A FINITE TRANSFORM INVOLVING SPHEROIDAL WAVE FUNCTION AND ITS APPLICATIONS

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In a recent paper Gupta (1964) has investigated a finite transform involving Mathieu functions analogous to finite Hankel transform and discussed its properties together with its application in solving the diffusion equation. Hitherto, however, a corresponding use of the spheroidal wave functions does not seem to have been made for the solution of boundary value problems involving spheroidal wave functions. The aim of the present paper is to supply this deficiency.

§ 1. In this paper we introduce a transform applicable to spheroidal wave functions analogous to finite Fourier's transform and give an account of some of the simple properties of this new transform as well as its application to the solution of a few boundary value problems relating to spheroids. As regards the spheroidal wave functions the notation used is as given in Flammer (1957).

Definition I: If any function of (ξ, η) is continuous and single-valued within the spheroid and vanishes on the boundary, then its finite transform analogous to the finite Hankel Transform in the range $1 \leqslant \xi \leqslant \xi_0$; $-1 \leqslant \eta \leqslant 1$, is defined to be

$$K_{mn}(C_{n,p}) = \int_1^{\xi_0} \int_{-1}^{+1} (\xi^2-\eta^2) K_{mn}(\xi, \eta) R_{mn}^{(1)}(C_{n,p}, \xi) S_{mn}^{(1)}(C_{n,p}, \eta)\, d\xi\, d\eta, \quad \ldots \quad (1.1)$$

where $C_{n,p}$ is the root of the equation

$$R_{mn}^{(1)}(C_n, \xi_0) = 0, \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (1.2)$$

and following Flammer (1957, p. 31)

$$R_{mn}^{(1)}(C_n, \xi) = \frac{1}{\sum\limits_{r=0,1}^{\infty}{}' d_r^{mn}(C_n) \frac{(2m+r)!}{(r)!}} \cdot \left(\frac{\xi^2-1}{\xi^2}\right)^{\frac{m}{2}} \sum_{r=0,1}^{\infty}{}' i^{r-n+m} \frac{(2m+r)!}{(r)!} \sqrt{\frac{\pi}{2C_n\xi}}\, J_{n+m+\frac{1}{2}}(C_n\xi),$$

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or following the series expansion (Flammer 1957, p. 34)

$$R^{(1)}_{mn}(C_n, \xi) = [k_{mn}(C_n)]^{-1}(\xi^2-1)^{\frac{m}{2}} \sum_{k=0}^{\infty} (-1)^k C^{mn}_{2k}(\xi^2-1)^k, \quad (n-m) \text{ even}$$

$$R^{(1)}_{mn}(C_n, \xi) = [k_{mn}(C_n)]^{-1}\xi(\xi^2-1)^{\frac{m}{2}} \sum_{k=0}^{\infty} (-1)^k C^{mn}_{2k}(\xi^2-1)^k, \quad (n-m) \text{ odd}$$

where

$$C^{mn}_{2k} = 2^{-m}\{(k)!\ (m+k)!\}^{-1} \sum_{r=k}^{\infty} \frac{(2m+2r)!}{(2r)!}(-r)_k(m+r+\tfrac{1}{2})_k d^{mn}_{2r}(C_n), \quad (n-m) \text{ even}$$

$$C^{mn}_{2k} = 2^{-m}\{(k)!\ (m+k)!\}^{-1} \sum_{r=k}^{\infty} \frac{(2m+2r+1)!}{(2r+1)!}(-r)_k(m+r+\tfrac{3}{2})_k d^{mn}_{2r+1}(C_n), \ (n-m) \text{ odd}$$

and

$$k^{(1)}_{mn}(C_n) = \frac{(2m+1)(n+m)!\sum_{r=0}^{\infty}{}' d^{mn}_r(C_n)\dfrac{\lfloor 2m+r}{\lfloor r}}{2^{n+m}d^{mn}_0(C_n)\,.\,C^m_n \lfloor m \left(\dfrac{n-m}{2}\right)!\left(\dfrac{n+m}{2}\right)!}, \quad (n-m) \text{ even}$$

$$k^{(1)}_{mn}(C_n) = \frac{(2m+3)\lfloor (m+n+1) \sum_{r=0}^{\infty}{}' d^{mn}_r(C_n)\dfrac{(2m+r)!}{\lfloor r}}{2^{n+m}d^{mn}_1(C_n)C^{m+1}_n \lfloor m \left(\dfrac{n-m-1}{2}\right)!\left(\dfrac{n+m+1}{2}\right)!}, \quad (n-m) \text{ odd}$$

