e^{-\frac{b^2}{4a^x}} L_m^n\left(\frac{b^2}{4a^x}\right) \doteq \frac{\Gamma(p'+m+1)\Gamma(p'+m+n+1)}{\Gamma(p')}\left(\frac{2}{b}\right)^{2p'+2m+2n+2},$$

Re $(p') > 0$, $b > 0$.

9. Let $f(x)U(x) = x^{-\mu} e^{-\frac{b^2}{4x}} W_{\mu,\nu}\left(\frac{b^2}{4x}\right) U(x) \doteq bp^{\mu+\frac{1}{2}} K_{2\nu}(b\sqrt{p}) = F(p)$,

Re $(p) > |\text{Re }(\nu)| - \mu - \tfrac{1}{2} > 0$, $b > 0$ (Van der Pol and Bremmer 1955, p. 217, No. 22).

$$a^{-\mu x} e^{-a^{-x}\frac{b^2}{8}} W_{\mu,\nu}(\tfrac{1}{4}b^2a^{-x}) \doteq \frac{\Gamma(p'+\mu-\nu+\frac{1}{2})\Gamma(p'+\mu+\nu+\frac{1}{2})}{\Gamma(p')}\left(\frac{2}{b}\right)^{2p'+2\mu},$$

Re $(p') > |\text{Re }(\nu)| - \mu - \tfrac{1}{2} > 0$.

10. Let $f(x)U(x) = (x^2-b^2)^{\nu} U(x-b) \doteq \dfrac{(2b)^{\nu+\frac{1}{2}}\Gamma(\nu+1)}{\sqrt{\pi}\, p^{\nu-\frac{1}{2}}} K_{\nu+\frac{1}{2}}(bp) = F(p)$,

Re $(p) > 0$, Re $(\nu) \geqslant -\tfrac{1}{2}$, $x > b$ (Erdelyi 1954, p. 14).

By Cor. 2, we get

$$(a^{2x}-b^2)^\nu U(a^x-b) \doteq \frac{\Gamma(\nu+1)\Gamma\left(\frac{p'+1}{2}\right)\Gamma\left(\frac{p'-2\nu}{2}\right)}{\sqrt{\pi}\Gamma(p')b^{p'-2\nu}2^{1-p'}},$$

$\operatorname{Re}(\nu) > -1$, $\operatorname{Re}(p'-2\nu) > 0$.

11. Let $f(x)U(x) = (x^2-1)^{-\frac{1}{2}\mu}P_\nu^\mu(x)U(x-1) \doteq \sqrt{\frac{2p}{\pi}}\, p^\mu K_{\nu+\frac{1}{2}}(p) = F(p)$,

$\operatorname{Re}(p) > 0$, $\operatorname{Re}(\mu) < 1$ (Watson 1944, p. 394).

$$(a^{2x}-1)^{-\frac{1}{2}\mu}P_\nu^\mu(a^x)U(a^x-1) \doteq \frac{2^{p'+\mu-1}\Gamma\left(\frac{p'+\mu-\nu}{2}\right)\Gamma\left(\frac{p'+\mu+\nu+1}{2}\right)}{\Gamma^{(p')}\sqrt{\pi}}$$

$\operatorname{Re}(p'+\mu-\nu) > 0$, $\operatorname{Re}(p'+\mu+\nu+1) > 0$, $\operatorname{Re}(\mu) < 1$.

12. Let

$$f(x)U(x) = (x-\tfrac{1}{2})^{\nu-\mu-\frac{1}{2}}(\tfrac{1}{2}+x)^{\nu+\mu-\frac{1}{2}}U(x-\tfrac{1}{2}) \doteq \Gamma(\nu-\mu+\tfrac{1}{2})p^{\frac{1}{2}-\nu}W_{\mu,\nu}(p) = F(p),$$

$\operatorname{Re}(p) > 0$, $\mu < \nu+\frac{1}{2}$ (Erdelyi 1954, I, p. 24, by shift rule).

