EXISTENCE OF THE PERIODIC ORBIT IN THE PROBLEM OF THE MOTION OF AN ARTIFICIAL SATELLITE

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In this paper the motion of an artificial satellite has been considered in the gravitational field of an axially symmetric oblate spheroid and disturbed by another satellite. The existence of periodic orbits of Poincaré type as well as of Schwarzschild's type has been investigated. All types of periodic orbits seem to exist in this problem.

1. INTRODUCTION

This paper is an extension of our earlier papers (*in press, a* and *b*). In the earlier papers we studied the solution of the problem of motion of an artificial satellite in the gravitational field of an oblate spheroid and of the atmospheric drag. We took into account spherical harmonics up to the fourth order. Next we investigated the case when the perturbation due to another body is also included.

The problem of motion of an artificial satellite has been studied extensively. Brouwer (1959), Brouwer and Hori (1960), Brouwer and Clemence (1961) as also Volkov (1961) restricted their considerations only to spherical harmonics of the second order. Here we have discussed the existence of different types of periodic solutions containing spherical harmonics up to the fourth order.

Poincaré (1892) showed that the actual path of a celestial body can always be approximated by periodic solutions. This possibility raised a keen interest in us to explore if periodic solutions of the Poincaré type exist in the satellitical problem. The question of a solution when various other types of forces, such as the rotation of the oblate spheroid, the shape of its satellite, decrease in the mass of the satellite whose motion is under our consideration, etc., are taken into account remains open.

In this paper we have used the method of finding the periodic solution as was used in Singh and Choudhry (*in press, b*). Throughout the paper Kepler's solution of the two-body problem has been taken for the generating solution.

In section II, we explain the notations which have been used in the paper. In section III, existence of periodic solution of the first kind is predicted on the line of von Zeipel (1915). In section IV, periodic solutions of the second kind of Poincaré type as well as of Schwarzschild's type are shown to exist and in section V, periodic solutions of the third kind have been studied.

2. Equations of Motion

The equations of motion of an artificial satellite in the gravitational field of an axially symmetric oblate spheroid are:

$$\frac{d^2x}{dt^2} = \frac{\partial U}{\partial x};\ \frac{d^2y}{dt^2} = \frac{\partial U}{\partial y};\ \frac{d^2z}{dt^2} = \frac{\partial U}{\partial z} \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

where U is the potential function

$$U = \frac{1}{r}+\frac{k_2}{r^3}(1-3\sin^2\beta)+\frac{k_4}{r^5}\left(1-10\sin^2\beta+\frac{35}{3}\sin^4\beta\right)+\ldots$$

The equatorial plane of the spheroid is taken as the xy-plane; β is the latitude and the unit of time has been taken such that the gravitational constant $k^2 = 1$.

Equations (1) have the particular solution corresponding to the motion of the satellite along a circle in the equatorial plane (Moulton 1920).

$$x_1 = \cos n_1 t;\ y_1 = \sin n_1 t$$

where $n_1 = 1+k_2+k_4+\ldots$ — (mean motion of the satellite).

We introduce the Delaunay variables:

$L = \sqrt{a}$, l = mean anomaly,

$G = \sqrt{a(1-e^2)}$, g = angular distance of ascending node from the pericentre,

$H = \sqrt{a(1-e^2)}\cos I$, h = longitude of the ascending node,

where a—semi-major axis; e—eccentricity; I—inclination.

The equations of motion in Delaunay variables are given by

$$\left.\begin{aligned} \frac{dL}{dt} &= \frac{\partial F}{\partial l};\ \frac{dl}{dt} = -\frac{\partial F}{\partial L} \\ \frac{dG}{dt} &= \frac{\partial F}{\partial g};\ \frac{dg}{dt} = -\frac{\partial F}{\partial G} \\ \frac{dH}{dt} &= \frac{\partial F}{\partial h};\ \frac{dh}{dt} = -\frac{\partial F}{\partial H} \end{aligned}\right\} \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

with

$$F = \frac{1}{2L^2}-\frac{1}{r}+U.$$

The solution of this problem has been found up to the second order, i.e. up to the terms k_4 (Kozai 1962) in the form

$$\left.\begin{aligned} \frac{dL''}{dt} &= \frac{\partial F^{**}}{\partial l} = 0\,; \quad \frac{dl''}{dt} = -\frac{\partial F^{**}}{\partial L''} = \text{const} \\ \frac{dG''}{dt} &= \frac{\partial F^{**}}{\partial g} = 0\,; \quad \frac{dg''}{dt} = -\frac{\partial F^{**}}{\partial G''} = \text{const} \\ \frac{dH''}{dt} &= \frac{\partial F^{**}}{\partial h} = 0\,; \quad \frac{dh''}{dt} = -\frac{\partial F^{**}}{dH''} = \text{const} \end{aligned}\right\} \quad \ldots \quad \ldots \quad (3)$$

with $F^{**}(L'', G'', H'')$, provided the inclination is not too near the critical inclination (Brouwer and Hori 1960). The relations that express (L, G, H, l, g, h) in terms of $(L'', G'', H'', l'', g'', h'')$ have already been found in (Kozai 1962). We shall drop the dashes and write L, G, H, l, g, h for $L'', G'', H'', l'', g'', h''$ and F_0 for F^{**}.

