# SUMMATION OF SOME SERIES OF PRODUCTS OF $H$-FUNCTIONS

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(Communicated by R. S. Varma, F.N.I.)

(*Received November 22, 1966*)

In this paper we establish some integrals of $H$-functions with respect to their parameters and employ them to sum certain series of products of two $H$-functions.

§ 1. Fox (1961, p. 408) introduced the $H$-function in the form of Mellin-Barnes type integral as

$$\frac{1}{2\pi i}\int_T \frac{\prod_{j=1}^{m}\Gamma(b_j-\beta_j s)\prod_{j=1}^{n}\Gamma(1-a_j+\alpha_j s)}{\prod_{j=m+1}^{q}\Gamma(1-b_j+\beta_j s)\prod_{j=n+1}^{p}\Gamma(a_j-\alpha_j s)}x^s ds, \quad \ldots \quad \ldots \quad (1.1)$$

where $x$ is not equal to zero and empty product is interpreted as unity; $p$, $q$, $m$ and $n$ are integers satisfying $1 \leqslant m \leqslant q$, $0 \leqslant n \leqslant p$; $\alpha_j(j = 1, \ldots, p)$, $\beta_j(j = 1, \ldots, q)$ are positive numbers and $a_j(j = 1, \ldots, p)$, $b_j(j = 1, \ldots, q)$ are complex numbers such that no pole of $\Gamma(b_h-\beta_h s)(h = 1, \ldots, m)$ coincides with any pole of $\Gamma(1-a_i+\alpha_i s)(i = 1, \ldots, n)$, i.e.

$$\alpha_i(b_h+\nu) \neq (a_i-\eta-1)\beta_h(\nu,\ \eta = 0, 1, \ldots;\ h = 1, \ldots, m;\ i = 1, \ldots, n) \quad (1.2)$$

Further the contour $T$ runs from $\sigma-i\infty$ to $\sigma+i\infty$ such that the points

$$s = \frac{b_h+\nu}{\beta_h}(h = 1, \ldots, m;\ \nu = 0, 1, \ldots), \quad \ldots \quad \ldots \quad (1.3)$$

which are poles of $\Gamma(b_h-\beta_h s)$ $(h = 1, \ldots, m)$ lie to the right and the points

$$s = \frac{(a_i-\eta-1)}{\alpha_i}(i = 1, \ldots, n;\ \eta = 0, 1, \ldots), \quad \ldots \quad \ldots \quad (1.4)$$

which are the poles of $\Gamma(1-a_i+\alpha_i s)(i = 1, \ldots, n)$ lie to the left of $T$. Such a contour is possible on account of (1.2). These assumptions for the $H$-function will be adhered to throughout this paper.

Recently Gupta and Jain (*in press*) have denoted (1.1) symbolically as

$$H_{p,q}^{m,n}\left[x\left|\begin{matrix}(a_1, \alpha_1), \ldots, (a_p, \alpha_p)\\(b_1, \beta_1), \ldots, (b_q, \beta_q)\end{matrix}\right.\right] \quad \ldots \quad \ldots \quad \ldots \quad (1.5)$$

and in a more compact form by

$$H_{p,q}^{m,n}\left[x \,\middle|\, \begin{matrix}\{(a_p,\alpha_p)\}\\ \{(b_q,\beta_q)\}\end{matrix}\right] \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (1.6)$$

where $\{(f_r,\gamma_r)\}$ stands for the set of the parameters $(f_1,\gamma_1),\ldots,(f_r,\gamma_r)$.

According to Braaksma (1963, p. 278)

$$H_{p,q}^{m,n}\left[x \,\middle|\, \begin{matrix}\{(a_p,\alpha_p)\}\\ \{(b_q,\beta_q)\}\end{matrix}\right] = 0(|x|^{\alpha}) \text{ for small } x,$$

where

$$\sum_1^p(\alpha_j)-\sum_1^q(\beta_j)\leqslant 0 \text{ and } \alpha=\operatorname{Re}\left(\frac{b_h}{\beta_h}\right)(h=1,\ldots,m)$$

