TRANSVERSE VIBRATIONS OF A SOLID RECTANGULAR BEAM

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(Communicated by R. S. Varma, F.N.I.)

(*Received November 25, 1966*)

Transverse vibrations of an isotropic, solid rectangular beam, in which the secondary effects of rotatory inertia and transverse-shear deformation are retained, are considered. The present theory is a modification over Timoshenko's theory of bending of beams by including shear terms of second order in thickness. The equations of motion are derived by Hamilton's principle and the solutions of these equations are compared with those of Timoshenko's equation and the exact equation of Lamb.

1. INTRODUCTION

The problem of vibrations of a beam is of intrinsic interest because the beam represents the simplest of all engineering structures. Although the structure itself is simple, the propagation of stress waves is complicated by the presence of the free boundaries. Reflection from the boundaries seriously modifies the propagation characteristics. The complexity of the problem calls for simplifying assumptions in many cases. Therefore, many approximate theories exist, differing in a greater or lesser degree from one another. In other cases exact solutions are available, but only for infinite trains of sinusoidal waves; and not even these solutions are available for beams of square or rectangular cross-section.

The elementary theory of bending of beams, due principally to Bernoulli and Euler, has been of great interest in the past. But the theory is too limited and has many shortcomings due to its over-simplified assumptions. Lamb (1917) was apparently the first to point out that the elementary beam theory is inadequate for impact type loads, since it leads to the physically impossible conclusion that disturbances are propagated instantaneously throughout the beam. Therefore, the theory requires some modifications. These modifications to the elementary theory were given by Rayleigh (1894) and Timoshenko (1921); the correction offered by Rayleigh was to account for rotation of beam cross-sections and that offered by Timoshenko was to account for the transverse deformation due to the shear force. The Timoshenko equation has received a great deal of attention in recent years. Solutions for different kinds of impacts have been found by Uflyand (1948), Dengler and Goland

(1951), Miklowitz (1953), Zajac (1954), Boley and Chao (1955), and Plass and Steyer (1956).

In the present paper the Timoshenko theory of beam bending is modified to include shear terms of second order in thickness. Making use of the standard strain-displacement relations and the stress-strain relations for plane strain case, the stress-displacement relations are developed. The displacement equations of motion are derived by application of Hamilton's principle. By the substitution of harmonic solutions, the characteristic equation for the phase velocity in terms of wavelength is obtained. The numerical results obtained are compared with those of Timoshenko's equation and the exact equation of Lamb. In fact, the case of plane strain has been considered to enable us to make direct comparison with Lamb's theory. The present theory provides a better agreement with the exact theory than the Timoshenko theory.

2. Deformations

Consider a solid rectangular beam of length l, breadth $2b$ and depth $2h$. Taking the middle point of one of the ends as origin, the beam is referred to the coordinate system x, y, z. These coordinates are being taken in the directions of length, breadth and depth (measured positive downward) respectively. The middle plane is taken as $z = 0$.

The components of the displacement u, v, w are approximated by $\bar{u}$, $\bar{v}$, $\bar{w}$, whose dependence on the coordinate z is given in the form

$$\left.\begin{aligned} \bar{u} &= \frac{z}{h}u_1(x,t)+\frac{z^3}{h^3}u_3(x,t) \\ \bar{v} &= 0 \\ \bar{w} &= w_0(x,t)+\frac{z^2}{h^2}w_2(x,t) \end{aligned}\right\} \quad \ldots \quad (1)$$

The elongations and shears are then approximated by

$$\left.\begin{aligned} \bar{\epsilon}_{xx} &= \frac{\partial\bar{u}}{\partial x} = \frac{z}{h}\frac{\partial u_1}{\partial x}+\frac{z^3}{h^3}\frac{\partial u_3}{\partial x} \\ \bar{\epsilon}_{yy} &= \frac{\partial\bar{v}}{\partial y} = 0 \\ \bar{\epsilon}_{zz} &= \frac{\partial\bar{w}}{\partial z} = 2\frac{z}{h^2}w_2 \\ \bar{\epsilon}_{xy} &= \frac{\partial\bar{v}}{\partial x}+\frac{\partial\bar{u}}{\partial y} = 0 \\ \bar{\epsilon}_{yz} &= \frac{\partial\bar{w}}{\partial y}+\frac{\partial\bar{v}}{\partial z} = 0 \\ \bar{\epsilon}_{zx} &= \frac{\partial\bar{u}}{\partial z}+\frac{\partial\bar{w}}{\partial x} = \frac{u_1}{h}+3\frac{z^2}{h^3}u_3+\frac{\partial w_0}{\partial x}+\frac{z^2}{h^2}\frac{\partial w_2}{\partial x} \end{aligned}\right\} \quad \ldots \quad (2)$$