whereas

$$(\alpha)_k = \alpha(\alpha+1)\ \dots\ (\alpha+k+-1), \text{ and } (\alpha)_0 = 1,$$

$$S^{(1)}_{nm}(C_n, \eta) = \sum_{r=0,1}^{\infty}{}' d^{mn}_r(C_n)P^m_{m+r}(\eta),$$

then at any point within the range

$$K_{mn}(\xi, \eta) = \sum_{n=1}^{\infty} B_n R^{(1)}_{nm}(C_n, \xi)S^{(1)}_{n,m}(C_n, \eta), \ \ n \text{ being odd or even}$$

$$= \sum_{n=1}^{\infty}\sum_{p=1}^{\infty} B_n R^{(1)}_{nm}(C_{n,p}, \xi)S^{(1)}_{nm}(C_{n,p}, \eta), \qquad \dots \qquad (1.3)$$

where constant coefficinets B_n are to be determined by the following method.

Multiplying equation (1.3) on both sides by $(\xi^2-\eta^2)R^{(1)}_{mn}(C_{n,p}, \xi)$ $S^{(1)}_{nm}(C_{n,p}, \eta)$ and integrating it with respect to η between the limits -1 to

1, and with respect to ξ, from 1 to ξ_0. Then by the orthogonal property (Flammer 1957, p. 22) we have

$$B_n = \frac{\int_1^{\xi_0}\int_{-1}^{1} K_{mn}(\xi,\eta)(\xi^2-n^2)R^{(1)}_{mn}(C_{n,p},\xi)S^{(1)}_{nm}(C_{n,p},\eta)\,d\xi\,d\eta}{N_{nm}\int_{+1}^{\xi_0}[R^{(1)}_{nm}]^2[\xi^2-\Theta_{n,p}]\,d\xi}, \quad (1.4)$$

where

$$\Theta_{n,p} = \frac{1}{N_{nm}}\int_{-1}^{1}\eta^2[S^{(1)}_{nm}(C_{n,p},\eta)]^2\,d\eta;$$

and

$$N_{nm} = 2\sum_{r=0,1}^{\infty}{}' \frac{(2m+r)!}{\underline{|r}(2m+2r+1)!}[d^{mn}_r(C_{n,p})]^2.$$

Hence, the inversion formula is given by

$$K_{mn}(\xi,\eta) = \sum_{n=1}^{\infty}\sum_{p=1}^{\infty}\frac{\bar{K}_{mn}(C_{n,p})R^{(1)}_{mn}(C_{n,p},\xi)S^{(1)}_{nm}(C_{n,p},\eta)}{N_{nm}\int_1^{\xi_0}\{R^{(1)}_{nm}(C_{n,p},\xi)\}^2[\xi^2-\Theta_{n,p}]\,d\xi}, \quad .. \quad (1.5)$$

where the sum is extended over all the positive roots of equation (1.2).

We find that the transform defined by Definition I is of use when the variable ξ lies in the interval $1 \leqslant \xi \leqslant \xi_0$ and the range of variation of ξ does not include $\xi = 1$, i.e. ξ lies in the interval $1 < \xi \leqslant \xi_0$. We make use of another transform defined by the following definitions.

Definition II: If $K_{mn}(\xi,\eta)$ is continuous and single-valued in the region $\xi_0 \leqslant \xi \leqslant \xi_1$, $-1 \leqslant \eta \leqslant 1$ and vanishes on $\xi = \xi_1$, then its finite transform in the range is defined by

$$\bar{K}_{mn}(C_{n,p}) = \int_{\xi_0}^{\xi_1}\int_{-1}^{1} K_{mn}(\xi,\eta)(\xi^2-\eta^2)B_{mn}(\xi)S^{(1)}_{nm}(C_{n,p},\eta)\,d\xi\,d\eta, \quad .. \quad (1.6)$$

where

$$B_{nm}(\xi) = R^{(1)}_{nm}(C_{n,p},\xi)R^{(2)'}_{nm}(C_{n,p},\xi_0)-R^{(1)'}_{nm}(C_{n,p},\xi_0)R^{(2)}_{nm}(C_{n,p},\xi),$$

and prime denotes differentiation with respect to ξ, then put $\xi = \xi_0$, while $C_{n,p}$ is the root of the equation.