and

$$H_{p,q}^{m,n}\left[x \,\middle|\, \begin{matrix}\{(a_p,\alpha_p)\}\\ \{(b_q,\beta_q)\}\end{matrix}\right] = 0(|x|^{\beta}) \text{ for large } x,$$

where

$$\sum_1^p(\alpha_j)-\sum_1^q(\beta_j)<0,\ \sum_1^n(\alpha_j)-\sum_{n+1}^p(\alpha_j)+\sum_1^m(\beta_j)-\sum_{m+1}^q(\beta_j)\equiv\lambda>0$$

$$|\arg x|<\tfrac{1}{2}\lambda\pi \text{ and } \beta=\operatorname{Re}\left(\frac{a_i-1}{\alpha_i}\right)(i=1,\ldots,n).$$

The relation between $H$- and $G$-functions is

$$H_{p,q}^{m,n}\left[x \,\middle|\, \begin{matrix}\{(a_p,1)\}\\ \{(b_q,1)\}\end{matrix}\right] = G_{p,q}^{m,n}\left(x \,\middle|\, \begin{matrix}a_1,\ldots,a_p\\ b_1,\ldots,b_q\end{matrix}\right). \qquad \ldots \quad \ldots \quad (1.7)$$

The object of this paper is to evaluate some integrals of $H$-functions with respect to their parameters and use them to sum certain series of products of two $H$-functions.

§ 2. In this section we establish the generalizations of known results (Erdélyi 1954, p. 417) and (Erdélyi 1954, p. 419), which will be used in §3.

$$\int_0^1 x^{-\alpha}(1-x)^{\alpha-\beta-1}H_{p,q}^{m,n}\left[zx^{r\delta} \,\middle|\, \begin{matrix}\{(a_p,\alpha_p)\}\\ \{(b_q,\beta_q)\}\end{matrix}\right]dx$$

$$=\Gamma(\alpha-\beta)H_{p+1,q+1}^{m,n+1}\left[z \,\middle|\, \begin{matrix}(\alpha,r\delta),\{(a_p,\alpha_p)\}\\ \{(b_q,\beta_q)\},(\beta,r\delta)\end{matrix}\right] \qquad \ldots \quad \ldots \quad (2.1)$$

where

$$\sum_1^n(\alpha_j)-\sum_{n+1}^p(\alpha_j)+\sum_1^m(\beta_j)-\sum_{m+1}^q(\beta_j)\equiv\lambda>0,\ |\arg z|<\tfrac{1}{2}\lambda\pi,$$

$$\operatorname{Re}(\alpha-\beta)>0,\ \operatorname{Re}(r\delta b_j/\beta_j-\alpha)>-1(j=1,\ldots,n)$$

$$\int_0^\infty x^{-\rho}e^{-\beta x}H_{p,q}^{m,n}\left[zx^{r\delta} \,\middle|\, \begin{matrix}\{(a_p,\alpha_p)\}\\ \{(b_q,\beta_q)\}\end{matrix}\right]dx$$

$$=\beta^{\rho-1}H_{p+1,q}^{m,n+1}\left[\frac{z}{\beta^{r\delta}} \,\middle|\, \begin{matrix}(\rho,r\delta),\{(a_p,\alpha_p)\}\\ \{(b_q,\beta_q)\}\end{matrix}\right] \qquad \ldots \quad \ldots \quad (2.2)$$

provided $\sum_{1}^{n}(\alpha_j)-\sum_{n+1}^{p}(\alpha_j)+\sum_{1}^{m}(\beta_j)-\sum_{m+1}^{q}(\beta_j)\equiv\lambda>0, |\arg z|<\frac{1}{2}\lambda\pi,$

$$\text{Re}\,(r\delta b_j/\beta_j-\rho)>-1(j=1,\ldots,m).$$

PROOF: To establish the integral (2.1), expressing the $H$-function on the left, as Mellin-Barnes type of integral (1.1) and interchanging the order of integration, which is justifiable due to the absolute convergence of the integrals involved in the process, we get

$$\frac{1}{2\pi i}\int_T \frac{\prod_{j=1}^{m}\Gamma(b_j-\beta_j\xi)\prod_{j=1}^{n}\Gamma(1-a_j+\alpha_j\xi)}{\prod_{j=m+1}^{q}\Gamma(1-b_j+\beta_j\xi)\prod_{j=n+1}^{p}\Gamma(a_j-\alpha_j\xi)}z^{\xi}\int_0^1 x^{-\alpha+r\delta\xi}(1-x)^{\alpha-\beta-1}dx\,d\xi$$

evaluating the inner integral with the help of the results (Erdélyi 1953, p. 9 (1), 9 (5)), using definition (1.1) of the $H$-function, the value of the integral is obtained.