By Cor. 1, we get

$$\frac{(a^x-\frac{1}{2})^{\nu-\mu-\frac{1}{2}}(a^x+\frac{1}{2})^{\nu+\mu-\frac{1}{2}}}{\Gamma(\nu-\mu+\frac{1}{2})}U(a^x-\tfrac{1}{2}) \doteq \frac{p'\Gamma(p'-2\nu+1)}{\Gamma(p'-\nu-\mu+\frac{3}{2})}$$

$$\times {}_2F_1(p'+1, p'-2\nu+1; p'-\nu-\mu+\tfrac{3}{2}; \tfrac{1}{2}),$$

$\operatorname{Re}(p') > -1$, $\operatorname{Re}(p') > 2\nu-1$, $\operatorname{Re}(\nu-\mu) > -\frac{1}{2}$.

13. Let $f(x)U(x) = J_\nu(2\sqrt{bx})J_\nu(2\sqrt{cx})U(x) \doteq e^{-\frac{b+c}{p}}I_\nu\left(\frac{2\sqrt{bc}}{p}\right) = F(p)$,

$\operatorname{Re}(a, b, c) > 0$, $\operatorname{Re}(\nu) > -1$ (Watson 1944, p. 39).

By Cor. 1, we get

$$J_\nu(2\sqrt{ba^x})J_\nu(2\sqrt{ca^x}) \doteq \frac{\sqrt{\frac{2}{\pi}}(2\sqrt{bc})^{p'}\cos(\nu\pi)Q_{\nu-\frac{1}{2}}^{-p'-\frac{1}{2}}\left(\frac{b+c}{2\sqrt{bc}}\right)}{\Gamma(p')\sinh^{-p'-\frac{1}{2}}\left\{\cosh^{-1}\frac{(b+c)}{2\sqrt{bc}}\right\}\sin(\nu-p')\pi},$$

$\operatorname{Re}(\nu-p') > 0$

and by Cor. 2, we get

$$e^{-(b+c)a^x}I_\nu(2\sqrt{bca^x}) \doteq \frac{\sqrt{\frac{2}{\pi}}\,p'\cos(\nu\pi)(2\sqrt{bc})^{p'}Q_{\nu-\frac{1}{2}}^{-p'-\frac{1}{2}}\left(\frac{b+c}{2\sqrt{bc}}\right)}{\sinh^{-p'-\frac{1}{2}}\left(\cosh^{-1}\frac{b+c}{2\sqrt{bc}}\right)\sin(\nu-p')\pi},$$

$0 < \operatorname{Re}(p') < \operatorname{Re}(\nu)$, $b > c > 0$.

14. Let $f(x)U(x) = \dfrac{(x-\frac{1}{2})^{\nu-1}}{\Gamma(\nu)}\, {}_2F_1(\tfrac{1}{2}+m-k,\ \tfrac{1}{2}-m-k\,;\ \nu\,;\ \tfrac{1}{2}-x)U(x-\tfrac{1}{2})$

$$\doteqdot p^{1-k-\nu}W_{k,\,m}(p) = F(p),\quad \mathrm{Re}\,(p) > 0,\ \mathrm{Re}\,(\nu) > 0$$

(Watson 1944, p. 398, by shift rule).

$$\frac{(a^x-\frac{1}{2})^{\nu-1}}{\Gamma(\nu)}\, {}_2F_1(\tfrac{1}{2}+m-k,\ \tfrac{1}{2}-m-k\,;\ \nu\,;\ \tfrac{1}{2}-a^x)U(a^x-\tfrac{1}{2}) \doteqdot \frac{\Gamma(p'-k-\nu+m+\frac{3}{2})\Gamma(p'-k-\nu-m+\frac{3}{2})}{\Gamma(p')\Gamma(p'-2k-\nu+2)}$$

$$\times {}_2F_1(p'-k-\nu+\tfrac{3}{2}+m,\ p'-k-\nu-m+\tfrac{3}{2}\,;\ p'-2k-\nu+2\,;\ \tfrac{1}{2}),$$

$$\mathrm{Re}\,(\tfrac{3}{2}-k\pm m+p'-\nu) > 0,\ \mathrm{Re}\,(\nu) > 0$$

and by Cor. 2, we get

$$a^{-x(1-k-\nu)}W_{k,\,m}(a^{-x}) \doteqdot \frac{p'\Gamma(p'-k-\nu+m+\frac{3}{2})\Gamma(p'-k-\nu-m+\frac{3}{2})}{\Gamma(p'-2k-\nu+2)}$$