$$R^{(1)}_{nm}(C_n,\xi_1)R^{(2)'}_{nm}(C_n,\xi_0)-R^{(1)'}_{nm}(C_n,\xi_0)R^{(2)}_{nm}(C_n,\xi_1) = 0, \quad .. \quad .. \quad (1.7)$$

whereas

$$R^{(2)}_{nm}(C_{n,p},\xi) = \frac{1}{k^{(2)}_{mn}(C_{n,p})}\left\{\sum_{r=2m,\,-2m+1}^{\infty}{}' d^{mn}_r(C_{n,p})Q^m_{r+m}(\xi) + \sum_{r=2m+2,\,2m+1}^{\infty}{}' d^{mn}_{\rho/r}P^{(m)}_{r-m-1}(\xi)\right\}$$

$$= \frac{1}{\sum_{r=0,1}^{\infty}{}' d^{mn}_r(C_{n,p})\frac{(2m+r)!}{\underline{|r}}}\left(\frac{\xi^2-1}{\xi^2}\right)^{\frac{m}{2}}\sum_{r=0}^{\infty}{}' i^{r+m-n}d^{mn}_r(C_{n,p})\frac{(2m+r)!}{\underline{|r}}(-1)^{n+1}\sqrt{\frac{\pi}{(2C_{n,p}\,\xi)}}\,J_{-n-m-\frac{1}{2}}(C_{n,p},\xi),$$

where

$$k^{(2)}_{mn}(C_{n,p}) = \frac{2^{n-m}\,\underline{|2m}\left(\frac{n-m}{2}\right)!\left(\frac{n+m}{2}\right)!}{(2m-1)\,\underline{|m}\,(n+m)!\;(C_{m,p})^{m-1}}\, d^{nm}_{-2m}(C_{n,p}) \sum_{r=0}^{\infty}{}' d^{mn}_{r} \frac{\underline{|2m+r}}{\underline{|r}},$$

$(n-m)$ even

and

$$k^{(2)}_{mn}(C_{n,p}) = \frac{2^{n-m}\,\underline{|2m}\left(\frac{n-m-1}{2}\right)!\left(\frac{n+m+1}{2}\right)!\, d^{nm}_{-2m+1}}{(2m-3)(2m-1)\,\underline{|m}\,(n+m+1)!\;(C_{n,p})^{m-2}} \sum_{r=1}^{\infty}{}' \cdot d^{mn}_{r}(C_{n,p}) \frac{\underline{|2m+r}}{\underline{|r}}.$$

$(n-m)$ odd

Obviously, the function K_{mn} (ξ, η) by the well-known theorem on Fourier's series within the region can be expressed as

$$\begin{aligned} K_{mn}(\xi, \eta) &= \sum_{n=1}^{\infty} D_n B_{nm}(\xi) S^{(1)}_{nm}(C_n, \eta) \\ &= \sum_{n=1}^{\infty}\sum_{p=1}^{\infty} D_n B_{nm}(C_{n,p}, \xi) S^{(1)}_{nm}(C_{n,p}, \eta). \end{aligned} \qquad \text{.. (1.8)}$$

The constants D_n can be determined by the same method as in Theorem 1, viz.

$$\begin{aligned} D_n &= \frac{\int_{\xi_0}^{\xi_1}\int_{-1}^{1} K_{mn}(\xi, \eta)(\xi^2-\eta^2) B_{nm}(C_{n,p}, \xi) S^{(1)}_{nm}(C_{n,p}, \eta)\, d\xi\, d\eta}{N_{nm}\int_{\xi_0}^{\xi_1} [B_{nm}(C_{n,p}, \xi)]^2 [\xi^2 - \Theta_{n,p}]\, d\xi} \\ &= \bar{K}_{nm}(C_{n,p})/N_{nm}\int_{\xi_0}^{\xi_1} [B_{nm}(C_{n,p}, \xi)]^2 [\xi^2 - \Theta_{n,p}]\, d\xi. \end{aligned} \qquad \text{.. .. (1.9)}$$

Hence, the inversion formula is given by

$$K_{nm}(\xi, \eta) = \sum_{n=1}^{\infty}\sum_{p=1}^{\infty} \frac{\bar{K}_{nm}(C_{n,p}) B_{nm}(C_{n,p}, \xi) S^{(1)}_{nm}(C_{n,p}, \eta)}{N_{nm}\int_{\xi_0}^{\xi_1} [B_{nm}(C_{n,p}, \xi)]^2 [\xi^2 - \Theta_{n,p}]\, d\xi}. \qquad \text{.. (1.10)}$$