Proceeding on similar lines, the integral (2.2) can be established with the help of (Erdélyi 1953, p. 1 (1)).

§ 3. The first integral to be established is

$$\frac{1}{2\pi i}\int_L \Gamma(s-k-\lambda)\Gamma(\lambda+\mu-s+\tfrac{1}{2})\Gamma(\lambda-\mu-s+\tfrac{1}{2})z^s$$
$$\times H^{2,\,q+m}_{q+m+1,\,q+2}\left[z\,\middle|\,\begin{matrix}(a_1-\delta s,\delta),\ldots,(a_{q+m}-\delta s,\delta),(1+k-\lambda,1)\\(\tfrac{1}{2}+\mu-\lambda,1),(\tfrac{1}{2}-\mu-\lambda,1),(b_1-\delta s,\delta),\ldots,(b_q-\delta s,\delta)\end{matrix}\right]ds$$
$$=\pi^{-\frac{1}{2}}\Gamma(\tfrac{1}{2}-k-\mu)\Gamma(\tfrac{1}{2}-k+\mu)$$
$$\times H^{4,\,m+q}_{2+m+q,\,4+q}\left[\frac{z^2}{4}\,\middle|\,\begin{matrix}(a_1,2\delta),\ldots,(a_{q+m},2\delta),(k+1,1),(1-k,1)\\(\tfrac{1}{2},1),(1,1),(\mu+\tfrac{1}{2},1),(\tfrac{1}{2}-\mu,1),(b_1,2\delta),\ldots,(b_q,2\delta)\end{matrix}\right] \quad (3.1)$$

provided $(\delta m+1)>0$, $|\arg z|<(\delta m+1)\frac{\pi}{2}$, Re $(a_r-b_r)>0$ $(r=1,2,\ldots,q)$ Re $[\delta(\frac{1}{2}\pm\mu-\lambda)-a_r]>-1(r=1,2,\ldots,q+m)$, Re $(k+\lambda)<0$, Re $\lambda>|\text{Re}\,\mu|-\frac{1}{2}$, $L$ the path of integration being as in (Erdélyi 1954, p. 302 (29)) with loops, if necessary, to ensure that $(\lambda+\mu+\frac{1}{2})$ and $(\lambda-\mu+\frac{1}{2})$ are to the right of the contour.

PROOF: From (2.1) and (2.2), the left-hand side of (3.1) can be put in the form

$$\frac{1}{2\pi i}\int_L \Gamma(s-k-\lambda)\Gamma(\lambda+\mu-s+\tfrac{1}{2})\Gamma(\lambda-\mu-s+\tfrac{1}{2})z^s\left[\prod_{r=1}^{q}\Gamma(a_r-b_r)\right]^{-1}$$
$$\times\prod_{r=1}^{q}\int_0^1 x_r^{-a_r+\delta s}(1-x_r)^{a_r-b_r-1}dx_r\prod_{r=q+1}^{q+m}\int_0^\infty x_r^{-a_r+\delta s}e^{-x_r}dx_r$$
$$\times H^{2,0}_{1,2}\left[z(x_1\ldots x_{q+m})^\delta\,\middle|\,\begin{matrix}(1+k-\lambda,1)\\(\tfrac{1}{2}+\mu-\lambda,1),(\tfrac{1}{2}-\mu-\lambda,1)\end{matrix}\right]ds$$

now changing the order of integration (putting the first integral in the last), which is justifiable due to the absolute convergence of the integrals involved in the process, we get

$$\left[\prod_{r=1}^{q}\Gamma(a_r-b_r)\right]^{-1}\prod_{r=1}^{q}\int_0^1 x_r^{-a_r}(1-x_r)^{a_r-b_r-1}\,dx_r$$