3. Effect of the Natural Satellite

Considering the extra effect due to the moon, the equations of motion (2) become:

$$\left.\begin{aligned} \frac{dL}{dt} &= \frac{\partial F}{\partial l}\,; \quad \frac{dl}{dt} = -\frac{\partial F}{\partial L} \\ \frac{dG}{dt} &= \frac{\partial F}{\partial g}\,; \quad \frac{dg}{dt} = -\frac{\partial F}{\partial G} \\ \frac{dH}{dt} &= \frac{\partial F}{\partial h}\,; \quad \frac{dh}{dt} = -\frac{\partial F}{\partial H} \end{aligned}\right\} \quad \ldots \quad \ldots \quad \ldots \quad (4)$$

with the new Hamiltonian

$$F = F_0 + m'F,$$

where

$$F_0 = F_0(L, G, H)$$

$$F_1 = \frac{1}{\sqrt{1-2r\cos H + r^2}} - r\cos H - \frac{1}{r}.$$

Here r—radius vector of the artificial satellite;

H—angle with the vertex at the centre of the spheroid between the natural and artificial satellite;

m'—mass of the natural satellite.

In order to find the necessary and sufficient conditions for the existence of periodic solution of the first kind we introduce new canonical variables of Poincaré

$$\begin{aligned} &L, && L-G, && G-H \\ &\lambda = l+g+h, && u = -g-h, && u' = -h \\ &L, && \xi = \rho\cos u, && \xi' = \rho'\cos u' \\ &\lambda, && \eta = \rho\sin u, && \eta' = \rho'\sin u' \end{aligned}$$

where

$$\rho = \sqrt{2(L-G)}, \quad \rho' = \sqrt{2(G-H)}.$$

The equations of motion of the satellite in the new variables will have the form

$$\left.\begin{aligned}\frac{dL}{dt}=\frac{\partial F}{\partial \lambda};\ \frac{d\lambda}{dt}=-\frac{\partial F}{\partial L}\\ \frac{d\xi}{dt}=\frac{\partial F}{\partial \eta};\ \frac{d\eta}{dt}=-\frac{\partial F}{\partial \xi}\\ \frac{d\xi'}{dt}=\frac{\partial F}{\partial \eta'};\ \frac{d\eta'}{dt}=-\frac{\partial F}{\partial \xi'}\end{aligned}\right\} \quad \ldots \quad \ldots \quad \ldots \quad (5)$$

with

$$F = F_0+m'F_1.$$

The eqns. (5) have Jacobi's integral, not depending explicitly on time. Thus the expansion of the function F_1 takes the form (cf. von Zeipel 1915)

$$F_1 = \Sigma C^{m,\,2f}_{i,\,j,\,2s}\rho^m \rho'^{2f} \cos (i\lambda+ju+2su'),$$

where the coefficients C depend only on L. Besides, s is non-negative; j—non-negative; where $s = 0$, j is non-negative; when $j = s = 0$, i is non-negative. The differences $m-|j|$ and $2f-|2s|$ are always even and positive.

Since the form of the expansion of F_1 in our problem coincides with that of the disturbing function in the restricted problem of three bodies, therefore the results of von Zeipel (1915) hold good in our problem. Consequently under the conditions stated by von Zeipel the periodic solution of the first kind exists in our problem as well. The study of the periodic orbit of the first kind is not pursued as it will only be repeating von Zeipel's paper step by step.

4. Periodic Motion of Second Kind

The differential equations in canonical elements of Delaunay giving the motion of the artificial satellite in the equatorial plane have the form

$$\left.\begin{aligned}\frac{dL}{dt}=\frac{\partial F}{\partial l};\ \frac{dl}{dt}=-\frac{\partial F}{\partial L}\\ \frac{dG}{dt}=\frac{\partial F}{\partial g};\ \frac{dg}{dt}=-\frac{\partial F}{\partial G}\end{aligned}\right\} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (6)$$

with

$$F = F_0+m'F_1$$

$$F_0 = F_0(L, G)$$

$$F_1 = \frac{1}{\sqrt{1-2r \cos H+r^2}} -r \cos H-\frac{1}{r}.$$

For symmetry we introduce the new variables

$$x_1 = L;\ y_1 = l$$

$$x_2 = G;\ y_2 = g.$$

The equations of motion (6) are now given by

$$\frac{dx_i}{dt} = \frac{\partial F}{\partial y_i};\ \frac{dy_i}{dt} = -\frac{\partial F}{\partial x_i} \quad (i = 1, 2) \quad \dots \quad \dots \quad \dots \quad (7)$$

for $m' = 0$, the eqns. (7) can be written as

$$\frac{dx_i}{dt} = 0;\ \frac{dy_i}{dt} = -\frac{\partial F_0}{\partial x_i} = n_i \text{ (const)} \quad (i = 1, 2).$$

Therefore the generating solutions are given by

$$x_i^{(0)} = a_i;\ y_i^{(0)} = n_i^{(0)}t + \omega_i \quad (i = 1, 2)$$

where a_i and ω_i are arbitrary constants and

$$n_i^{(0)} = -\left.\frac{\partial F_0}{\partial x_i}\right|_{x_i = a_i}$$

This solution is said to be periodic with period T_0 if

$$y_i(T_0) - y_i(0) = n_i^{(0)}T_0 = 2k_i\pi$$

$$x_i(T_0) - x_i(0) = 0 \qquad (i = 1, 2).$$

Here k_i $(i = 1, 2)$ are integers, so that $n_i^{(0)}$ are commensurable.

