ON SUMMATION OF MEIJER'S G-FUNCTION OF TWO VARIABLES *

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In this paper a series summation of Meijer's G-function of two variables is summed, expressing the G-function of two variables as Mellin-Barnes type of double integral and then interchanging the order of integration and summation.

1. INTRODUCTION

Agarwal (1965) defined the G-function of two variables which not only includes Meijer's G-function of a single variable as a particular case but also most of special functions of two variables, e.g. Appell's functions, the Whittaker functions of two variables. Bhise (1963) summed a number of finite and infinite series of Meijer's G-function of single variable, expressing the G-function as Mellin-Barnes type integral, though MacRobert, Ragab and others have summed many such series for E-function and its product. Recently I summed a number of infinite series involving Meijer's G-function of two variables, expressing the G-function as Mellin-Barnes type of double integral and then interchanging the order of summation and integration.

The object of this note is to establish a finite summation of Meijer's G-function of two variables and recurrence relation based upon it.

The following notation will be used throughout this note. The symbol (ϵ_p) denotes the sequence of elements $\epsilon_1, \epsilon_2, \ldots, \epsilon_p$ and $(a)_n = \dfrac{\Gamma(a+n)}{\Gamma(a)}$, i.e. $(a)_0 = 1$, $(a)_n = a(a+1)\ldots(a+n-1)$, $n = 1, 2, 3, \ldots$

Meijer's G-function of two variables due to Agarwal (1965) is expressed by the relation

$$G^{n_1, \nu_1, \nu_2, m_1, m_2}_{p, [t:t'], s, [q:q']}\left[\begin{array}{c|l} x & (\epsilon_p) \\ & (\gamma_t);\ (\gamma'_{t'}) \\ & (\delta_s) \\ y & (\beta_q);\ (\beta'_{q'}) \end{array}\right] = \frac{1}{(2\pi i)^2}\int_{-i\infty}^{+i\infty}\int_{-i\infty}^{+i\infty} \Phi(\xi+\eta)\psi(\xi, \eta)x^{\xi}y^{\eta}\, d\xi\, d\eta, \quad \ldots (1.1)$$

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where

$$\Phi(\xi+\eta) = \prod_{j=1}^{n_1} (1-\epsilon_j+\xi+\eta) \Big/ \prod_{j=n_1+1}^{p} \Gamma(\epsilon_j-\xi-\eta) \prod_{j=1}^{s} \Gamma(\delta_j+\xi+\eta),$$

$$\Psi(\xi, \eta) = \frac{\prod_{j=1}^{m_1} \Gamma(\beta_j-\xi) \prod_{j=1}^{\nu_1} \Gamma(\gamma_j+\xi) \prod_{j=1}^{m_2} \Gamma(\beta_j'-\eta) \prod_{j=1}^{\nu_2} \Gamma(\gamma_j'+\eta)}{\prod_{j=m_1+1}^{q} \Gamma(1-\beta_j+\xi) \prod_{j=\nu_1+1}^{t} \Gamma(1-\gamma_j-\xi) \prod_{j=m_2+1}^{q'} \Gamma(1-\beta_j'+\eta) \prod_{j=\nu_2+1}^{t'} \Gamma(1-\gamma_j'-\eta)}$$

and

$$0 \leqslant m_1 \leqslant q,\ 0 \leqslant m_2 \leqslant q',\ 0 \leqslant \nu_1 \leqslant t,\ 0 \leqslant \nu_2 \leqslant t',\ 0 \leqslant n_1 \leqslant p.$$

The sequence of parameters (β_{m_1}), (β'_{m_2}), (γ_{ν_1}), (γ'_{ν_2}) and (ϵ_{n_1}) is such that none of the poles of the integrand coincides. The paths of integration are indented, if necessary, in such a manner that all the poles of $\Gamma(\beta_j-\xi)$, $j = 1, 2, \ldots, m_1$, and $\Gamma(\beta_k'-\eta), k = 1, 2, \ldots, m_2$, lie to the right and those of $\Gamma(\gamma_j+\xi)$, $j = 1, 2, \ldots, \nu_1$, $\Gamma(\gamma_k'+\eta)$, $k = 1, 2, \ldots, \nu_2$, and $\Gamma(1-\epsilon_j+\xi+)\eta$, $j = 1, 2, \ldots, n_1$, lie to the left of the imaginary axis.

The integral (1.1) converges if

$$p+q+s+t < 2(m_1+\nu_1+n_1),$$
$$p+q'+s+t' < 2(m_2+\nu_2+n_1),$$

and

$$|\arg x| < \pi[m_1+\nu_1+n_1-\tfrac{1}{2}(p+q+s+t)],$$
$$|\arg y| < \pi[m_2+\nu_2+n_1-\tfrac{1}{2}(p+q'+s+t')].$$

2. Finite Summation Formula

We establish the formula

$$\sum_{r=0}^{n} (-1)^r n_{c_r} \frac{(1-\alpha-\beta_1-\gamma_t)_r}{(1-\alpha-n)_r} G^{n_1, \nu_1, \nu_2, m_1, m_2}_{p, [t:t'], s, [q:q']} \left[\begin{array}{c|l} x & (\epsilon_p) \\ & \gamma_1, \ldots, \gamma_{t-1}, \gamma_t-r;\ (\gamma'_{t'}) \\ & (\delta_s) \\ y & \beta_1+r, \beta_2, \ldots, \beta_q;\ (\beta'_{q'}) \end{array} \right]$$