$$\times\prod_{r=q+1}^{q+m}\int_0^\infty x_r^{-a_r}e^{-x_r}H_{1,2}^{2,0}\left[z(x_1\ldots x_{q+m})^\delta\,\middle|\,\begin{matrix}(1+k-\lambda,1)\\(\tfrac12+\mu-\lambda,1),(\tfrac12-\mu-\lambda,1)\end{matrix}\right]dx_r$$

$$\times\frac{1}{2\pi i}\int_L\Gamma(s-k-\lambda)\Gamma(\lambda+\mu-s+\tfrac12)\Gamma(\lambda-\mu-s+\tfrac12)(x_1\ldots x_{q+m})^{\delta s}z^s\,ds$$

substituting for the last integral from (Erdélyi 1954, p. 302 (29)), using (1.7) and (Erdélyi 1954, p. 435 (3)), the expression becomes

$$\Gamma(\tfrac12-k-\mu)\Gamma(\tfrac12-k+\mu)\left[\prod_{r=1}^{q}\Gamma(a_r-b_r)\right]^{-1}\prod_{r=1}^{q}\int_0^1 x_r^{-a_r}(1-x_r)^{a_r-b_r-1}dx_r$$

$$\times\prod_{r=q+1}^{q+m}\int_0^\infty x_r^{-a_r}e^{-x_r}W_{-k,\mu}\{z(x_1\ldots x_{q+m})^\delta\}W_{k,\mu}\{z(x_1\ldots x_{q+m})^\delta\}dx_r$$

now substituting from (Erdélyi 1954, p. 443 (5)) and with the help of (1.7) it reduces to

$$\pi^{-\frac12}\Gamma(\tfrac12-k-\mu)\Gamma(\tfrac12-k+\mu)\left[\prod_{r=1}^{q}\Gamma(a_r-b_r)\right]^{-1}\prod_{r=1}^{q}\int_0^1 x_r^{-a_r}(1-x_r)^{a_r-b_r-1}dx_r$$

$$\times\prod_{r=q+1}^{q+m}\int_0^\infty x_r^{-a_r}e^{-x_r}H_{2,4}^{4,0}\left[\frac{z^2}{4}(x_1\ldots x_{q+m})^{2\delta}\,\middle|\,\begin{matrix}(1+k,1),(1-k,1)\\(\tfrac12,1),(1,1),(\tfrac12+\mu,1),(\tfrac12-\mu,1)\end{matrix}\right]dx_r$$

on integrating with the help of (2.2) and (2.1) the formula (3.1) is obtained.

The second integral to be evaluated is

$$\frac{1}{2\pi i}\int_L\Gamma(s-k-\lambda)\Gamma(\lambda+\mu-s+\tfrac12)\Gamma(\lambda-\mu-s+\tfrac12)z^s$$

$$\times H_{q+m+1,q+2}^{1,q+m+1}\left[z\,\middle|\,\begin{matrix}(1+k-\lambda,1),(a_1-\delta s,\delta),\ldots,(a_{q+m}-\delta s,\delta)\\(\tfrac12+\mu-\lambda,1),(b_1-\delta s,\delta),\ldots,(b_q-\delta s,\delta),(\tfrac12-\mu-\lambda,1)\end{matrix}\right]ds$$

$$=\pi^{-\frac12}\Gamma(\tfrac12-k-\mu)\Gamma(\tfrac12-k+\mu)$$

$$\times H_{2+q+m,4+q}^{3,q+m+1}\left[\tfrac14 z^2\,\middle|\,\begin{matrix}(1+k,1),(a_1,2\delta),\ldots,(a_{q+m},2\delta),(1-k,1)\\(\tfrac12,1),(1,1),(\tfrac12+\mu,1),(b_1,2\delta),\ldots,(b_q,2\delta),(\tfrac12-\mu,1)\end{matrix}\right]\quad(3.2)$$

where $(\delta m+1)>0$, $|\arg z|<(\delta m+1)\frac{\pi}{2}$, Re $(a_r-b_r)>0 (r=1,\ldots,q)$ Re $[\delta(\frac12-\lambda+\mu)-a_r]>-1$ $(r=1,2,\ldots,q+m)$, Re $(k+\lambda)<0$, Re $\lambda>|\text{Re}\,\mu|-\frac12$; $L$ the path of integration being as in (Erdélyi 1954, p. 302 (29)) with loops, if necessary, to ensure that $\lambda+\mu+\frac12$ and $\lambda-\mu+\frac12$ are to the right of the contour.