Definition III : If any function of (ξ, η) is continuous and single-valued in the region $\xi_0 \leqslant \xi \leqslant \xi_1$, $-1 \leqslant \eta \leqslant 1$ vanishing on both the boundaries $\xi = \xi_0$ and $\xi = \xi_1$, then its infinite transform in the region is defined,

$$\bar{K}_{nm}(C_{n,p}) = \int_{\xi_0}^{\xi_1}\int_{-1}^{1} K_{mn}(\xi, \eta)(\xi^2-\eta^2) B_{nm}(C_{n,p}, \xi) S^{(1)}_{nm}(C_{n,p}, \eta)\, d\xi\, d\eta, \qquad (1.11)$$

where

$$\begin{aligned} B_{nm}(C_{n,p}, \xi) = \big[&\{R^{(2)}_{nm}(C_{n,p}, \xi_0) - R^{(2)}_{nm}(C_{n,p}, \xi_1)\} R^{(1)}_{nm}(C_{n,p}, \xi) \\ &- R^{(2)}_{nm}(C_{n,p}, \xi)\{R^{(1)}_{nm}(C_{n,p}, \xi_0) - R^{(1)}_{nm}(C_{n,p}, \xi_1)\}\big], \end{aligned} \qquad \text{.. (1.12)}$$

and $C_{n,\,p}$ is the pth root $(m = 1, \ldots)$ of the equation

$$R^{(1)}_{nm}(C_n, \xi_1)R^{(2)}_{nm}(C_n, \xi_0) - R^{(1)}_{nm}(C_n, \xi_0) \times R^{(2)}_{nm}(C_n, \xi_1) = 0. \quad \ldots (1.13)$$

Adopting the same method as in Theorems I and II above we have

$$K_{nm}(\xi, \eta) = \sum_{n=1}^{\infty} \sum_{p=1}^{\infty} D_n B_{nm}(C_{n,\,p}, \xi) S^{(1)}_{nm}(C_{n,\,p}, \eta),$$

where

$$D_n = \frac{\bar{K}_{nm}(C_n, p)}{N_{nm} \int_{\xi_0}^{\xi_1} [B_{nm}(C_{n,\,p}, \xi)]^2 [\xi^2 - \Theta_{n,\,p}]\, d\xi}.$$

Hence, the inversion formula is given by

$$K_{nm}(\xi, \eta) = \sum_{n=1}^{\infty} \sum_{p=1}^{\infty} \frac{\bar{K}_{mn}(C_n, p) B_{n,\,m}(C_{n,\,p}, \xi) S^{(1)}_{nm}(C_{n,\,p}, \eta)}{N_{nm} \int_{\xi_0}^{\xi_1} [B_{nm}(C_{n,\,p}, \xi)]^2 [\xi^2 - \Theta_{n,\,p}]\, d\xi}. \quad (1.14)$$

§ 2. Properties of the Finite Transforms defined by Definitions I, II, III investigated before

In the application of the theory of the transform we have discussed above, we shall require some simple properties of this transform for the solution of special problems. Hence, in the following section we shall discuss them.

Case I: Let $C_{n,\,p}$ be a root of the eqn. (1.2) and

(i) $$\left\{(1-\eta^2)\left[S^{(1)}_{nm} \frac{\partial K_{n,\,m}}{\partial \eta} - K_{nm} \frac{\partial S_{nm}}{\partial \eta}\right]\right\}_{-1}^{1} = 0.$$

(ii) Let $K_{mn}(\xi, \eta)$ along with its first and second derivatives be continuous functions of ξ and η. Also let $K_{nm}(\xi, \eta)$ satisfy the equation,

$$\left[L_\xi + L_\eta + C_n^2(\xi^2 - \eta^2)\right] K_{mn}(\xi, \eta) = 0, \quad \ldots \quad \ldots (2.1)$$

in the domain of the ξ-plane which includes the interval $1 \leqslant \xi \leqslant \xi_0$ and in the domain of η-plane that includes the interval $-1 \leqslant \eta \leqslant 1$,
where

$$L_\xi = \frac{\partial}{\partial \xi}(\xi^2 - 1)\frac{\partial}{\partial \xi} + \frac{m^2}{\xi^2 - 1},$$

$$L_\eta = \frac{\partial}{\partial \eta}(1 - \eta^2)\frac{\partial}{\partial \eta} - \frac{m^2}{1 - \eta^2}.$$

Now

$$\int_{+1}^{\xi_0} \int_{-1}^{1} R^{(1)}_{nm}(C_{n,\,p}, \xi) S_{nm}(C_{n,\,p}, \eta)[L_\xi + L_\eta] K_{nm}(\xi, \eta)\, d\xi\, d\eta = I_1 + I_2 \quad (2.2)$$

where

$$I_1 = \int_1^{\xi_0} \int_{-1}^{1} R^{(1)}_{nm} S^{(1)}_{nm} L\xi K_{nm}(\xi, \eta)\, d\xi\, d\eta,$$

$$I_2 = \int_1^{\xi_0} \int_{-1}^{1} R^{(1)}_{nm} S^{(1)}_{nm} L\eta K_{nm}(\xi, \eta)\, d\xi\, d\eta.$$