Let the general solution in the neighbourhood of the generating solution be periodic with the period $T_0(1+\alpha)$ where α is a negligible quantity of order m'. Introducing new independent variables ζ by the formula

$$\zeta = \frac{t}{1+\alpha}$$

for which the period of the general solution will be T, the equations of motion (7) can be written as

$$\frac{dx_i}{d\zeta} = (1+\alpha)\frac{\partial F}{\partial y_i};\ \frac{dy_i}{d\zeta} = -(1+\alpha)\frac{\partial F}{\partial x_i} \quad (i = 1, 2) \quad \dots \quad \dots \quad (8)$$

and the general solution in the neighbourhood of the generating solution as

$$x_i = a_i + \beta_i + \xi_i(\zeta),\ \ y_i = n_i^{(0)}\zeta\omega_i + \gamma_i + \eta_i(\zeta) \quad (i = 1, 2).$$

The eqns. (8) in terms of the new variables ξ_i and η_i are

$$\frac{d\xi_i}{d\zeta} = \frac{\partial k}{\partial \eta_i};\ \frac{d\eta_i}{d\zeta} = -\frac{\partial k}{\partial \xi_i} \quad (i = 1, 2) \quad \dots \quad \dots \quad (9)$$

where

$$k = (1+\alpha)F(\zeta,\ \xi_i + a_i + \beta_i,\ \eta_i + n_i^{(0)}\zeta + \omega_i + \gamma_i) - F(\zeta,\ a_i + \beta_i,\ n_i^{(0)}\zeta + \omega_i + \gamma_i)$$

$$+ \sum_{i=1}^{2}\left[\xi_i n_i^{(0)} - n_i^{(0)} \times 0\right]$$

$$= \xi_1 n_1^{(0)} + \xi_2 n_2^{(0)} + \alpha F(\zeta,\ a_i + \beta_i,\ n_i^{(0)}\zeta + \omega_i + \gamma_i) + (1+\alpha)\sum_{i=1}^{2}\left[\frac{\partial F}{\partial a_i}\xi_i + \frac{\partial F}{\partial \omega_i}\eta_i\right]$$

$$+ \frac{(1+\alpha)}{2!}\sum_{i=1}^{2}\sum_{i=1}^{2}\left[\xi_i\xi_j\frac{\partial^2 F}{\partial a_i \partial a_j} + \xi_i\eta_j\frac{\partial^2 F}{\partial a_i \partial \omega_j} + \eta_i\eta_j\frac{\partial^2 F}{\partial \omega_i \partial \omega_j}\right] + O(\xi_i^3). \quad \dots \quad (10)$$

The necessary and sufficient conditions for the existence of the periodic solution are

$$x_i(T)-x_i(0) = \xi_i(T_0) = 0 \quad \ldots \quad (11)$$

$$y_i(T)-y_i(0)-2k_i\pi = \eta_i[T_0(1+\alpha)] = 0 \quad (i = 1, 2). \quad \ldots \quad (12)$$

Restricting our solution only to the first order infinitesimals, the eqns. (9) may be written as

$$\frac{d\xi_i}{d\zeta} = (1+\alpha)\frac{\partial F}{\partial \omega_i}\,;\; \frac{d\eta_i}{d\zeta} = -n_i^{(0)}-(1+\alpha)\frac{\partial F}{\partial a_i} \quad (i = 1, 2). \quad \ldots \quad (13)$$

Expanding $F(\zeta,\, a_i+\beta_i,\, n_i^{(0)}(\zeta)+\omega_i+\gamma_i)$ in ascending powers of $\beta_1, \beta_2, \gamma_1, \gamma_2, m'$, the eqns. (11) and (12) become

$$\frac{\xi_k(T_0, \beta_i, \gamma_i, m')}{m'T_0} = \frac{\partial[F_1]}{\partial\omega_k}+\left\{\sum_{j=1}^{2}\frac{\partial^2[F_1]}{\partial\omega_k\partial a_j}\beta_j+\sum_{j=1}^{2}\frac{\partial^2[F_1]}{\partial\omega_k\partial\omega_j}\gamma_j\right\}+\ldots$$

$$\eta_k(T_0, \beta_i, \gamma_i, m') = -T_0\alpha\frac{\partial F_0}{\partial a_k}-m'\left[T_0\frac{\partial[F_1]}{\partial a_k}-\int_0^{T_1}\frac{\partial^2 F_0}{\partial a_k^2}\int_0^t\frac{\partial F_1}{\partial\omega_k}\zeta\, d\rho\, dt\right.$$

$$-\beta_1 T_0\left[\frac{\partial^2 F_0}{\partial a_k\partial a_1}+m'\frac{\partial^2[F_1]}{\partial a_k\partial a_1}+\ldots\right]$$

$$-\beta_2 T_0\left[\frac{\partial^2 F_0}{\partial a_k\partial a_2}+m'\frac{\partial^2[F_1]}{\partial a_k\partial a_2}+\ldots\right]$$

$$-m'T_0\left[\gamma_1\frac{\partial^2[F_1]}{\partial a_k\partial\omega_1}+\gamma_2\frac{\partial^2[F_1]}{\partial a_k\partial\omega_2}\right]+\ldots \quad \ldots \quad \ldots \quad (14)$$

where $(k = 1, 2)$

$$[F_1] = \frac{1}{T_0}\int_0^{T_0}F_1(\zeta,\, a_i,\, n_i^{(0)}\rho+\omega_i)\, d\rho \quad (i = 1, 2).$$

(a) Periodic Orbit of Schwarzschild Type

In the problem of the existence of the periodic solution of Schwarzschild's type, for which the period of the solution differs from the period of the generating solution, we have $\alpha \neq 0$.

