SPINOR REPRESENTATION OF $2n$-DIMENSIONAL CONTACT TRANSFORMATION

by N. N. GHOSH, *Indian Association for the Cultivation of Science, Calcutta 32*

(Communicated by S. N. Bose, F.N.I.)

(*Received January 20, 1967*)

In a complex n-dimensional non-analytic manifold S_n characterized by a general non-analytic coordinate transformation scheme, a special transformation called *unitary non-analytic spinor transformation* has been framed. It is shown that a mixed spinor of the second rank undergoing this transformation is associated with a real vector undergoing, in general, a $(2n^2-n)$-dimensional Lorentz transformation which leaves invariant a quadratic form with n^2 positive squares and $n(n-1)$ negative squares. In the auxiliary space R_{2n} this unitary spinor transformation, however, induces a $2n$-dimensional contact transformation having $(1-1)$ correspondence, thus constituting the spinor representation in S_n of the contact transformation in R_{2n}.

INTRODUCTION

In a previous paper (Ghosh 1966) a generalized theory of spinors has been developed in a non-analytic complex n-space S_n having an auxiliary real space R_{2n} associated with it. It has been shown that there exists a (1—1) correspondence between similar entities in S_n and R_{2n}, the relevant connection formulae being expressible by means of a set of two-index spin-tensors with characteristic structure which is preserved if S_n or R_{2n} is specialized. The object of the present paper is to discuss a special transformation in S_n, called unitary non-analytic spinor transformation which leaves invariant an elementary spinor $\eta_{\mu\nu}$ with $2n$ non-vanishing components ($\eta_{\mu\dot{\nu}} = i$, $\eta_{\dot{\mu}\nu} = -i$, with $\mu = \nu$) and to show that its 'analogue' in R_{2n} is a $2n$-dimensional contact transformation. In an earlier paper (Ghosh 1958) this relationship between the two transformations was first noticed. It is further noted that a mixed spinor undergoing this unitary transformation in S_n or its analogue, a mixed tensor, undergoing a contact transformation in R_{2n} induces, in general, a $(2n^2—n)$-dimensional Lorentz transformation to a vector associated with it.

§ 1. Referring to § 1 of my paper (Ghosh 1966) we write the transformation formula for $\eta_{\mu\nu}$ in the form

$$\eta'_{\mu\nu} = \eta_{\alpha\beta}(\xi'_{\mu}X^{\alpha})(\xi'_{\nu}X^{\beta}), \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (1.1)$$

where the dummy indices run over the values $1, 2, \ldots n, \dot{1}, \dot{2}, \ldots \dot{n}$. Substituting the non-vanishing components of $\eta_{\alpha\beta}$ in (1.1) we get

$$\eta'_{\mu\nu} = i[(\xi'_{\mu}X^{\epsilon})(\xi'_{\nu}X^{\dot{\epsilon}})-(\xi'_{\mu}X^{\dot{\epsilon}})(\xi'_{\nu}X^{\epsilon})], \quad \ldots \quad \ldots \quad (1.2)$$

where $(\epsilon, \dot{\epsilon})$ are summed over the values 1, 2, . . . n. Postulating the invariance of $\eta_{\mu\nu}$ we obtain the conditions for a unitary non-analytic spinor transformation in S_n as follows:

$$(\xi'_\lambda X^\epsilon)(\xi'_\sigma X^{\dot{\epsilon}})-(\xi'_\lambda X^{\dot{\epsilon}})(\xi'_\sigma X^\epsilon) = 0,$$

$$(\xi'_\lambda X^\epsilon)(\xi'_{\dot{\sigma}} X^{\dot{\epsilon}})-(\xi'_\lambda X^{\dot{\epsilon}})(\xi'_{\dot{\sigma}} X^\epsilon) = 1, \text{ if } \lambda = \sigma,$$
$$= 0, \text{ if } \lambda \neq \sigma. \quad \ldots \quad \ldots \quad (1.3)$$