The integral can be established by applying the same procedure as in (3.1) and using the results (Erdélyi 1954, p. 302 (29), p. 442 (7), p. 443 (3)) and (1.7).

The third integral to be established is

$$\frac{1}{2\pi i}\int_L \frac{\Gamma(\lambda+\mu-s+\frac{1}{2})\Gamma(\lambda-\mu-s+\frac{1}{2})}{\Gamma(\lambda-k-s+1)} z^s$$

$$\times H^{2,\,q+m+1}_{q+m+1,\,q+2}\left[z\,\middle|\,\begin{matrix}(1-k-\lambda,\,1),\,(a_1-\delta s,\,\delta),\,\ldots,\,(a_{q+m}-\delta s,\,\delta)\\(\mu-\lambda+\frac{1}{2},\,1),\,(\frac{1}{2}-\mu-\lambda,\,1),\,(b_1-\delta s,\,\delta),\,\ldots,\,(b_q-\delta s,\,\delta)\end{matrix}\right]ds$$

$$=\pi^{-\frac{1}{2}}\Gamma(\tfrac{1}{2}+k+\mu)\Gamma(\tfrac{1}{2}+k-\mu)$$

$$\times H^{4,\,q+m}_{q+m+2,\,4+q}\left[\tfrac{1}{4}z^2\,\middle|\,\begin{matrix}(a_1,\,2\delta),\,\ldots,\,(a_{q+m},\,2\delta),\,(k+1,\,1),\,(1-k,\,1)\\(\frac{1}{2},\,1),\,(1,\,1),\,(\mu+\frac{1}{2},\,1),\,(\frac{1}{2}-\mu,\,1),\,(b_1,\,2\delta),\,\ldots,\,(b_q,\,2\delta)\end{matrix}\right] \quad (3.3)$$

provided $(\delta m+3)>0$, $|\arg z|<(\delta m+3)\frac{\pi}{2}$, Re $(a_r-b_r)>0$ $(r=1,\ldots,q)$ Re $[\delta(\mu-\lambda-\frac{1}{2})-a_r]>-1$, Re $[\delta(\frac{1}{2}-\lambda-\mu)-a_r]>-1$ $(r=1,2,\ldots,q+m)$, Re $(\lambda)>|\text{Re }\mu|-\frac{1}{2}$, $L$ the path of integration being as in (Erdélyi 1954, p. 302 (30)) with loops, if necessary, to ensure that $(\lambda+\mu+\frac{1}{2})$, $(\lambda-\mu+\frac{1}{2})$ are to the right of the contour.

The integral can be established by applying the same procedure as in (3.1) and using the results (Erdélyi 1954, p. 302 (30), p. 435 (5), p. 443 (5)) and (1.7).

The fourth integral to be evaluated is

$$\frac{1}{2\pi i}\int_L \frac{\Gamma(s-k-\lambda)\Gamma(\lambda+\mu-s+\frac{1}{2})}{\Gamma(\mu-\lambda+s+\frac{1}{2})} z^s$$

$$\times H^{2,\,q+m+1}_{q+m+1,\,q+2}\left[z\,\middle|\,\begin{matrix}(1+k-\lambda,\,1),\,(a_1-\delta s,\,\delta),\,\ldots,\,(a_{q+m}-\delta s,\,\delta)\\(\frac{1}{2}+\mu-\lambda,\,1),\,(\frac{1}{2}-\mu-\lambda,\,1),\,(b_1-\delta s,\,\delta),\,\ldots,\,(b_q-\delta s,\,\delta)\end{matrix}\right]ds$$