Since ζ is not involved explicitly in the characteristic function F, so $F = C$ is an integral. Hence any one of the above equations may be considered as a consequence of the remaining three. Let us assume that $\xi_1 = 0$ is the consequence of the remaining three equations. Thus for five unknowns β_i, γ_i and α, we have only three equations. Therefore, two unknowns (say γ_1 and β_2) can be taken arbitrarily. Thus let $\gamma_1 = \beta_2 = 0$ and again, as the choice of the origin of time is arbitrary, we may take $\omega_1 = 0$. Then the eqns. (14) will give β_1, γ_2, α as holomorphic functions of m', reducing to zero with m' if the following conditions are satisfied:

$$\frac{\partial[F_1]}{\partial\omega_k} = 0 \quad (k = 1, 2) \quad \ldots \quad \ldots \quad (15)$$

and

$$\frac{\partial(\xi_2, \eta_1, \eta_2)}{\partial(\gamma_2, \beta_1, \alpha)} \neq 0. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (16)$$

The Jacobian (16) can be written as

$$\begin{vmatrix} \frac{\partial \xi_2}{\partial \gamma_2} & \frac{\partial \xi_2}{\partial \beta_1} & \frac{\partial \xi_2}{\partial \alpha} \\ \frac{\partial \eta_1}{\partial \gamma_2} & \frac{\partial \eta_1}{\partial \beta_1} & \frac{\partial \eta_1}{\partial \alpha} \\ \frac{\partial \eta_2}{\partial \gamma_2} & \frac{\partial \eta_2}{\partial \beta_1} & \frac{\partial \eta_1}{\partial \alpha} \end{vmatrix} \neq 0,$$

i.e.

$$\begin{vmatrix} \frac{\partial^2[F_1]}{\partial \omega_2^2} & 0 & 0 \\ 0 & \frac{\partial^2 F_0}{\partial a_1^2} & \frac{\partial F_0}{\partial a_1} \\ 0 & \frac{\partial^2 F_0}{\partial a_2 \partial a_1} & \frac{\partial F_0}{\partial a_2} \end{vmatrix} \neq 0,$$

i.e.

$$\frac{\partial^2[F_1]}{\partial \omega_2^2} \cdot \begin{vmatrix} \frac{\partial^2 F_0}{\partial a_1^2} & \frac{\partial F_0}{\partial a_1} \\ \frac{\partial^2 F_0}{\partial a_2 \partial a_1} & \frac{\partial F_0}{\partial a_2} \end{vmatrix} \neq 0. \quad \ldots \quad \ldots \quad \ldots \quad (17)$$

Also if we take $\beta_1 = 0$ instead of β_2, the Jacobian (16) will be of the form

$$\frac{\partial^2[F_1]}{\partial \omega_2^2} \cdot \begin{vmatrix} \frac{\partial^2 F_0}{\partial a_1 \partial a_2} & \frac{\partial F_0}{\partial a_1} \\ \frac{\partial^2 F_0}{\partial a_2^2} & \frac{\partial F_0}{\partial a^2} \end{vmatrix} \neq 0. \quad \ldots \quad \ldots \quad \ldots \quad (18)$$

Thus we have deduced that our problem will have periodic solution of the second kind for a period not coinciding with the generating solution if conditions (15) and any one of the conditions (17) or (18) are satisfied.

As

$$[F_1] = \frac{1}{T_0} \int_0^{T_0} F_1 \, dt;$$

where T_0 is the period of the generating solution, we shall have

$$[F_1] = \Sigma A \cos (i\omega_1 + i'\omega_2),$$

where the summation is over all the combinations $in - i' = 0$. Here $n = \frac{p}{q}$, p and q being mutually prime integers. Further $i\frac{p}{q} - i' = 0$ or $ip - i'q = 0$.

Hence $i = sq$, $i' = sp$, where s is an arbitrary integer.

Whence
$$[F_1] = \sum_{s=0}^{\infty} A \cos s(q\omega_1+p\omega_2).$$

Conditions (15) show that
$$\frac{\partial[F_1]}{\partial\omega_1} = 0 = \sin s(q\omega_1+p\omega_2) = \frac{\partial[F_1]}{\partial\omega_2}.$$

So either $\omega_1 = \omega_2 = 0$; or $\omega_1 = 0$, $\omega_2 = \pi$; or $\omega_1 = \omega_2 = \pi$; or $\omega_1 = \pi$, $\omega_2 = 0$. Expanding $[F_1]$ in the neighbourhood of $\omega_1 = 0 = \omega_2$,
$$[F_1](\omega_1, \omega_2)-[F_1](0, 0) = \frac{1}{2}\frac{\partial^2[F_1]}{\partial\omega_1^2}\omega_1^2+\frac{\partial^2[F_1]}{\partial\omega_1\partial\omega_2}\omega_1\omega_2+\frac{1}{2}\frac{\partial^2[F_1]}{\partial\omega_2^2}.$$

Since $[F_1](0, 0)$ is a stationary value, the L.H.S. must be of constant sign and so
$$\frac{\partial^2[F_1]}{\partial\omega_1^2}\cdot\frac{\partial^2[F_1]}{\partial\omega_2^2}-\left(\frac{\partial^2[F_1]}{\partial\omega_1\partial\omega_2}\right)^2 > 0$$
as well as
$$\frac{\partial^2[F_1]}{\partial\omega_2^2} \neq 0.$$