Expressing the above in matrix form one can see that the determinant $\triangle$ of the unitary transformation coefficients satisfy the equation

$$\triangle^2 = 1. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (1.4)$$

The associated contravariant spinor $\eta^{\mu\nu}$ will be defined by the $2n$ non-vanishing components

$$\eta^{\mu\dot{\nu}} = -i, \; \eta^{\dot{\mu}\nu} = i, \text{ with } \mu = \nu. \quad \ldots \quad \ldots \quad \ldots \quad (1.5)$$

The spinors $\eta^{\mu\nu}$, $\eta_{\mu\nu}$ serve as metric spinors in S_n. The raising and lowering of spinor indices may be performed under the scheme

$$\psi^\mu = \eta^{\mu\nu}\psi_\nu, \; \psi_\mu = \psi^\nu\eta_{\nu\mu}, \quad \ldots \quad \ldots \quad \ldots \quad (1.6)$$

where

$$\eta^{\mu\nu}\eta_{\mu\rho} = \eta^{\nu\mu}\eta_{\rho\mu} = \delta^\nu_\rho.$$

Thus

$$\psi^\alpha = -i\psi_{\dot{\alpha}}, \; \psi_\alpha = -i\psi^{\dot{\alpha}}. \quad \ldots \quad \ldots \quad \ldots \quad (1.7)$$

It may be noted that the contravariant and the covariant transformation coefficients are connected by means of the equation

$$\xi_\alpha X'^\mu = \eta^{\mu\nu}\eta_{\alpha\beta}\xi'_\nu X^\beta, \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (1.8)$$

whence we obtain the relations

$$\xi_\lambda X'^\epsilon = \xi'_{\dot{\epsilon}} X^{\dot{\lambda}}, \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (1.9)$$
$$\xi_\lambda X'^{\dot{\epsilon}} = -\xi'_\epsilon X^{\dot{\lambda}}.$$

To define an infinitesimal unitary non-analytic spinor transformation we take the infinitesimal transformation equations (Ghosh 1966)

$$x'^k = x^k + \epsilon f^k(x^0, x^1, \ldots x^{2n-1}) \quad \ldots \quad \ldots \quad \ldots \quad (1.10)$$

with the transformation coefficients

$$\xi_\alpha X'^\mu = \delta^\mu_\alpha + \epsilon\xi_\alpha F^\mu, \; \xi_\alpha X'^{\dot{\mu}} = \epsilon\xi_\alpha F^{\dot{\mu}},$$
$$\xi'_\alpha X^\mu = \delta^\mu_\alpha - \epsilon\xi_\alpha F^\mu, \; \xi'_\alpha X^{\dot{\mu}} = -\epsilon\xi_\alpha F^{\dot{\mu}}, \quad \ldots \quad \ldots \quad (1.11)$$

where F^μ denotes $f^{\mu-1} + if^{2n-\mu}$.

Making use of (1.8) we then derive the set of conditions

$$\left.\begin{aligned} \xi_\alpha F^\mu &= -\xi_{\dot{\mu}} F^{\dot{\alpha}}, \\ \xi_\alpha F^{\dot{\mu}} &= \xi_\mu F^{\dot{\alpha}}, \end{aligned}\right\} \quad \ldots \quad \ldots \quad \ldots \quad (1.12)$$

which the f's must satisfy. In real terms the above gives

$$\frac{\partial f^{\epsilon-1}}{\partial x^{\lambda-1}} = -\frac{\partial f^{2n-\lambda}}{\partial x^{2n-\epsilon}},$$

$$\frac{\partial f^{\epsilon-1}}{\partial x^{2n-\lambda}} = \frac{\partial f^{\lambda-1}}{\partial x^{2n-\epsilon}}, \quad \frac{\partial f^{2n-\epsilon}}{\partial x^{\lambda-1}} = \frac{\partial f^{2n-\lambda}}{\partial x^{\epsilon-1}}. \qquad \text{.. (1.13)}$$