$$= \frac{(1-\gamma_t-\beta_1)_n}{(\alpha)_n} G^{n_1, \nu_1, \nu_2, m_1+1, m_2}_{p, [t+1:t'], s, [q+1:q']} \left[\begin{array}{c|l} x & (\epsilon_p) \\ & \gamma_1, \ldots, \gamma_{t-1}, \gamma_t-n, 1-\beta_1-\alpha;\ \gamma'_1, \ldots, \gamma'_{t'} \\ & (\delta_s) \\ y & \beta_1+\alpha+n, (\beta_q);\ (\beta'_{q'}) \end{array} \right], \quad \ldots (2.1)$$

where

$R(\alpha) > 0$, $R(1-\alpha-\beta_1-\gamma_t) > 0$, $p+q+s+t < 2(m_1+\nu_1+n_1)$,
$p+q'+s+t' < 2(m_2+\nu_2+n_1)$,
$|\arg x| < \pi[m_1+\nu_1+n_1-\frac{1}{2}(p+q+s+t)]$ and
$|\arg y| < \pi[m_2+\nu_2+n_1-\frac{1}{2}(p+q'+s+t')]$, n, r being positive integers.

PROOF: Expressing the G-function on the left-hand side of (2.1) by Mellin-Barnes type of double integral (1.1), and changing the order of summation and integration as permissible by absolute convergence for $R(\alpha) > 0$, $R(1-\alpha-\beta_1-\gamma_t) > 0$ and the stated conditions in the formula, the series becomes

$$\frac{1}{(2\pi i)^2}\int_{-i\infty}^{+i\infty}\int_{-i\infty}^{+i\infty}\Phi(\xi+\eta)\psi(\xi,\eta)\left[\sum_{r=0}^{n}\frac{(-n)_r(\beta_1-\xi)_r(1-\alpha-\beta_1-\gamma_t)_r}{r!\,(1-\gamma_t-\xi)_r(1-\alpha-n)_r}\right]x^\xi y^\eta\,d\xi\,d\eta$$

$$=\frac{1}{(2\pi i)^2}\int_{-i\infty}^{+i\infty}\int_{-i\infty}^{+i\infty}\Phi(\xi+\eta)\psi(\xi,\eta)\,{}_3F_2\left(\begin{matrix}-n,\,\beta_1-\xi,\,1-\alpha-\beta_1-\gamma_t;\,1\\ 1-\gamma_t-\xi,\,1-\alpha-n\end{matrix}\right)x^\xi y^\eta\,d\xi\,d\eta.$$

Now we simplify the above expression with the help of the result (Saalschütz's theorem):

$${}_3F_2\left(\begin{matrix}-n,\,a,\,b;\,1\\ c,\,1-c+a+b-n\end{matrix}\right)=\frac{(c-a)_n(c-b)_n}{(c)_n(c-a-b)_n}, \qquad \ldots \quad \ldots \quad (2.2)$$

$n = 0, 1, 2, \ldots$, and a, b, c are independent of n. Thus the left-hand side of (2.1) becomes

$$=\frac{(1-\gamma_t-\beta_1)_n}{(\alpha)_n}\frac{1}{(2\pi i)^2}\int_{-i\infty}^{+i\infty}\int_{-i\infty}^{+i\infty}\Phi(\xi+\eta)\psi(\xi,\eta)\frac{\Gamma(\beta_1+\alpha+n-\xi)}{\Gamma(\beta_1+\alpha-\xi)}$$

$$\times\frac{\Gamma(1-\gamma_t-\xi)}{\Gamma(1-\gamma_t+n-\xi)}x^\xi y^\eta\,d\xi\,d\eta,$$

and this completes the proof of (2.1).

Setting $n = 1$ and multiplying through by α, (2.1) yields

$$\alpha G^{n_1,\nu_1,\nu_2,m_1,m_2}_{p,[t:t'],s,[q:q']}\left[\begin{matrix}x\\ \\ \\ y\end{matrix}\left|\begin{matrix}(\epsilon_p)\\ (\gamma_t);\ (\gamma'_{t'})\\ (\delta_s)\\ (\beta_q);\ (\beta'_{q'})\end{matrix}\right.\right]$$

$$+(1-\alpha-\beta_1-\gamma_t)G^{n_1,\nu_1,\nu_2,m_1,m_2}_{p,[t:t'],s,[q:q']}\left[\begin{matrix}x\\ \\ \\ y\end{matrix}\left|\begin{matrix}(\epsilon_p)\\ \gamma_1,\ldots,\gamma_{t-1},\gamma_t-1;\ (\gamma'_{t'})\\ (\delta_s)\\ \beta_1+1,\beta_2,\ldots,\beta_q;\ (\beta'_{q'})\end{matrix}\right.\right]$$

$$=(1-\gamma_t-\beta_1)G^{n_1,\nu_1,\nu_2,m_1+1,m_2}_{p,[t+1:t'],s,[q+1:q']}\left[\begin{matrix}x\\ \\ \\ y\end{matrix}\left|\begin{matrix}(\epsilon_p)\\ \gamma_1,\gamma_2,\ldots,\gamma_{t-1},\gamma_t-1,1-\beta_1-\alpha;\ \gamma'_1,\gamma'_2,\ldots,\gamma'_{t'})\\ (\delta_s)\\ \beta_1+\alpha+1,(\beta_q);\ (\beta'_{q'})\end{matrix}\right.\right]$$

$$\ldots \quad (2.3)$$

$R(1-\alpha-\beta_1-\gamma_t) > 0$, $p+q+s+t < 2(m_1+\nu_1+n_1)$, $p+q'+s+t' < 2(m_2+\nu_2+n_1)$,
$|\arg x| < \pi[m_1+\nu_1+n_1-\frac{1}{2}(p+q+s+t)]$ and
$|\arg y| < \pi[m_2+\nu_2+n_1-\frac{1}{2}(p+q'+s+t')]$.

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