Thus the condition (17) or (18) may be satisfied if one of the determinants
$$\begin{vmatrix} \dfrac{\partial^2 F_0}{\partial a_1^2} & \dfrac{\partial F_0}{\partial a_1} \\ \dfrac{\partial^2 F_0}{\partial a_1\partial a_2} & \dfrac{\partial F_0}{\partial a_2} \end{vmatrix} \quad \text{or} \quad \begin{vmatrix} \dfrac{\partial^2 F_0}{\partial a_2\partial a_1} & \dfrac{\partial F_0}{\partial a_1} \\ \dfrac{\partial^2 F_0}{\partial a_2^2} & \dfrac{\partial F_2}{\partial a_2} \end{vmatrix}$$
does not vanish.

Let us suppose that both of these determinants vanish. Then we have the condition
$$\frac{\partial^2 F_0/\partial a_1^2}{\partial^2 F_0/\partial a_1\partial a_2} = \frac{\partial^2 F_0/\partial a_2\partial a_1}{\partial^2 F_0/\partial a_2^2},$$

i.e.
$$\frac{\partial^2 F_0}{\partial a_1^2}\cdot\frac{\partial^2 F_0}{\partial a_2^2}-\frac{\partial^2 F_0}{\partial a_1\partial a_2}\cdot\frac{\partial^2 F_0}{\partial a_2\partial a_1} = 0,$$

i.e.
$$\begin{vmatrix} \dfrac{\partial^2 F_0}{\partial a_1^2} & \dfrac{\partial^2 F_0}{\partial a_1\partial a_2} \\ \dfrac{\partial^2 F_0}{\partial a_2\partial a_1} & \dfrac{\partial^2 F_0}{\partial a_2^2} \end{vmatrix} = 0. \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (19)$$

But as F_0 is a continuous function and since a continuous function always attains its upper and lower bounds, the determinant (19) can never be equal to zero and, therefore, our supposition is fallacious. This proves the existence of the periodic solution of second kind of Schwarzschild's type for which the period of general solution does not coincide with the period of generating solution.

(*b*) *Periodic Solution of Poincaré Type*

In the case of the periodic solution of Poincaré type of the second kind for which the period of the general solution and generating solution coincide, we have $\alpha = 0$ and, therefore, we shall have to take only $\gamma_1 = 0$. Then determination of β_1, β_2, γ_2 as holomorphic functions of m', reducing to zero with m', is possible only when

$$\frac{\partial[F_1]}{\partial \omega_k} = 0 \quad (k = 1, 2). \qquad \ldots \quad \ldots \quad \ldots \quad (20)$$

And the condition for the solvability of the equations will be that the Jacobian

$$\frac{\partial(\xi_2, \eta_1, \eta_2)}{\partial(\gamma_2, \beta_1, \beta_2)} \neq 0,$$

i.e.
$$\begin{vmatrix} \frac{\partial \xi_2}{\partial \gamma_2} & \frac{\partial \xi_2}{\partial \beta_1} & \frac{\partial \xi_2}{\partial \beta_2} \\ \frac{\partial \eta_1}{\partial \gamma_2} & \frac{\partial \eta_1}{\partial \beta_1} & \frac{\partial \eta_1}{\partial \beta_2} \\ \frac{\partial \eta_2}{\partial \gamma_2} & \frac{\partial \eta_2}{\partial \beta_1} & \frac{\partial \eta_2}{\partial \beta_2} \end{vmatrix} \neq 0,$$

i.e.
$$\begin{vmatrix} \frac{\partial^2[F_1]}{\partial \omega_2^2} & 0 & 0 \\ 0 & \frac{\partial^2 F_0}{\partial a_1^2} & \frac{\partial^2 F_0}{\partial a_2 \partial a_1} \\ 0 & \frac{\partial^2 F_0}{\partial a_1 \partial a_2} & \frac{\partial^2 F_0}{\partial a_2^2} \end{vmatrix} \neq 0,$$

i.e.
$$\frac{\partial^2[F_1]}{\partial \omega_2^2} \begin{vmatrix} \frac{\partial^2 F_0}{\partial a_1^2} & \frac{\partial^2 F_0}{\partial a_2 \partial a_1} \\ \frac{\partial^2 F_0}{\partial a_1 \partial a_2} & \frac{\partial^2 F_0}{\partial a_2^2} \end{vmatrix} \neq 0. \qquad \ldots \quad \ldots \quad \ldots \quad (21)$$

But the conditions (20) and (21) are satisfied from the preceding case and hence the problem has periodic solution of the second kind of Poincaré type.