Consider now a mixed spinor K^{μ}_{ν} satisfying the structural equation

$$K^{\mu}_{\nu}\eta_{\mu\rho}\eta^{\nu\sigma} = K^{\sigma}_{\rho} \qquad \text{.. .. .. .. (1.14)}$$

so that

$$K^{\epsilon}_{\lambda} = K^{\dot{\lambda}}_{\dot{\epsilon}}, \; K^{\dot{\epsilon}}_{\lambda} = -K^{\dot{\lambda}}_{\epsilon}.$$

Applying the rule (1.7) of raising and lowering of indices one can see that K^{μ}_{ν} will have the bilinear form

$$K^{\mu}_{\nu} = \phi^{\mu}\chi_{\nu} - \chi^{\mu}\phi_{\nu} \qquad \text{.. .. .. .. (1.15)}$$

if constructed with ϕ^{μ}, χ_{ν}, two arbitrary spinors of rank 1.

In general, K^{μ}_{ν} has $2n^2-n$ mutually independent components, the structure (1.14) being preserved when K^{μ}_{ν} undergoes the unitary non-analytic spinor transformation. Expressing the components of K^{μ}_{ν} in terms of $(2n^2-n)$ real quantities k_t suitably chosen in conformity with the structural relation (1.14) and expressing the invariant $K^{\mu}_{\nu}K^{\nu}_{\mu}$ in terms of k_t one can remark that when K^{μ}_{ν} undergoes a unitary non-analytic spinor transformation, the vector k_t undergoes a real Lorentz transformation leaving invariant a quadratic form with n^2 positive squares and $n(n-1)$ negative squares.

§ 2. Let us now frame the $2n$-dimensional contact transformation in R_{2n} which is the analogue of the unitary non-analytic spinor transformation in S_n. Let P_{kl} denote the tensor in R_{2n} corresponding to $\eta_{\mu\nu}$ in S_n then (Ghosh 1966)

$$P_{kl} = S^{\mu}_{k}S^{\nu}_{l}\eta_{\mu\nu}, \qquad \text{.. .. .. .. (2.1)}$$

where the spin-tensors S^{μ}_{k} are defined by the non-vanishing components

$$S^{\lambda}_{\lambda-1} = 1, \; S^{\lambda}_{2n-\lambda} = i, \; S^{\dot{\lambda}}_{\lambda-1} = 1, \; S^{\dot{\lambda}}_{2n-\lambda} = -i. \qquad \text{.. .. (2.2)}$$

Evaluating (2.1), the $2n$ non-vanishing components of P_{kl} are given by

$$P_{\lambda-1,\,2n-\lambda} = 2, \; P_{2n-\lambda,\,\lambda-1} = -2, \; (\lambda = 1, 2, \ldots n). \qquad \text{.. .. (2.3)}$$

Postulating its invariance in R_{2n} under a $2n$-dimensional contact transformation we obtain the conditions expressed as follows:

$$\binom{\epsilon-1}{\lambda-1}\binom{2n-\epsilon}{\sigma-1} - \binom{2n-\epsilon}{\lambda-1}\binom{\epsilon-1}{\sigma-1} = 0,$$

$$\binom{\epsilon-1}{2n-\lambda}\binom{2n-\epsilon}{2n-\sigma} - \binom{2n-\epsilon}{2n-\lambda}\binom{\epsilon-1}{2n-\sigma} = 0,$$

$$\binom{\epsilon-1}{\lambda-1}\binom{2n-\epsilon}{2n-\sigma} - \binom{2n-\epsilon}{\lambda-1}\binom{\epsilon-1}{2n-\sigma} = 1, \text{ if } \lambda = \sigma$$

$$= 0, \text{ if } \lambda \neq \sigma \qquad \text{.. .. (2.4)}$$

where we have used the symbol $\binom{\epsilon}{\lambda}$ to denote $\partial x^{\epsilon}/\partial x'^{\lambda}$, the set of basic transformation equations defining the contact transformation being of the type