3. Stress-strain Relations

The motion is supposed to take place in two dimensions x and z, therefore the stress-strain relations for plane-strain case are then approximated by

$$\left.\begin{aligned} T_{xx} &= (\lambda+2\mu)\bar{\epsilon}_{xx}+\lambda\bar{\epsilon}_{zz} \\ T_{zz} &= (\lambda+2\mu)\bar{\epsilon}_{zz}+\lambda\bar{\epsilon}_{xx} \\ T_{xy} &= T_{yz} = 0 \\ T_{zx} &= \mu\bar{\epsilon}_{zx} \end{aligned}\right\}, \quad \ldots \quad \ldots \quad \ldots \quad (3)$$

where λ and μ are Lamb's elastic constants.

4. Energy Considerations

The strain-energy density

$$W^* = \tfrac{1}{2}(T_{xx}\epsilon_{xx}+T_{yy}\epsilon_{yy}+T_{zz}\epsilon_{zz}+T_{xy}\epsilon_{xy}+T_{yz}\epsilon_{yz}+T_{zx}\epsilon_{zx}) \quad \ldots \quad (4)$$

is therefore approximated by

$$\overline{W}^* = \tfrac{1}{2}(T_{xx}\bar{\epsilon}_{xx}+T_{zz}\bar{\epsilon}_{zz}+T_{zx}\bar{\epsilon}_{zx}). \quad \ldots \quad \ldots \quad \ldots \quad (5)$$

Now the stresses are replaced by the approximate expressions (3) and then the approximate expressions for strains are substituted from (2). The strain energy density thus obtained in terms of displacements is integrated over the cross-sectional area, the element of area being $dx\,.\,dz$, and over the length l, from zero to l. The coefficient of integral containing the expression for ϵ_{zx} is multiplied by a constant k', which is known as shear coefficient. The strain energy of the beam and its first variation are found to be

$$\begin{aligned} \overline{W} = 2b\int_0^l \Bigg[& (\lambda+2\mu)\left\{\frac{1}{3}h\left(\frac{\partial u_1}{\partial x}\right)^2+\frac{2h}{5}\,\frac{\partial u_1}{\partial x}\,\frac{\partial u_3}{\partial x}+\frac{h}{7}\left(\frac{\partial u_3}{\partial x}\right)+\frac{4}{3h}w_2^2\right\} \\ & +\lambda\left(\frac{4}{3}w_2\frac{\partial u_1}{\partial x}+\frac{4}{5}w_2\frac{\partial u_3}{\partial x}\right)+k'\mu\left\{\frac{u_1^2}{h}+\frac{9}{5h}u_3^2+h\left(\frac{\partial w_0}{\partial x}\right)^2+\frac{h}{5}\left(\frac{\partial w_2}{\partial x}\right)^2\right. \\ & +\frac{2}{h}u_1u_3+2u_1\frac{\partial w_0}{\partial x}+\frac{2}{3}u_1\frac{\partial w_2}{\partial x}+2u_3\frac{\partial w_0}{\partial x} \\ & \left.+\frac{6}{5}u_3\frac{\partial w_2}{\partial x}+\frac{2h}{3}\,\frac{\partial w_0}{\partial x}\,\frac{\partial w_2}{\partial x}\right\}\Bigg]\,dx, \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (6) \end{aligned}$$