5. Periodic Motion of the Third Kind

For the periodic solution of the third kind, proceeding as in that for the second kind, we obtain the eqns. (11) and (12) in the form

$$x_i(T)-x_i(0) = \xi_i(T_0) = 0 \quad \ldots \quad (22)$$

$$y_i(T)-y_i(0)-2k_i\pi = \eta_i T_0(1+\alpha) = 0 \quad (i = 1, 2, 3) \quad \ldots \quad (23)$$

and the eqns. (14) take the form

$$\frac{\xi_k(T_0, \beta_i, \gamma_i, m')}{m'T_0} = \frac{\partial[F_1]}{\partial\omega_k}+\sum_{j=1}^{3}\frac{\partial^2[F_1]}{\partial\omega_k\partial a_j}\beta_j+\sum_{j=1}^{3}\frac{\partial^2[F_1]}{\partial\omega_k\partial\omega_j}\gamma_j+\ldots \quad \ldots \quad (24)$$

$$\eta_k(T_0, \beta_i, \gamma_i, m') = -T_0\alpha\frac{\partial F_0}{\partial a_k}-m'\left[T_0\frac{\partial[F_1]}{\partial a_k}-\int_0^{T_1}\frac{\partial^2 F_0}{\partial a_k^2}\int_0^t\frac{\partial F_1}{\partial\omega_k}\zeta\, d\rho\, dt\right]$$

$$-\beta_1 T_0\left[\frac{\partial^2 F_0}{\partial a_k\partial a_1}+m'\frac{\partial^2[F_1]}{\partial a_k\partial a_1}+\ldots\right]$$

$$-\beta_2 T_0\left[\frac{\partial^2 F_0}{\partial a_k\partial a_2}+m'\frac{\partial^2[F_1]}{\partial a_k\partial a_2}+\ldots\right]$$

$$-\beta_3 T_0\left[\frac{\partial^2 F_0}{\partial a_k\partial a_3}+m'\frac{\partial^2[F_1]}{\partial a_k\partial a_3}+\ldots\right]$$

$$-m'T_0\left[\frac{\partial^2[F_1]}{\partial a_k\partial\omega_1}\gamma_1+\frac{\partial^2[F_1]}{\partial a_k\partial\omega_2}\gamma_2+\frac{\partial^2[F_1]}{\partial a_k\partial\omega_3}\gamma_3\right]+\ldots \quad (25)$$

where $(k = 1, 2, 3)$.

(a) Periodic Solution of Schwarzschild Type

For the problem of the existence of the periodic solution of the third kind of the Schwarzschild type for which the period of general solution differs from the period of the generating solution, we have $\alpha \neq 0$.

Now since ζ is not involved explicitly in the characteristic function F, so $F = C$ is an integral, hence the first of the above six equations may be considered as the consequence of the remaining five. Thus for seven unknowns β_i, γ_i and α, we have only five equations, therefore two unknowns (say γ_1 and β_3) may be taken arbitrarily. Let $\gamma_1 = \beta_3 = 0$. Again the choice of the origin of time is arbitrary so we may take $\omega_1 = 0$, then eqns. (24) give $\gamma_2, \gamma_3, \beta_1, \beta_2, \alpha$ as a holomorphic function of m', reducing to zero with m' if the following conditions are satisfied:

$$\frac{\partial[F_1]}{\partial\omega_k} = 0 \quad (k = 1, 2, 3) \quad \ldots \quad (26)$$

and

$$\frac{\partial(\xi_2, \xi_3, \eta_1, \eta_2, \eta_3)}{\partial(\gamma_2, \gamma_3, \beta_1, \beta_2, \alpha)} \neq 0. \quad \ldots \quad (27)$$

The Jacobian (27) can be written as

$$\begin{vmatrix} \dfrac{\partial^2[F_1]}{\partial\omega_2^2} & \dfrac{\partial^2[F_1]}{\partial\omega_2\partial\omega_3} & 0 & 0 & 0 \\ \dfrac{\partial^2[F_1]}{\partial\omega_3\partial\omega_2} & \dfrac{\partial^2[F_1]}{\partial\omega_3^2} & 0 & 0 & 0 \\ 0 & 0 & \dfrac{\partial^2 F_0}{\partial a_1^2} & \dfrac{\partial^2 F_0}{\partial a_1\partial a_2} & \dfrac{\partial^2 F_0}{\partial a_1} \\ 0 & 0 & \dfrac{\partial^2 F_0}{\partial a_1\partial a_2} & \dfrac{\partial^2 F_0}{\partial a_2^2} & \dfrac{\partial F_0}{\partial a_2} \\ 0 & 0 & \dfrac{\partial^2 F_0}{\partial a_1\partial a_3} & \dfrac{\partial^2 F_0}{\partial a_2\partial a_3} & \dfrac{\partial F_0}{\partial a_3} \end{vmatrix} \neq 0$$

$$\begin{vmatrix} \dfrac{\partial^2[F_1]}{\partial\omega_2^2} & \dfrac{\partial^2[F_1]}{\partial\omega_2\partial\omega_3} \\ \dfrac{\partial^2[F_1]}{\partial\omega_3\partial\omega_2} & \dfrac{\partial^2[F_1]}{\partial\omega_3^2} \end{vmatrix} \times \begin{vmatrix} \dfrac{\partial^2 F_0}{\partial a_1^2} & \dfrac{\partial^2 F_0}{\partial a_1\partial a_2} & \dfrac{\partial F_0}{\partial a_1} \\ \dfrac{\partial^2 F_0}{\partial a_1\partial a_2} & \dfrac{\partial^2 F_0}{\partial a_2^2} & \dfrac{\partial F_0}{\partial a_3} \\ \dfrac{\partial^2 F_0}{\partial a_1\partial a_3} & \dfrac{\partial^2 F_0}{\partial a_2\partial a_3} & \dfrac{\partial F_0}{\partial a_3} \end{vmatrix} \neq 0. \qquad \text{(28)}$$

The existence of the periodic solution follows if the two determinants in (28) are not zero.