$$=\pi^{-\frac{1}{2}}\Gamma(\tfrac{1}{2}+\mu-k)\Gamma(\tfrac{1}{2}-\mu-k)$$

$$\times H^{3,\,1+q+m}_{q+m+2,\,4+q}\left[\tfrac{1}{4}z^2\,\middle|\,\begin{matrix}(1+k,\,1),\,(a_1,\,2\delta),\,\ldots,\,(a_{q+m},\,2\delta),\,(1-k,\,1)\\(\frac{1}{2},\,1),\,(1,\,1),\,(\frac{1}{2}+\mu,\,1),\,(b_1,\,2\delta),\,\ldots,\,(b_q,\,2\delta),\,(\frac{1}{2}-\mu,\,1)\end{matrix}\right] \quad .. \quad (3.4)$$

where $(\delta m+3)>0$, $|\arg z|<(\delta m+3)\frac{\pi}{2}$, Re $(a_r-b_r)>0$ $(r=1,\ldots,q)$ Re $[\delta(\mu-\lambda+\frac{1}{2})-a_r]>-1$, Re $[\delta(\frac{1}{2}-\lambda-\mu)-a_r]>-1$ $(r=1,2,\ldots,q+m)$, Re $(k+\lambda)<0$, Re $(\lambda+\mu)>-\frac{1}{2}$, $L$ the path of integration being as in (Erdélyi 1954, p. 302 (31)) with loops, if necessary, to ensure that $\lambda+\mu+\frac{1}{2}$ is to the right of the contour.

The integral can be established by applying the same procedure as in (3.1) and using the results (Erdélyi 1954, p. 302 (31), p. 435 (5), p. 443 (3)) and (1·7).

## § 4. Summation of Series

The first summation is

$$\sum_{r=0}^{\infty} \frac{1}{r!\,(1+b_1-a_2)_r} H_{2,2}^{2,2}\left[z \,\middle|\, \begin{matrix}(a_1,\sigma),(a_2,\sigma)\\(b_1+r,\sigma),(b_2,\sigma)\end{matrix}\right]$$

$$\times H_{2,2}^{2,1}\left[z \,\middle|\, \begin{matrix}(1+a_2-b_1-b_2,\sigma),(1+a_1-b_1-b_2,\sigma)\\(1-b_2+r,\sigma),(1-b_1,\sigma)\end{matrix}\right]$$

$$= \pi^{-\frac{1}{2}}\sigma^{-2}\Gamma(1+b_1-a_2)\Gamma(1-a_1+b_1)\Gamma(1-a_1+b_2)$$

$$\times H_{3,4}^{4,1}\left[\tfrac{1}{4}z^{\frac{2}{\sigma}} \,\middle|\, \begin{matrix}(1+a_2-b_2,2),\left(\frac{1+2a_1-b_1-b_2}{2},1\right),\left(\frac{3-2a_1+b_1+b_2}{2},1\right)\\(\frac{1}{2},1),(1,1),\left(\frac{1+b_1-b_2}{2},1\right),\left(\frac{1+b_2-b_1}{2},1\right)\end{matrix}\right] \tag{4.1}$$

where $|\arg z| < \sigma\pi$, Re $(1+b_1-a_2) > 0$, Re $(1-a_1+b_1) > 0$, Re $(1-a_1+b_2) > 0$.

Proof: To prove (4.1), substitute on the left from (1.1), so getting

$$\sum_{r=0}^{\infty} \frac{1}{r!\,(1+b_1-a_2)_r}\cdot\frac{1}{2\pi i}\int_L \Gamma(b_1+r-\sigma s)\Gamma(b_2-\sigma s)\Gamma(1-a_1+\sigma s)\Gamma(1-a_2+\sigma s)z^s\,ds$$

$$\times\frac{1}{2\pi i}\int_L \frac{\Gamma(1-b_2+r-\sigma\omega)\Gamma(1-b_1-\sigma\omega)\Gamma(b_1+b_2-a_2+\sigma\omega)}{\Gamma(1+a_1-b_1-b_2-\sigma\omega)}\,z^{\omega}d\omega,$$

replacing $s$ by $\frac{s}{\sigma}$ and $w$ by $\frac{\omega}{\sigma}$ and changing the order of integration and summation, in view of (Bromwich 1926, p. 500) the expression becomes

