ANOTHER METHOD OF STUDYING THE STABILITY OF THE SOLUTIONS OF NEUTRAL FUNCTIONAL DIFFERENTIAL SYSTEMS

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Lyapunov functions have been conveniently used to study the stability properties of neutral functional differential equations by studying their anticipated systems. We follow the method of B. S. Razumihin.

B. S. Razumihin (1966) has studied the problem of stability of differential equations with retarded arguments. We follow his method and study the stability of neutral differential difference equations of the form

$$\frac{dx(t)}{dt} = f(t,\, x(t-\theta),\, \dot{x}(t-\theta)), \quad \theta < 0, \qquad (1)$$

where x and f are n-dimensional vectors in a real Euclidean n-space E^n. It is quite interesting and important to note that many of the stability and boundedness properties of (1) could be derived by studying the system with anticipation which corresponds to this system (1). First we introduce a new independent variable α by putting $t = t^*-\alpha$, where $t^* > t_0$ is a known parameter. Then we get $\frac{dx}{dt} = -\frac{dx}{d\alpha}$. The system (1) can now be written as

$$\frac{dx(t^*-\alpha)}{d\alpha} = f(t^*-\alpha,\, x(t-\theta-\alpha),\, \dot{x}(t-\theta-\alpha)), \qquad (2)$$

where the dots denote the right-hand derivatives. Suppose

$$\left.\begin{aligned} x(t^*-\alpha) &= y(\alpha) \\ f(t^*-\alpha) &= -g(\alpha) \end{aligned}\right\}, \qquad (3)$$

then (2) can be written as

$$\frac{dy}{d\alpha} = g(\alpha,\, y(\alpha+\theta),\, \dot{y}(\alpha+\theta)). \qquad (4)$$

The derivative of $y(\alpha+\theta)$ is assumed to exist. We call (4) as the system with anticipation which corresponds to the system (1). Suppose $V(t, x)$ is a continuous scalar Lyapunov function with continuous partial derivatives with respect to its arguments. Its derivative with respect to system (1) is

$$\left.\begin{aligned}\frac{dV}{dt} &= \frac{\partial V}{\partial t}+\frac{\partial V}{\partial x}\cdot f(t, x(t-\theta), \dot{x}(t-\theta)) \\ &= W(t, x(t-\theta), \dot{x}(t-\theta))\end{aligned}\right\} \qquad (5)$$

and is a continuous function defined on the solutions of system (1). The dot in (5) represents scalar product. Thus the Lyapunov function $V(t, x)$ has as many values as there are integral curves at the point $x(t)$. Suppose

$$W(t^*-\alpha, x(t^*-\alpha-\theta), \dot{x}(t^*-\alpha-\theta)) = U(\alpha, y(\alpha+\theta), \dot{y}(\alpha+\theta)) \qquad (6)$$

defined on the solutions of the system with anticipation (4). We now extend the technique of Lyapunov functions to a system of the form (1).

THEOREM 1: If for the differential equation (1) there exists a Lyapunov function, $V(t, x)$ whose derivative $\frac{dV}{dt}$ relative to this system is such that the functional $U(0, y(\theta), \dot{y}(\theta))$ is negative or $\equiv 0$ along every continuous solution of the anticipated system (4) satisfying the conditions

$$y(0) = x$$

$$V(t-a, y(a)) \leqq V(t, x) \text{ for } a \geqq t \qquad (7)$$

$$0 \leqq a \leqq t-t_0 \qquad (8)$$

then the differential difference equation (1) is stable.

PROOF: Let ϵ be an arbitrarily given positive number $< \bar{c} = \inf V(t, x)$ for $t \geqq t_0$ and for $x(t)$ belonging to the boundary of the region $\|x\| \leqq H$ in which the conditions of the existence and uniqueness theorems for (1) are satisfied.