Since

$$[F_1] = \sum_{i_1, i_2, i_3} C_{i_1, i_2, i_3}(a_1, a_2, a_3) \cos\left[i_1(n_1^{(0)}\zeta+\omega_1)+i_2(n_2^{(0)}\zeta+\omega_2)+i_3(n_3^{(0)}\zeta+\omega_3)\right]$$

$$\therefore \quad [F_1] = \sum_{i_1, i_2, i_3} C_{i_1, i_2, i_3}(a_1, a_2, a_3) \cos(i_1\omega_1+i_2\omega_2+i_3\omega_3).$$

From the conditions (25) we have

$$\frac{\partial[F_1]}{\partial\omega_2} = 0 = \sin(i_2\omega_2+i_3\omega_3) = \frac{\partial[F_1]}{\partial\omega_3}.$$

So either $\omega_2 = \omega_3 = 0$; or $\omega_2 = 0$, $\omega_3 = \pi$; or $\omega_2 = \pi$, $\omega_3 = \pi$; or $\omega_2 = \pi$, $\omega_3 = 0$, which show that the conditions of symmetric conjugations or oppositions are the necessary conditions for existence of the periodic solutions and which coincide with the conditions already obtained. For definiteness, let $\omega_2 = \omega_3 = 0$ in which case we find that

$$\left.\frac{\partial^2[F_1]}{\partial\omega_2^2}\right|_{\omega_2=\omega_3=0} \neq 0, \quad \left.\frac{\partial^2[F_1]}{\partial\omega_i\partial\omega_j}\right|_{\omega_2=\omega_3=0} \neq 0.$$

$$[i = 1, 2, 3;\ j = 1, 2, 3;\ i \neq j]$$

Now expanding $[F_1]$ in the neighbourhood of $\omega_2 = 0 = \omega_3$, we find that

$$[F_1](\omega_2, \omega_3) - [F_1](0, 0) = \frac{1}{2}\frac{\partial^2[F_1]}{\partial\omega_2^2}\omega_2^2 + \frac{\partial^2[F_1]}{\partial\omega_2\partial\omega_3}\omega_2\omega_3 + \frac{1}{2}\frac{\partial^2[F_1]}{\partial\omega_3^2}\omega_3^2.$$

Since $[F_1](0, 0)$ is a stationary value so the L.H.S. must be of constant sign and so

$$\frac{\partial^2[F_1]}{\partial\omega_2^2}\cdot\frac{\partial^2[F_1]}{\partial\omega_3^2} - \left(\frac{\partial^2[F_1]}{\partial\omega_2\partial\omega_3}\right)^2 > 0. \qquad \ldots \quad \ldots \quad (28A)$$

So the first determinant of the L.H.S. of (28) does not vanish. It remains to show that the second determinant also does not vanish. On the contrary let us assume that it vanishes

i.e.
$$\begin{vmatrix} \frac{\partial^2 F_0}{\partial a_1^2} & \frac{\partial^2 F_0}{\partial a_1\partial a_2} & \frac{\partial F_0}{\partial a_1} \\ \frac{\partial^2 F_0}{\partial a_2\partial a_1} & \frac{\partial^2 F_0}{\partial a_2^2} & \frac{\partial F_0}{\partial a_2} \\ \frac{\partial^2 F_0}{\partial a_3\partial a_1} & \frac{\partial^2 F_0}{\partial a_3\partial a_2} & \frac{\partial F_0}{\partial a_3} \end{vmatrix} = 0,$$

i.e.
$$\frac{\partial F_0}{\partial a_1}\begin{vmatrix} \frac{\partial^2 F_0}{\partial a_1\partial a_2} & \frac{\partial^2 F_0}{\partial a_2^2} \\ \frac{\partial^2 F_0}{\partial a_1\partial a_3} & \frac{\partial^2 F_0}{\partial a_2\partial a_3} \end{vmatrix} - \frac{\partial F_0}{\partial a_2}\begin{vmatrix} \frac{\partial^2 F_0}{\partial a_1^2} & \frac{\partial^2 F_0}{\partial a_1\partial a_2} \\ \frac{\partial^2 F_0}{\partial a_1\partial a_3} & \frac{\partial^2 F_0}{\partial a_2\partial a_3} \end{vmatrix} + \frac{\partial F_0}{\partial a_3}\begin{vmatrix} \frac{\partial^2 F_0}{\partial a_1^2} & \frac{\partial^2 F_0}{\partial a_1\partial a_2} \\ \frac{\partial^2 F_0}{\partial a_1\partial a_2} & \frac{\partial^2 F_0}{\partial a_2^2} \end{vmatrix} = 0. \qquad \ldots \quad (29)$$

Let us denote the minors of $\frac{\partial^2 F_0}{\partial a_r \partial a_s}$ by A_{rs} in the determinant

$$D = \begin{vmatrix} \frac{\partial^2 F_0}{\partial a_1^2} & \frac{\partial^2 F_0}{\partial a_1\partial a_2} & \frac{\partial^2 F_0}{\partial a_1\partial a_3} \\ \frac{\partial^2 F_0}{\partial a_2\partial a_1} & \frac{\partial^2 F_0}{\partial a_2^2} & \frac{\partial^2 F_0}{\partial a_2\partial a_3} \\ \frac{\partial^2 F_0}{\partial a_3\partial a_1} & \frac{\partial^2 F_0}{\partial a_3\partial a_2} & \frac{\partial^2 F_0}{\partial a_3^2} \end{vmatrix} \qquad \ldots \quad \ldots \quad (30)$$