$$\begin{aligned} \delta\overline{W} = 4b\int_0^l \Bigg[& (\lambda+2\mu)\left\{\frac{h}{3}\,\frac{\partial u_1}{\partial x}\,\frac{\partial}{\partial x}\,\delta u_1+\frac{h}{5}\left(\frac{\partial u_1}{\partial x}\,\frac{\partial}{\partial x}\,\delta u_3+\frac{\partial u_3}{\partial x}\,\frac{\partial}{\partial x}\,\delta u_1\right)\right. \\ & \left.+\frac{h}{7}\,\frac{\partial u_3}{\partial x}\,\frac{\partial}{\partial x}\,\delta u_3+\frac{4}{3h}w_2\,\delta w_2\right\}+\lambda\left\{\frac{2}{3}\left(\frac{\partial u_1}{\partial x}\,\delta w_2+w_2\frac{\partial}{\partial x}\,\delta u_1\right)\right. \\ & \left.+\frac{2}{5}\left(\frac{\partial u_3}{\partial x}\,\delta w_2+w_2\frac{\partial}{\partial x}\,\delta u_3\right)\right\}+k'\mu\left\{\frac{u_1}{h}\,\delta u_1+\frac{9}{5h}u_3\,\delta u_3\right. \end{aligned}$$

$$+h\frac{\partial w_0}{\partial x}\frac{\partial}{\partial x}\delta w_0+\frac{h}{5}\frac{\partial w_2}{\partial x}\frac{\partial}{\partial x}\delta w_2+\frac{1}{h}(u_1\,\delta u_3+u_3\,\delta u_1)$$

$$+u_1\frac{\partial}{\partial x}\delta w_0+\frac{\partial w_0}{\partial x}\delta u_1+\frac{1}{3}\left(u_1\frac{\partial}{\partial x}\delta w_2+\frac{\partial w_2}{\partial x}\delta u_1\right)+u_3\frac{\partial}{\partial x}\delta w_0$$

$$+\frac{\partial w_0}{\partial x}\delta u_3+\frac{3}{5}\left(u_3\frac{\partial}{\partial x}\delta w_2+\frac{\partial w_2}{\partial x}\delta u_3\right)$$

$$+\frac{h}{3}\left(\frac{\partial w_0}{\partial x}\frac{\partial}{\partial x}\delta w_2+\frac{\partial w_2}{\partial x}\frac{\partial}{\partial x}\delta w_0\right)\Bigg\}\Bigg]\,dx. \quad \ldots \quad (7)$$

The expression for the kinetic energy density T^*, according to the general linear theory, is

$$T^*=\frac{\rho}{2}\left[\left(\frac{\partial u}{\partial t}\right)^2+\left(\frac{\partial v}{\partial t}\right)^2+\left(\frac{\partial w}{\partial t}\right)^2\right] \quad \ldots \quad (8)$$

where ρ is the density of the mass.

Employing relations (1) and integrating over the volume of the beam, the approximate kinetic energy $\overline{T}$ of the beam takes the form

$$\overline{T}=2\rho b\int_0^l\left[\frac{h}{3}\left(\frac{\partial u_1}{\partial t}\right)^2+\frac{2h}{5}\frac{\partial u_1}{\partial t}\frac{\partial u_3}{\partial t}+\frac{h}{7}\left(\frac{\partial u_3}{\partial t}\right)^2+h\left(\frac{\partial w_0}{\partial t}\right)^2\right.$$

$$\left.+\frac{2h}{3}\frac{\partial w_0}{\partial t}\frac{\partial w_2}{\partial t}+\frac{h}{5}\left(\frac{\partial w_2}{\partial t}\right)^2\right]dx. \quad \ldots \quad (9)$$

The first variation of this expression is thus given by

$$\delta\overline{T}=4\rho b\int_0^l\left[\frac{h}{3}\frac{\partial u_1}{\partial t}\frac{\partial}{\partial t}\delta u_1+\frac{h}{5}\left(\frac{\partial u_1}{\partial t}\frac{\partial}{\partial t}\delta u_3+\frac{\partial u_3}{\partial t}\frac{\partial}{\partial t}\delta u_1\right)+\frac{h}{7}\frac{\partial u_3}{\partial t}\frac{\partial}{\partial t}\delta u_3\right.$$

$$\left.+h\frac{\partial w_0}{\partial t}\frac{\partial}{\partial t}\delta w_0+\frac{h}{3}\left(\frac{\partial w_0}{\partial t}\frac{\partial}{\partial t}\delta w_2+\frac{\partial w_2}{\partial t}\frac{\partial}{\partial t}\delta w_0\right)+\frac{h}{5}\frac{\partial w_2}{\partial t}\frac{\partial}{\partial t}\delta w_2\right]dx. \quad (10)$$