Integrating (2.2) by parts we get

$$I_1 = \int_{-1}^{1} S^{(1)}_{nm}(C_{n,p}, \eta) \left[(\xi^2 - 1) \left\{ R^{(1)}_{nm} \frac{\partial K_{nm}}{\partial \xi} - K_{nm} \frac{\partial R^{(1)}_{nm}}{\partial \xi} \right\} \right]_1^{\xi_0} d\eta$$

$$+ \int_1^{\xi_0} \int_{-1}^{1} K_{nm} \frac{\partial}{\partial \xi} \left\{ (\xi^2 - 1) \frac{\partial R^{(1)}_{nm}}{\partial \xi} \right\} S^{(1)}_{nm} d\xi\, d\eta$$

$$+ \int_1^{\xi_0} \int_{-1}^{1} K_{nm} R^{(1)}_{nm} S^{(1)}_{nm} \frac{m^2}{\xi^2 - 1}\, d\xi\, d\eta,$$

the first integral vanishes at $\xi = \xi_0$ and $\xi = 1$ provided if we assume $K_{mn} = 0$ at $\xi = \xi_0$ then

$$I_1 = \int_1^{\xi_0} \int_{-1}^{1} K_{nm}(\xi, \eta) \left[\frac{\partial}{\partial \xi} (\xi^2 - 1) \frac{\partial R^{(1)}_{nm}}{\partial \xi} + \frac{m^2 R^{(1)}_{nm}}{\xi^2 - 1} \right] S^{(1)}_{nm} d\xi\, d\eta,$$

similarly

$$I_2 = \int_1^{\xi_0} \left[(1 - \eta^2) \left\{ S^{(1)}_{nm} \frac{\partial K_{nm}}{\partial \eta} - K_{nm} \frac{\partial S^{(1)}_{nm}}{\partial \eta} \right\} \right]_{-1}^{1} R^{(1)}_{nm}(C_{n,p}, \xi)\, d\xi$$

$$+ \int_{-1}^{1} \int_1^{\xi_0} \left[\frac{\partial}{\partial \eta} (1 - \eta^2) \frac{\partial S^{(1)}_{nm}}{\partial \eta} - \frac{m^2 S^{(1)}_{nm}}{1 - \eta^2} \right] K_{nm} R^{(1)}_{nm}(C_{n,p}, \xi)\, d\xi\, d\eta.$$

Obviously, the first integral vanishes on both the limits, hence

$$I_2 = \int_1^{\xi_0} \int_{-1}^{1} K_{nm} \left[\frac{\partial}{\partial \eta} (1 - \eta^2) \frac{\partial S^{(1)}_{nm}}{\partial \eta} - \frac{m^2}{1 - \eta^2} S^{(1)}_{n,m} \right] R^{(1)}_{nm}(\xi)\, d\xi\, d\eta.$$

Hence, (2.2) transforms into

$$I_1 + I_2 = -C^2_{n,p} \int_1^{\xi_0} \int_{-1}^{1} K_{nm}(\xi, \eta)(\xi^2 - \eta^2) R^{(1)}_{nm}(C_{n,p}, \xi) S^{(1)}_{n,m}(C_{n,p}, \eta)\, d\xi\, d\eta$$

$$- \int_{-1}^{1} S^{(1)}_{nm} \left[(\xi_0^2 - 1) K_{nm} \frac{\partial R^{(1)}_{nm}}{\partial \xi} \right]_{\xi = \xi_0} d\eta,$$

by virtue of (2.1).

Hence, we see that

$$\int_1^{\xi_0} \int_{-1}^{1} R^{(1)}_{nm}(C_{n,p}, \xi) S^{(1)}_{nm}(C_{n,p}, \eta) [L\xi + L\eta] K_{nm}(\xi, \eta)\, d\xi\, d\eta$$

$$= -C^2_{n,p} \bar{K}_{n,p}(C_n, p) - \int_{-1}^{1} (\xi_0^2 - 1) \left[K_{nm} \frac{\partial R^{(1)}_{nm}}{\partial \xi} \right]_{\xi = \xi_0} S^{(1)}_{nm}(C_{n,p}, \eta)\, d\eta. \quad \ldots \quad (2.3)$$

Case II: If $C_{n,\,p}$ is a root of eqn. (1.7) and

$$\text{(i)}\quad \left[(1-\eta^2)\left\{S^{(1)}_{nm}\frac{\partial K_{nm}}{\partial \eta}-K_{nm}\frac{\partial S^{(1)}_{nm}}{\partial \eta}\right\}\right]_{-1}^{1}=0,$$