$$\frac{1}{\sigma^2}\cdot\frac{1}{2\pi i}\int_L \Gamma(b_1-s)\Gamma(b_2-s)\Gamma(1-a_1+s)\Gamma(1-a_2+s)z^{\frac{s}{\sigma}}\,ds$$

$$\times\frac{1}{2\pi i}\int_L \frac{\Gamma(1-b_2-\omega)\Gamma(1-b_1-\omega)\Gamma(b_1+b_2-a_2+\omega)}{\Gamma(1+a_1-b_1-b_2-\omega)}\,{}_2F_1\left(\begin{matrix}b_1-s,\,1-b_2-\omega\,;\,1\\1+b_1-a_2\end{matrix}\right)z^{\frac{\omega}{\sigma}}\,d\omega$$

now applying Gauss' theorem and substituting $a_1 = 1+k+\lambda$, $a_2 = \frac{3}{2}-\gamma+\lambda+\mu$, $b_1 = \lambda+\mu+\frac{1}{2}$ and $b_2 = \lambda-\mu+\frac{1}{2}$, it reduces to

$$\frac{\sigma^{-2}\Gamma(\gamma)}{2\pi i}\int_L \Gamma(s-k-\lambda)\Gamma(\lambda+\mu+\tfrac{1}{2}-s)\Gamma(\lambda-\mu+\tfrac{1}{2}-s)$$

$$\times H_{2,2}^{2,1}\left[z^{\frac{1}{\sigma}} \,\middle|\, \begin{matrix}(2-\gamma+2\mu-s,1),(1+k-\lambda,1)\\(\frac{1}{2}+\mu-\lambda,1),(\frac{1}{2}-\mu-\lambda,1)\end{matrix}\right]z^{\frac{s}{\sigma}}\,ds$$

using (3.1) with $z$ replaced by $z^{\frac{1}{\sigma}}$, $\delta = 1$, $q = 0$, $m = 1$, $a_1 = 2-\gamma+2\mu$ we get the expression equal to

$$\Gamma(\gamma)\sigma^{-2}\pi^{-\frac{1}{2}}\Gamma(\tfrac{1}{2}-k-\mu)\Gamma(\tfrac{1}{2}-k+\mu)\,H_{3,4}^{4,1}\left[\tfrac{1}{4}z^{\frac{2}{\sigma}} \,\middle|\, \begin{matrix}(2-\gamma+2\mu,2),(1+k,1),(1-k,1)\\(\frac{1}{2},1),(1,1),(\frac{1}{2}+\mu,1),(\frac{1}{2}-\mu,1)\end{matrix}\right]$$

now substituting the values of $k$, $\gamma$, $\mu$ in terms of $a_1$, $a_2$, $b_1$ and $b_2$ we get the result.

$$\sum_{r=0}^{\infty} \frac{1}{r!\,(1+b_1-a_2)_r} H_{2,2}^{2,2}\left[z \,\middle|\, \begin{matrix}(a_1,\sigma),(a_2,\sigma)\\(b_1+r,\sigma),(b_2,\sigma)\end{matrix}\right]$$
$$\times H_{2,2}^{1,2}\left[z \,\middle|\, \begin{matrix}(1+a_1-b_1-b_2,\sigma),(1+a_2-b_1-b_2,\sigma)\\(1-b_2+r,\sigma),(b_1,\sigma)\end{matrix}\right]$$
$$= \pi^{-\frac{1}{2}}\sigma^{-2}\Gamma(1+b_1-a_2)\,\Gamma(1-a_1+b_1)\,\Gamma(1-a_1+b_2)$$
$$\times H_{3,4}^{3,2}\left[\tfrac{1}{4}z^{\frac{2}{\sigma}} \,\middle|\, \begin{matrix}(1+a_2-b_2,2),\left(\frac{1+2a_1-b_1-b_2}{2},1\right),\left(\frac{3-2a_1+b_1+b_2}{2},1\right)\\(\frac{1}{2},1),(1,1),\left(\frac{1+b_1-b_2}{2},1\right),\left(\frac{1-b_1+b_2}{2},1\right)\end{matrix}\right] \tag{4·2}$$

provided $|\arg z| < \sigma\pi$, Re $(1+b_1-a_2) > 0$, Re $(1-a_1+b_1) > 0$, and Re $(1-a_1+b_2) > 0$.