5. Equations of Motion

The equations of motion may now be deduced from the Hamilton principle, which is expressed by the equation

$$\delta\int L\,dt=0, \quad \ldots \quad (11)$$

the variation being taken between fixed initial and final values (t_1 and t_2) for t, and the kinetic potential L is approximated by

$$\overline{L}=\overline{T}-\overline{W}. \quad \ldots \quad (12)$$

After partial integration of all terms in $\delta\overline{T}$ with respect to time t and of some terms in $\delta\overline{W}$ with respect to x, the variation of the approximate kinetic potential yields the results

$$\delta\int_{t_1}^{t_2}\bar{L}\,dt=\int_{t_1}^{t_2}\int_0^l\Bigg[\Big\{(\lambda+2\mu)\Big(\frac{h}{3}u_1''+\frac{h}{5}u_3''\Big)-\lambda\frac{2}{3}w_2'$$
$$+k'\mu\Big(\frac{u_1}{h}+\frac{u_3}{h}+w_0'+\frac{1}{3}w_2'\Big)-\frac{\rho h}{3}\ddot{u}_1-\frac{\rho h}{5}\ddot{u}_3\Big\}\delta u_1$$
$$+\Big\{(\lambda+2\mu)h\Big(\frac{u_1''}{5}+\frac{u_3''}{7}\Big)-k'\mu\Big(\frac{9}{5h}u_3+\frac{u_1}{h}+w_0'+\frac{3}{5}w_2'\Big)$$
$$+\frac{2}{5}\lambda w_2'-\rho h\Big(\frac{\ddot{u}_1}{5}+\frac{\ddot{u}_3}{7}\Big)\Big\}\delta u_3+\Big\{k'\mu\Big(hw_0''+u_1'+u_3'+\frac{h}{3}w_2''\Big)$$
$$-\rho h\Big(\ddot{w}_0+\frac{1}{3}\ddot{w}_2\Big)\Big\}\delta w_0+\Big\{-\frac{4}{3h}(\lambda+2\mu)w_2$$
$$+k'\mu\Big(\frac{h}{5}w_2''+\frac{u_1'}{3}+\frac{3}{5}u_3'+\frac{h}{3}w_0''\Big)-2\lambda\Big(\frac{u_1'}{3}+\frac{u_3'}{5}\Big)$$
$$-\rho h\Big(\frac{\ddot{w}_0}{3}+\frac{\ddot{w}_2}{5}\Big)\Big\}\delta w_2\Bigg]dx\,dt+\int_0^l\Bigg[\rho h\Big\{\Big(\frac{\dot{u}_1}{3}+\frac{\dot{u}_3}{5}\Big)\delta u_1$$
$$+\Big(\frac{\dot{u}_1}{5}+\frac{\dot{u}_3}{7}\Big)\delta u_3+\Big(\dot{w}_0+\frac{\dot{w}_2}{3}\Big)\delta w_0+\Big(\frac{\dot{w}_0}{3}+\frac{\dot{w}_2}{5}\Big)\delta w_2\Big\}\Bigg]_{t_1}^{t_2}dx$$
$$-\int_{t_1}^{t_2}\Bigg[\Big\{(\lambda+2\mu)h\Big(\frac{u_1'}{3}+\frac{u_3'}{5}\Big)+\frac{2}{3}\lambda w_2\Big\}\delta u_1$$
$$+\Big\{(\lambda+2\mu)h\Big(\frac{u_1'}{5}+\frac{u_3'}{7}\Big)+\lambda\frac{2}{5}w_2\Big\}\delta u_3+\Big\{k'\mu\Big(hw_0'+\frac{h}{3}w_2'$$
$$+u_1+u_3\Big)\Big\}\delta w_0+\Big\{k'\mu\Big(\frac{h}{5}w_2'+\frac{h}{3}w_0'+\frac{u_1}{5}+\frac{3}{5}u_3\Big)\Big\}\delta w_2\Bigg]_0^l dt, \tag{13}$$

where dots and dashes denote differentiations with respect to t and x.