Then from (29) we have

$$\frac{\partial F_0}{\partial a_1}A_{13} - \frac{\partial F_0}{\partial a_2}A_{23} + \frac{\partial F_0}{\partial a_3}A_{33} = 0. \qquad \ldots \quad \ldots \quad (31)$$

Similarly, if we take β_1 and β_2 zeros instead of β_3 successively and if they vanish as well, we shall get

$$\frac{\partial F_0}{\partial a_1}A_{12} - \frac{\partial F_0}{\partial a_2}A_{22} + \frac{\partial F_0}{\partial a_3}A_{32} = 0, \qquad \ldots \quad \ldots \quad \ldots \quad (32)$$

$$\frac{\partial F_0}{\partial a_1}A_{11} - \frac{\partial F_0}{\partial a_2}A_{21} - \frac{\partial F_0}{\partial a_3}A_{31} = 0. \qquad \ldots \quad \ldots \quad \ldots \quad (33)$$

Eliminating $\frac{\partial F_0}{\partial a_1}, \frac{\partial F_0}{\partial a_2}, \frac{\partial F_0}{\partial a_3}$ from (31), (32), (33) we get

$$\begin{vmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{vmatrix} = 0, \quad \text{i.e. } D^2 = 0.$$

But D is not zero as F_0 is a continuous function in a_1, a_2 and a_3 if a_1, a_2, a_3 are not very close to the critical case. So in the region of continuity of F_0, it must attain its bounds, both upper as well as lower. Thus the determinant D must be of constant sign. It shows that $D \neq 0$. Hence our supposition that all the three determinants vanish leads to a contradiction. Hence periodic solution of this type exists.

(b) Periodic Motion of Poincaré Type

Now let us consider the problem of the existence of the periodic solution of the third kind of Poincaré type for which the period of the general solution coincides with the period of the generating solution. Here $\alpha = 0$ and thus we have to choose only one variable of the six β_i and γ_i arbitrarily (say, $\gamma_1 = 0$).

Then the eqns. (24) give $\gamma_2, \gamma_3, \beta_1, \beta_2, \beta_3$ as a holomorphic function of m', reducing to zero with m' if the following conditions are satisfied:

$$\frac{\partial[F_1]}{\partial \omega_k} = 0 \quad (k = 1, 2, 3) \qquad \ldots \qquad (34)$$

and

$$\frac{\partial(\xi_2, \xi_3, \eta_1, \eta_2, \eta_3)}{\partial(\gamma_2, \gamma_3, \beta_1, \beta_2, \beta_3)} \neq 0. \qquad \ldots \qquad (35)$$

The Jacobian can be written as

$$\begin{vmatrix} \frac{\partial^2[F_1]}{\partial \omega_2^2} & \frac{\partial^2[F_1]}{\partial \omega_2 \partial \omega_3} & 0 & 0 & 0 \\ \frac{\partial^2[F_1]}{\partial \omega_2 \partial \omega_3} & \frac{\partial^2[F_1]}{\partial \omega_3^2} & 0 & 0 & 0 \\ 0 & 0 & \frac{\partial^2 F_0}{\partial a_1^2} & \frac{\partial^2 F_0}{\partial a_1 \partial a_2} & \frac{\partial^2 F_0}{\partial a_1 \partial a_2} \\ 0 & 0 & \frac{\partial^2 F_0}{\partial a_2 \partial a_1} & \frac{\partial^2 F_0}{\partial a_2^2} & \frac{\partial^2 F_0}{\partial a_2 \partial a_3} \\ 0 & 0 & \frac{\partial^2 F_0}{\partial a_3 \partial a_1} & \frac{\partial^2 F_0}{\partial a_3 \partial a_2} & \frac{\partial^2 F_0}{\partial a_3^2} \end{vmatrix} \neq 0,$$

i.e.
$$\begin{vmatrix} \frac{\partial^2[F_1]}{\partial\omega_2^2} & \frac{\partial^2[F_1]}{\partial\omega_2\partial\omega_3} \\ \frac{\partial^2[F_1]}{\partial\omega_2\partial\omega_3} & \frac{\partial^2[F_1]}{\partial\omega_3^2} \end{vmatrix} \times \begin{vmatrix} \frac{\partial^2F_0}{\partial a_1^2} & \frac{\partial^2F_0}{\partial a_1\partial a_2} & \frac{\partial^2F_0}{\partial a_1\partial a_3} \\ \frac{\partial^2F_0}{\partial a_2\partial a_1} & \frac{\partial^2F_0}{\partial a_2^2} & \frac{\partial^2F_0}{\partial a_2\partial a_3} \\ \frac{\partial^2F_0}{\partial a_3\partial a_1} & \frac{\partial^2F_0}{\partial a_3\partial a_2} & \frac{\partial^2F_0}{\partial a_3^2} \end{vmatrix} \neq 0. \qquad (36)$$

The non-vanishing of the first determinant follows from (28A) and the second determinant coincides with D which has already been shown to be non-vanishing in the previous section 5(a).

Hence the condition (36) is also satisfied. Thus our problem has periodic solution of the third kind of Poincaré type, which proves that all kind of periodic solutions exists for our problem. Therefore, we have established the fact that $(L'', G'', H'', l'', g'', h'')$ are periodic. But $(L'', G'', H'', l'', g'', h'')$ are given in terms of (L, G, H, l, g, h) by Kozai (1962, formulas (6.32), (6.33), (6.34) and (6.35)), and hence (L, G, H, l, g, h) are also periodic.

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