$$x'^{k} = \phi^{k}(x^{0}, x^{1}, \ldots x^{2n-1}) \quad (k = 0, 1, \ldots 2n-1) \qquad \ldots \qquad (2.5)$$

with its inverse.
Setting

$$x^{\epsilon-1} = q^{\epsilon},\ x^{2n-\epsilon} = p^{\epsilon},$$

$$\frac{\partial}{\partial x'^{\lambda-1}} = \frac{\partial}{\partial q'^{\lambda}},\ \frac{\partial}{\partial x'^{2n-\lambda}} = \frac{\partial}{\partial p'^{\lambda}},$$

one can verify that these are the conditions for $2n$-dimensional contact transformation in Lagrangian form. The contravariant tensor P^{kl} associated to P_{kl} will have $2n$ non-vanishing components

$$P^{\lambda-1,\,2n-\lambda} = \tfrac{1}{2}, \quad P^{2n-\lambda,\,\lambda-1} = -\tfrac{1}{2}. \qquad \ldots \qquad (2.6)$$

The tensors P^{kl}, P_{kl} may be regarded as metric tensors, the raising and lowering of indices being performed according to the scheme

$$A_{l}P^{kl} = A^{k}, \quad A^{k}P_{kl} = A_{l}, \qquad \ldots \qquad (2.7)$$

so that

$$P^{kl}P_{km} = P^{lk}P_{mk} = \delta^{l}_{m}.$$

It may be noted that the contravariant and covariant transformation coefficients with regard to a contact transformation are connected by the equation

$$\frac{\partial x'^{h}}{\partial x^{l}} = P_{kl}P^{mh}\frac{\partial x^{k}}{\partial x'^{m}}. \qquad \ldots \qquad (2.8)$$

This gives

$$\frac{\partial x'^{\sigma-1}}{\partial x^{\lambda-1}} = \frac{\partial x^{2n-\lambda}}{\partial x'^{2n-\sigma}}, \qquad \frac{\partial x'^{2n-\sigma}}{\partial x^{2n-\lambda}} = \frac{\partial x^{\lambda-1}}{\partial x'^{\sigma-1}},$$
$$\frac{\partial x'^{2n-\sigma}}{\partial x^{\lambda-1}} = -\frac{\partial x^{2n-\lambda}}{\partial x'^{\sigma-1}}, \quad \frac{\partial x'^{\sigma-1}}{\partial x^{2n-\lambda}} = -\frac{\partial x^{\lambda-1}}{\partial x'^{2n-\sigma}}. \qquad \ldots \qquad (2.9)$$

Using (2.9) in (2.4) we obtain the Poisson conditions in the usual form.

The analogue of the spinor K^{μ}_{ν} in S_{n} is the mixed tensor C^{m}_{h} in R_{2n} satisfying the structural equation

$$C^{m}_{h} = C^{k}_{l}P^{lm}P_{kh}, \qquad \ldots \qquad (2.10)$$

the form being preserved when C^{k}_{l} undergoes a contact transformation. Tensors of higher rank having this property will have special structure which can be formulated by suitably introducing P^{kl}, P_{kl}.

The bilinear differential form invariant under a contact transformation may be written as

$$P_{kl}dx'^{k}\delta x'^{l} = P_{kl}dx^{k}\delta x^{l}, \qquad \ldots \qquad (2.11)$$

d and δ being two differential symbols.

Introducing the metric tensor P_{kl}, the canonical equations of motion take the covariant vector form

$$P_{kl}\frac{dx^l}{dt} = \frac{\partial}{\partial x^k}(2H). \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (2.12)$$

ACKNOWLEDGEMENT

The author wishes to thank Professor S. N. Bose, F.N.I., for his kind interest and helpful comments.

REFERENCES

Ghosh, N. N. (1958). Studies in contact transformation over a complex space. *Bull. Calcutta math. Soc.*, **50**, 94.

———— (1966). Generalized spinors in a non-analytic complex n-space. *Proc. natn. Inst. Sci. India*, A **32**, 342–348.