Since the variation of the kinetic potential must vanish for arbitrary values of δu_1, δu_3, etc., the corresponding coefficients of these in (13) must vanish separately. The vanishing of the coefficients of δu_1, δu_3, etc., in the first part of the above expression (i.e. the terms under the double integral) gives the following four equations of motion:

$$\left.\begin{aligned}
&\frac{h}{3}\left\{\rho\ddot{u}_1-(\lambda+2\mu)u_1''\right\}+\frac{h}{5}\left\{\rho\ddot{u}_3-(\lambda+2\mu)u_3''\right\}+\mu k'\left(\frac{u_1}{h}+\frac{u_3}{h}+w_0'\right)+\frac{k'\mu-2\lambda}{3}w_2'=0\\
&\frac{h}{5}\left\{\rho\ddot{u}_1-(\lambda+2\mu)u_1''\right\}+\frac{h}{7}\left\{\rho\ddot{u}_3-(\lambda+2\mu)u_3''\right\}+\mu k'\left(\frac{u_1}{h}+\frac{9}{5h}u_3+w_0'\right)+\frac{3k'\mu-2\lambda}{5}w_2'=0\\
&h(\rho\ddot{w}_0-k'\mu w_0'')+\frac{h}{3}(\rho\ddot{w}_2-k'\mu w_2'')-k'\mu(u_1'+u_3')=0\\
&\frac{h}{3}(\rho\ddot{w}_0-k'\mu w_0'')+\frac{h}{5}(\rho\ddot{w}_2-k'\mu w_2'')+\frac{4}{3h}(\lambda+2\mu)w_2-\left(\frac{\mu k'-2\lambda}{3}u_1'+\frac{3\mu k'-2\lambda}{5}u_3'\right)=0
\end{aligned}\right\}. \tag{14}$$

The vanishing of the second part (i.e. the terms already integrated with respect to x and t) provides us the appropriate boundary and initial conditions.

6. Wave Solutions

The propagation of free harmonic waves in the beam is considered by seeking solutions of eqn. (14) in the form

$$\left.\begin{aligned} u_1 &= Ae^{i\omega t}e^{ipx} \\ u_3 &= Be^{i\omega t}e^{ipx} \\ w_0 &= Ce^{i\omega t}e^{ipx} \\ w_2 &= De^{i\omega t}e^{ipx} \end{aligned}\right\}, \qquad (15)$$

where ω is the angular frequency, p is the wave number which is equal to $\dfrac{2\pi}{\eta}$ and η is wavelength.

The resulting equations are

$$\left.\begin{aligned} & h\left(\frac{A}{3}+\frac{B}{5}\right)\{-\rho\omega^2+p^2(\lambda+2\mu)\}+k'\mu\left(\frac{A}{h}+\frac{B}{h}+ipC\right)+\frac{i(k'\mu-2\lambda)pD}{3}=0 \\ & h\left(\frac{A}{5}+\frac{B}{7}\right)\{-\rho\omega^2+p^2(\lambda+2\mu)\}+k'\mu\left(\frac{A}{h}+\frac{9}{5h}B+ipC\right)+\frac{i(3k'\mu-2\lambda)}{5}pD=0 \\ & h\left(C+\frac{D}{3}\right)\{-\rho\omega^2+k'\mu p^2\}-ik'\mu p(A+B)=0 \\ & h\left(\frac{C}{3}+\frac{D}{5}\right)\{-\rho\omega^2+k'\mu p^2\}+\frac{4}{3h}(\lambda+2\mu)D-ip\left(\frac{k'\mu-2\lambda}{3}A+\frac{3k'\mu-2\lambda}{5}B\right)=0 \end{aligned}\right\}. \qquad (16)$$

After substituting

$$\left.\begin{aligned} \mu &= \frac{E}{2(1+\nu)} \\ \lambda &= \frac{\nu E}{(1+\nu)(1-2\nu)} \\ \frac{\omega}{p} &= c \text{ (phase velocity)} \\ \sqrt{\frac{E}{\rho}} &= c_0 \text{ (the velocity of propagation of longitudinal waves)} \end{aligned}\right\}, \qquad (17)$$

where E is Young's modulus of elasticity and
ν is Poisson's ratio,

we get

$$\left.\begin{aligned} & \{10G^2(VC_r^2-W)-15k'\}A+\{6G^2(VC_r^2-W)-15k'\}B-5iG\{3k'C+(k'-2U)D\}=0 \\ & \{14G^2(VC_r^2-W)-35k'\}A+\{10G^2(VC_r^2-W)-63k'\}B-7iG\{5k'C+(3k'-2U)D\}=0 \\ & 3ik'(A+B)+G(2VC_r^2-k')(3C+D)=0 \\ & 5iG(k'-2U)A+3iG(3k'-2U)B+5G^2(2VC_r^2-k')C+\{3G^2(2VC_r^2-k')-40W\}D=0 \end{aligned}\right\} \qquad (18)$$