The series can be established by applying the same procedure as in (4.1) and using (3.2).

$$\sum_{r=0}^{\infty} \frac{1}{r!\,(1-a_1+b_1)_r} H_{2,2}^{2,1}\left[z \,\middle|\, \begin{matrix}(1-2b_1+a_1,\sigma),(a_2-b_1-b_2,\sigma)\\(1-b_1+r,\sigma),(-b_2,\sigma)\end{matrix}\right]$$
$$\times H_{2,2}^{2,2}\left[z \,\middle|\, \begin{matrix}(a_1,\sigma),(a_2,\sigma)\\(b_1+r,\sigma),(b_2,\sigma)\end{matrix}\right]$$
$$= \pi^{-\frac{1}{2}}\sigma^{-2}\Gamma(1-a_1+b_1)\,\Gamma(2-a_2+b_2)\,\Gamma(1-a_2+b_1)$$
$$\times H_{3,4}^{4,1}\left[\tfrac{1}{4}z^{\frac{2}{\sigma}} \,\middle|\, \begin{matrix}(1+a_1-b_1,2),\left(\frac{4-2a_2+b_1+b_2}{2},1\right),\left(\frac{2a_2-b_1-b_2}{2},1\right)\\(\frac{1}{2},1),(1,1),\left(\frac{2-b_1+b_2}{2},1\right),\left(\frac{b_1-b_2}{2},1\right)\end{matrix}\right] \quad .. \tag{4·3}$$

where $|\arg z| < \sigma\pi$, Re $(1-a_1+b_1) > 0$, Re $(2-a_2+b_2) > 0$ and Re $(1-a_2+b_1) > 0$.

This series can be established by applying the same procedure as in (4.1), substituting $a_1 = \frac{3}{2}-\mu-\lambda-\gamma$, $a_2 = 1-k-\lambda$, $b_1 = \frac{1}{2}-\mu-\lambda$, $b_2 = -\frac{1}{2}+\mu-\lambda$ and using (3.3).

$$\sum_{r=0}^{\infty} \frac{1}{r!\,(1+b_1-a_1)_r} H_{2,2}^{1,2}\left[z \,\middle|\, \begin{matrix}(1+a_1-2b_1,\sigma),(1+a_2-b_1-b_2,\sigma)\\(1-b_1+r,\sigma),(1-b_2,\sigma)\end{matrix}\right]$$
$$\times H_{2,2}^{2,2}\left[z \,\middle|\, \begin{matrix}(a_1,\sigma),(a_2,\sigma)\\(b_1+r,\sigma),(b_2,\sigma)\end{matrix}\right]$$
$$= \pi^{-\frac{1}{2}}\sigma^{-2}\Gamma(1+b_1-a_1)\Gamma(1-a_2+b_1)\Gamma(1-a_2+b_2)$$
$$\times H_{3,4}^{3,2}\left[\tfrac{1}{4}z^{\frac{2}{\sigma}} \,\middle|\, \begin{matrix}\left(\frac{1+2a_2-b_1-b_2}{2},1\right),(1+a_1-b_1,2),\left(\frac{3-2a_2+b_1+b_2}{2},1\right)\\(\frac{1}{2},1),(1,1),\left(\frac{1-b_1+b_2}{2},1\right),\left(\frac{1+b_1-b_2}{2},1\right)\end{matrix}\right] \tag{4.4}$$

provided $|\arg z| < \sigma\pi$, Re $(1+b_1-a_1) > 0$, Re $(1-a_2+b_1) > 0$, Re $(1-a_2+b_2) > 0$.

The series can be established by applying the same procedure as in (4.1), substituting $a_1 = \frac{3}{2}-\mu-\lambda-\gamma$, $a_2 = 1+k-\lambda$, $b_1 = \frac{1}{2}-\mu-\lambda$, $b_2 = \frac{1}{2}+\mu-\lambda$ and using (3.4).

## Acknowledgement

I wish to express my sincere thanks to Dr. R. K. Saxena of the G.S. Technological Institute, Indore, for his kind help and guidance during the preparation of this paper.

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