(ii) the same as in case II, then

$$\int_{\xi_0}^{\xi_1}\int_{-1}^{1}[(L_\xi+L_\eta)K_{nm}]S^{(1)}_{nm}B_{nm}d\xi\, d\eta \qquad \ldots \qquad \ldots \qquad (2.4)$$

by the same procedure as in case (2.4) transforms to

$$=-(\xi_0^2-1)B_n(C_{n,\,p},\,\xi_0)\int_{-1}^{1}\left(\frac{\partial K_{nm}}{\partial \xi}\right)_{\xi=\xi_0}S^{(1)}_{nm}(C_{n,\,p},\,\eta)\,d\eta$$
$$+\int_{\xi_0}^{\xi_1}\int_{-1}^{1}K_{nm}\frac{\partial}{\partial \xi}\left\{(\xi^2-1)\frac{\partial B_n}{\partial \xi}\right\}S^{(1)}_{nm}(C_{n,\,p},\,\eta)\,d\xi\,d\eta$$
$$+\int_{\xi_0}^{\xi_1}\int_{-1}^{1}\frac{m^2}{\xi^2-1}K_{nm}(\xi,\,\eta)R^{(1)}_{nm}S^{(1)}_{nm}d\xi\,d\eta+\int_{\xi_0}^{\xi_1}\int_{-1}^{1}K_{nm}\left\{\frac{\partial}{\partial \eta}(1-\eta^2)\frac{\partial S^{(1)}_{nm}}{\partial \eta}\right\}$$
$$\times B_{nm}(\xi)d\xi\,d\eta+\int_{\xi_0}^{\xi_1}\int_{-1}^{1}\frac{m^2}{1-\eta^2}K_{nm}R^{(1)}_{nm}S^{(1)}_{nm}d\xi\,d\eta. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (2.5)$$

Hence, (2.4) becomes

$$=-(\xi_0^2-1)\int_{-1}^{1}B_n(C_{n,\,p},\,\xi_0)\left(\frac{\partial K_{nm}}{\partial \xi}\right)_{\xi=\xi_0}S^{(1)}_{nm}(C_{n,\,p},\,\eta)\,d\eta$$
$$+\int_{\xi_0}^{\xi_1}\int_{-1}^{1}K_{nm}(L_\xi+L_\eta)B_{nm}(C_{n,\,p},\,\xi)S^{(1)}_{nm}(\eta)\,d\xi\,d\eta$$
$$=-(\xi_0^2-1)\,B_n(C_{n,\,p},\,\xi_0)\int_{-1}^{1}\left(\frac{\partial K_{nm}}{\partial \xi}\right)_{\xi=\xi_0}S^{(1)}_{nm}(C_{n,\,p},\,\eta)\,d\eta$$
$$-C^2_{n,\,p}\bar{K}_{nm}(C_n,\,p)\,; \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (2.6)$$

where $B_{nm}(C_{n,\,p},\,\xi)S^{(1)}_{nm}(C_{n,\,p},\,\eta)$ is the solution of

$$[L_\xi+L_\eta+C^2_{n,\,p}(\xi^2-\eta^2)]K_{nm}(\xi,\,\eta)=0.$$

Case III: If $C_{n,\,p}$ is a root of eqn. (1.3) and (i) the same as in Cases I and II,

then proceeding exactly as has been done in deriving eqn. (2.6) we get

$$\int_{\xi_0}^{\xi_1}\int_{-1}^{1}B_{nm}S_{nm}(L_\xi+L_\eta)K_{nm}(\xi,\,\eta)\,d\xi\,d\eta$$
$$=\int_{\xi_0}^{\xi_1}\left[(1-\eta^2)\left\{S^{(1)}_{mn}\frac{\partial K_{nm}}{\partial \eta}-K_{nm}\frac{\partial S^{(1)}_{nm}}{\partial \eta}\right\}\right]_{-1}^{1}B_n(C_{n,\,p},\,\xi)\,d\xi$$
$$+\int_{\xi_0}^{\xi_1}\int_{-1}^{1}K_{nm}(\xi,\,\eta)\left\{\frac{\partial}{\partial \eta}(1-\eta^2)\frac{\partial S^{(1)}_{nm}}{\partial \eta}\right\}B_{nm}(C_{n,\,p},\,\xi)\,d\xi\,d\eta$$