where

$$\left.\begin{aligned} \frac{c}{c_0} &= C_r \text{ (velocity parameter)} \\ \frac{h}{\eta} &= H \text{ (wavelength parameter)} \\ \frac{2\pi h}{\eta} &= G \\ 1+\nu &= V \\ \frac{1-\nu}{1-2\nu} &= W \\ \frac{2\nu}{1-2\nu} &= U \end{aligned}\right\} . \qquad \ldots \qquad \ldots \quad (19)$$

Eliminating the constants A, B, C and D from (18) we obtain a fourth order equation in C_r^2:

$$a_1 C_r^8 + a_2 C_r^6 + a_3 C_r^4 + a_5 = 0, \qquad \ldots \qquad \ldots \qquad \ldots \quad (20)$$

where

$$a_1 = 8V^4G^6,$$

$$a_2 = -4V^3G^4[15(3k'+2W)+2G^2(k'+2W)],$$

$$a_3 = 2V^2G^2[15k'(7k'+90W)+15(-U^2+3k'^2+8k'W+8W^2)G^2 \\ +(k'^2+8k'W+4W^2)G^4],$$

$$a_4 = -V[3150Wk'^2+k'\{-210U(k'+3U)+150W(7k'+18W)\}G^2 \\ +\{-15U^2(k'+2W)+30W(4Wk'+3k'^2+4W^2)\}G^4+4Wk'(k'+2W)G^6],$$

$$a_5 = k'G^2[7k'(-37{\cdot}5U^2+150W^2)+15W(4W^2-U)G^2+2k'W^2G^4].$$

7. Numerical Results and Discussions

The values of the velocity parameter C_r can be found for different values of wavelength parameter H by solving eqn. (20) numerically. It will give four values of C_r for every value of H. When a graph is plotted for C_r versus H, we get four separate branches of the curve. The value of the shear coefficient k' is taken to be equal to $\pi^2/12$ (the value of k' calculated by Mindlin (1951) for Timoshenko's equation by considering thickness shear motion). The value of Poisson's ratio is taken to be equal to 0·29.

The velocity parameters calculated on the basis of three theories (Timoshenko's theory, Lamb's exact theory and the present theory) are given in Tables I–III. These have also been plotted in Fig. 1. We see that the two branches of the curve given by the first two roots of the eqn. (20) are in better agreement with the corresponding branches given by the exact equation than those given by the Timoshenko equation. The remaining two branches are also in good agreement with the exact theory.