Here $$\|x\| = \sum_{i=1}^{n} |x_i|;$$

let $c = \inf V(t, x)$ on the sphere $\|x\| = \epsilon$ and for $t \geqq t_0$ and let $\delta(\epsilon) = \|x\|$ on the set $t \geqq t_0$, $V(t, x(t)) = c$. Suppose $x(t)$ is the solution of (1) corresponding to an arbitrary system of initial functions $\theta(s)$ which satisfy the condition

$$\|\theta\| \leqq \delta_1(\epsilon), \qquad (9)$$

where $\delta_1(\epsilon) < \delta(\epsilon)$ is small enough, so that the corresponding solution does not leave the region $\|x\| \leqq \delta(\epsilon)$ in the interval $t_0 \leqq t \leqq t_0+h$. We use the method of contradiction. Suppose the assertion of the theorem is not true. Let there exist at least one system of initial functions such that $\|\theta\| < \delta_1(\epsilon)$ and the corresponding solution does leave the sphere $\|x\| = \epsilon$, that there exists an instant $t_1 > t_0+h$ such that

$$\|x(t)\| \begin{cases} \lesseqgtr \epsilon & \text{for } t < t_1 \\ > \epsilon & \text{for } t_1+\beta > t > t_1 \end{cases} \qquad (10)$$

where β is a non-zero positive number or is ∞.

Since $c = \inf V(t, x)$ on $\| x(t) \| = \epsilon$ there exists an instant t_2, $t_0+h \leqq t_2 \leqq t_1$ such that along this solution

$$V(t, x(t)) \begin{cases} < C \text{ for } t < t_2 \\ > C \text{ for } t_2+\delta > t > t_2 \end{cases} \quad \ldots \quad \ldots \quad \ldots \quad (11)$$

where δ is a non-zero positive number or is ∞.

Condition (11) implies that

$$\left.\frac{dv}{dt}\right|_{t=t_2} > 0.$$

However, under a change of the independent variable, the integral curve we have been considering is an integral curve of system (4) satisfying conditions (7) and (8). Hence the inequality

$$\left.\frac{dv}{dt}\right|_{t=t_2} > 0$$

is impossible along this curve. The proof of theorem 1 is complete.

THEOREM 2: If the conditions of theorem 1 are satisfied for (1) and if the function admits of an infinitesimal upper bound while the functional $U(0, y(\theta), \dot{y}(\theta))$ is negative definite under the conditions (7) and (8), then the system (1) is asymptotically stable.

Proof of this theorem is analogous to the proof of theorem 2 in Razumihin's work (1966).

A simpler problem is the problem of integration on the set of solutions of system (4) satisfying the conditions (7) and (8) on a segment of length T, where $T \geqq a \geqq 0$. Let $\theta(s)$ be an initial vector function in the interval $t-T-h \leqq s \leqq t-T$, $T > h$, $t > t_0+T$ and satisfying the condition $V(s, \theta(s) \leqq V(t, x(t))$, where θ and x are vector functions with n components. The corresponding solution on the interval $(t-T, t)$ can be found by the method. Let $E_0(t, T)$ be the set of initial functions satisfying the condition $V(s, \theta(s)) \leqq V(t, x(t))$ and let $E(t, T)$ be the set of the corresponding solutions. Let $E^*(t, T) \subset E(t, T)$ be the subset of solutions $x^*(a)$ which satisfy

$$V(a, x^*(a)) \leqq V(t, x(t))$$
$$t-T \leqq a \leqq t$$
$$x^*(t) = x(t)$$

THEOREM 3: If for the system (1) there exists a positive number $T > h$ and a positive definite function $V(t, x(t))$ whose derivative $\frac{dV}{dt} = W(t, x(t-\theta), \dot{x}(t-\theta))$ relative to this system is negative or $\equiv 0$, functional on the set of functions $E^1(t, T)$ for every $t > t_0+T$, then the system (1) is stable.

Remark 1: The above technique can be employed to study the total stability or stability under persistent disturbances of (1) by considering the system

$$\frac{dx}{dt} = f(t, x(t-\theta), \dot{x}(t-\theta)) + F(t, x(t-\theta), \dot{x}(t-\theta)), \quad \ldots \quad \ldots \quad (12)$$

where F is a perturbed function. We can also extend I. G. Malkin's theorem for ordinary differential equations that uniform asymptotic stability implies total stability.

Remark 2: It is natural to ask whether it might be more convenient to use a vector Lyapunov function rather than a scalar function. Bellman (1962) has dealt with the advantages of using vector Lyapunov functions to study the stability properties of ordinary differential equations. Vector Lyapunov functions can also be used to study the stability properties of neutral functional differential equations.

References

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