$$+\int_{-1}^{1} S_{nm}^{(1)}(C_{n,p},\eta)\left[(\xi^2-1)\left\{B_{nm}(\xi)\frac{\partial K_{n,m}}{\partial\xi}-K_{nm}\frac{\partial B_{nm}}{\partial\xi}\right\}\right]_{\xi_0}^{\xi_1} d\eta$$

$$+\int_{\xi_0}^{\xi_1}\int_{-1}^{1} K_{nm}(\xi,\eta)\left\{\frac{\partial}{\partial\xi}(\xi^2-1)\frac{\partial B_{n,m}}{\partial\xi}\right\} S_{nm}^{(1)}(C_{n,p},\eta)\,d\xi\,d\eta$$

$$+\int_{\xi_0}^{\xi_1}\int_{-1}^{1}\left(\frac{m^2}{\xi^2-1}-\frac{m^2}{1-\eta^2}\right) K_{nm}B_{nm}S_{nm}^{(1)}(C_{n,p},\eta)\,d\xi\,d\eta$$

$$=\int_{\xi_0}^{\xi_1}\int_{-1}^{1} K_{nm}(L_\xi+L_\eta)B_{nm}(\xi)S_{nm}^{(1)}\,d\xi\,d\eta\,; \qquad \ldots \qquad (2.7)$$

where we have assumed that K_{nm} vanishes both at $\xi=\xi_0$ and $\xi=\xi_1$. Hence, (2.7) becomes

$$\int_{\xi_0}^{\xi_1}\int_{-1}^{1} B_{nm}S_{nm}^{(1)}(L_\xi+L_\eta)K_{nm}d\xi\,d\eta = -C_{n,p}^2\int_{\xi_0}^{\xi_1}\int_{-1}^{1}(\xi^2-\eta^2)K_{nm}B_{nm}S_{nm}^{(1)}\,d\xi\,d\eta$$

$$= -C_{n,p}^2\,\bar{K}_{nm}(C_{n,p})\,;$$

in virtue of theorem (3), where $B_{nm}(C_{n,p},\xi)S_{nm}(C_{n,p},\eta)$ being the solution of

$$\left[L_\xi+L_\eta+C_{n,p}^2(\xi^2-\eta^2)\right]K_{mn}(\xi,\eta)=0.$$

It can be readily seen that if $m^2 = 1/4$ these transforms reduce to that of Mathieu's functions.

§ 3. Slowing Down of Neutrons

In this section we shall discuss the slowing down of neutrons in which the slowing-down medium is in the form of a prolate spheroid, as well as the slowing-down density and the source function T are functions of (ξ, η) the spheroidal coordinates. The governing equation in this form is represented by

$$\left[\frac{\partial}{\partial\xi}(\xi^2-1)\frac{\partial}{\partial\xi}+\frac{\partial}{\partial\eta}(1-\eta^2)\frac{\partial}{\partial\eta}-C^2(\xi^2-\eta^2)\frac{\partial}{\partial t}\right]\chi = -C^2(\xi^2-\eta^2)S(\xi,\eta)\delta(t), \qquad (3.1)$$

when the boundary condition

$$\chi(\xi,\eta)=0, \text{ on } \xi=\xi_0,$$

where $d = 2C$ is the inter-focal distance and $\delta(t)$ is the Dirac delta function.

Here, if $C_{n,p}$ is the root of the equation

$$R_{0n}^{(1)}(C_n,\xi)=0,$$

it follows from eqn. (2.3) that

$$\int_{1}^{\xi_0}\int_{-1}^{1} R_{0n}^{(1)}(C_{n,p},\xi)S_{0n}^{(1)}(C_{n,p},\eta)\left(L'_\xi+L'_\eta\right)K_{0n}(\xi,\eta)\,d\xi\,d\eta$$

$$= -C_{n,p}^2\int_{1}^{\xi_0}\int_{-1}^{1} K_{0n}(\xi,\eta)(\xi^2-\eta^2)R_{0n}^{(1)}(C_{n,p},\xi)S_{0n}^{(1)}(C_{n,p},\eta)\,d\xi\,d\eta$$

$$= -C_{n,p}^2\bar{K}_{0n}(\xi,\eta)\,;$$

where

$$L'_{\xi} = \frac{\partial}{\partial \xi}(\xi^2-1)\frac{\partial}{\partial \xi};\; L'_{\eta} = \frac{\partial}{\partial \eta}(1-\eta^2)\frac{\partial}{\partial \eta}.$$