$$\times {}_2F_1(p'-k-\nu+m+\tfrac{3}{2},\ p'-k-\nu-m+\tfrac{3}{2}\,;\ p'-2k-\nu+2\,;\ \tfrac{1}{2}), \quad .. \quad \text{(A)}$$

$$\mathrm{Re}\,(\tfrac{3}{2}+p'-k-\nu\pm m) > 0,\ \mathrm{Re}\,(\nu) > 0.$$

15. Again since

$$P_n^{(\alpha,\,\beta)}(x) = \frac{(-1)^n\Gamma(n+\alpha,\,+1)}{n!\ \Gamma(\alpha+1)}\, {}_2F_1\left(-n,\ n+\alpha+\beta+1\,;\ \alpha+1\,;\ \frac{1-x}{2}\right),$$

where $P_n^{(\alpha,\,\beta)}(x)$ denotes Jacobi's polynomial of integral order n, we can write (A) in the form

$$\frac{(-1)^n n!}{\Gamma(\alpha+n+1)}(a^x-\tfrac{1}{2})^\alpha P_n^{(\alpha,\,\beta)}(2a^x)U(a^x-\tfrac{1}{2}) \doteqdot \frac{\Gamma(p'-\alpha-n)\Gamma(p'+n+\beta+1)}{\Gamma(p')\Gamma(p'+\beta+1)}$$

$$\times {}_2F_1(p'-\alpha-n,\ p'+n+\beta+1\,;\ p'+\beta+1\,;\ \tfrac{1}{2}),$$

$$\mathrm{Re}\,(\alpha) > -1,\ \mathrm{Re}\,(p') > \max\,(\alpha+n,\ -1-n-\beta).$$

Now in particular $\alpha = \beta = 0$, we get

$$P_n(2a^x)U(a^x-\tfrac{1}{2}) \doteqdot \frac{(-1)^n\Gamma(p'-n)\Gamma(p'+n+1)}{\Gamma(p')\Gamma(p'+1)}\, {}_2F_1(p'-n,\ p'+n+1\,;\ p'+1\,;\ \tfrac{1}{2}),$$

$$\mathrm{Re}\,(p') > n,$$

$P_n(x)$ denotes the Legendre polynomial of order n.

Next since

$$E(\alpha,\,\beta :: x) = \Gamma(\alpha)\Gamma(\beta)e^{\frac{1}{2}x}x^{\frac{1}{2}(\alpha+\beta-1)}W_{\frac{1}{2}(1-\alpha-\beta),\,\frac{1}{2}(\beta-\alpha)}(x),\quad \alpha,\beta < 0,$$

where $E(\alpha,\ \beta :: x)$ denotes MacRobert's E-function and since

$$k_{2n}(x) = \frac{1}{\Gamma(n+1)}W_{n,\,\frac{1}{2}}(2x),$$

where $k_{2n}(x)$ denoted Bateman's k-function, therefore in particular from (A), we get

$$e^{-\frac{1}{2}a^{-x}}a^{-(1-\nu)x}E(\alpha,\,\beta :: a^{-x}) \doteqdot \frac{p'\Gamma(\alpha)\Gamma(\beta)\Gamma(p'-\nu+\alpha\mp 1)\Gamma(p'-\nu+\beta+1)}{\Gamma(p'-\nu+\alpha+\beta+1)}$$

$$\times {}_2F_1(p'-\nu+\alpha+1,\ p'-\nu+\beta+1\,;\ p'-\nu+\alpha+1+\beta;\ \tfrac{1}{2}),$$

$$\alpha,\beta > 0,\ \mathrm{Re}\,(\nu) > 0,\ \mathrm{Re}\,(p') > \max\,(\nu-\alpha-1,\ \nu+\beta-1)$$

and

$$\frac{k_{2n}(\frac{1}{2}a^{-x})}{a^{(1-n-\nu)x}} \doteqdot \frac{p'\Gamma(p'-n-\nu+2)\Gamma(p'-n-\nu+1)}{\Gamma(p'-2n-\nu+2)\Gamma(n+1)} \, {}_2F_1(p'-n-\nu+2,\, p'-n-\nu+1\,;\; p'-2n-\nu+2\,;\, \tfrac{1}{2}), \ \mathrm{Re}\,(p') > n+\nu-1,\ \mathrm{Re}\,(\nu) > 0.$$

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