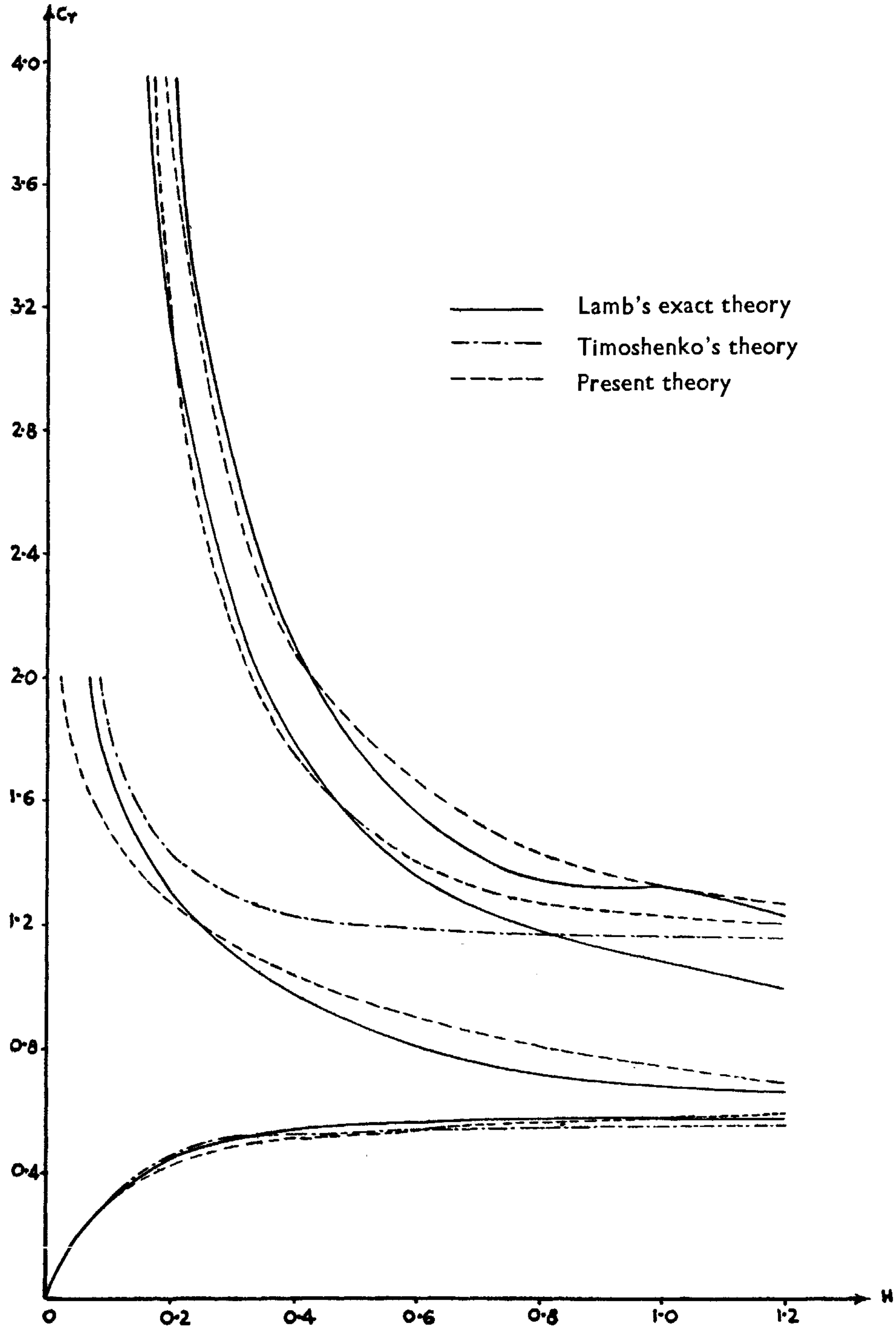

FIG. 1. C_r versus H for first four modes of transverse vibrations of a rectangular beam.

TABLE I

Velocity parameter C_r obtained from Timoshenko's theory

H	C_r	
	C_{r1}	C_{r2}
0·2	0·4538169	1·4242247
0·4	0·5272975	1·2257537
0·6	0·5467325	1·1821813
0·8	0·5542603	1·1661254
1·0	0·5578941	1·1585300
1·2	0·5599106	1·1543577

TABLE II

Velocity parameter C_r obtained from Lamb's exact theory

H	C_r			
	C_{r1}	C_{r2}	C_{r3}	C_{r4}
0·2	0·4452025	1·2979624	3·0944114	3·9560639
0·4	0·5344691	0·9810107	1·7770315	2·1005694
0·6	0·5601530	0·8055349	1·3511539	1·5664824
0·8	0·5697330	0·7185728	1·1780364	1·3382566
1·0	0·5736681	0·6772566	1·0816259	1·3273951
1·2	0·5753027	0·6562912	0·9892671	1·2287133

TABLE III

Velocity parameter C_r obtained from the present theory

H	C_r			
	C_{r1}	C_{r2}	C_{r3}	C_{r4}
0·2	0·4239699	1·2622665	3·0811222	3·6751619
0·4	0·5082052	1·0321639	1·7492228	2·0624574
0·6	0·5414388	0·9082718	1·3961060	1·6067869
0·8	0·5605319	0·8052788	1·2785251	1·4162766
1·0	0·5736399	0·7299280	1·2320727	1·3187844
1·2	0·5845415	0·6761358	1·2106315	1·2615936

ACKNOWLEDGEMENTS

The author is highly grateful to Prof. C. Prasad for the encouragement and guidance during the course of investigation.

Thanks are also due to the staff of Computer Centre, Structural Engineering Research Centre, Roorkee, for their co-operation during the computations involved in this paper.

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