If, therefore, we multiply both the sides of the equation (3.1) by

$$R^{(1)}_{0n}(C_{n,p},\,\xi)S^{(1)}_{0n}(C_{n,\,p},\,\eta)$$

and integrate with respect to η from -1 to 1 and with respect to ξ between 1 to ξ_0 we find that $\bar{\chi}$ satisfies the following differential equation,

$$\frac{d\bar{\chi}}{dt}+\frac{C^2_{n,\,p}\bar{\chi}}{C^2} = \bar{S}\delta(t)C^2_{n,\,p}, \qquad \qquad (3.2)$$

where

$$\bar{S}(C_{n,\,p}) = \int_1^{\xi_0}\int_{-1}^{1} S(\xi,\,\eta)(\xi^2-\eta^2)R^{(1)}_{0n}(\xi)S^{(1)}_{0n}(\eta)\,d\xi\,d\eta.$$

The solution of eqn. (3.2) is obviously given by

$$\bar{\chi} = C^2_{n,\,p}\bar{S}\exp\left[-C^2_{n,\,p}t/C^2\right], \qquad \qquad (3.3)$$

which, on inversion formulae as in eqn. (1.5), gives the slowing-down density as

$$\chi(\xi,\,\eta) = \sum_{n=1}^{\infty}\sum_{p=1}^{\infty}\frac{C^2_{n,\,p}\bar{S}\exp\left[-C^2_{n,\,p}t/C^2\right]R^{(1)}_{0n}(C_{n,\,p},\,\xi)S^{(1)}_{0n}(C_{n,\,p},\,\eta)}{N_{0n}\int_1^{\xi_0}\left[R^{(1)}_{0n}(C_{n,\,p},\,\xi)\right]^2[\xi^2-\Theta_{n,\,p}]\,d\xi}$$

$$= \sum_{n=1}^{\infty}\sum_{p=1}^{\infty}\frac{C^2_{n,\,p}\exp\left[-C^2_{n,\,p}t/C^2\right]R^{(1)}_{0n}S^{(1)}_{0n}\int_{-1}^{1}\int_1^{\xi_0}S(\xi,\,\eta)(\xi^2-\eta^2)R^{(1)}_{0n}S^{(1)}_{0n}d\eta\,d\xi}{N_{0n}\int_1^{\xi_0}\left[R^{(1)}_{0n}(C_{n,\,p},\,\xi)\right]^2[\xi^2-\Theta_{n,\,p}]\,d\xi}. \qquad (3.4)$$

It is quite evident that as $C\to 0$, $\xi\to\infty$ the result agrees with that for the case of sphere as given in Sneddon (1951).

§ 4. Multiplication of Neutrons in a Medium without Sources

In the section discussed previously we have made no assumptions as to the nature of the processes resulting from the capture of a neutron by a nucleus of the slowing-down material. The process which is of most interest in the theory of nuclear reactors is that in which the capture of the neutron is followed by the emission of new fast neutrons. With this modification which is imposed on the equations, assuming for simplicity that each capture results in the emission of k-fast neutrons. In the diffusion theory for thermal neutrons $1/\tau$ denotes the probability of capture per unit time, so that, as a consequence of capture, $\frac{\rho k}{\tau}$ new fast neutrons will be produced per unit time in unit volume. Now in this section we shall try to determine, for a simple system, the conditions under which a distribution of neutrons is self-sustaining without any type of sources. Here we investigate the case of a prolate spheroid with the boundary conditions, the vanishing of ρ and χ when $\xi = \xi_0$ and 1.

Hence, the governing equations of ρ, χ are given by

$$\left[\frac{\partial}{\partial \xi}(\xi^2-1)\frac{\partial}{\partial \xi}+\frac{\partial}{\partial \eta}(1-\eta^2)\frac{\partial}{\partial \eta}-C'^2(\xi^2-\eta^2)\right]\rho = \frac{-\tau}{\Lambda^2}C'^2(\xi^2-\eta^2)\chi, \qquad (4.1)$$

where

$$C' = \tfrac{1}{2}\frac{d}{\Lambda},\ \Lambda = (D\tau)^{\frac{1}{2}} = \left(\frac{L_c L_t}{3}\right)^{\frac{1}{2}} \text{ is the diffusion length; and}$$

$$\left[\frac{\partial}{\partial \xi}(\xi^2-1)\frac{\partial}{\partial \xi}+\frac{\partial}{\partial \eta}(1-\eta^2)\frac{\partial}{\partial \eta}-C'^2(\xi^2-\eta^2)\frac{\partial}{\partial \theta}\right]\chi = -\frac{k\rho}{\tau}C^2(\xi^2-\eta^2)\delta(\theta) \qquad (4.2)$$

where $C = \frac{1}{2}d$; and $\delta(\theta)$ is the Dirac delta function, with the boundary conditions

$$\rho = 0 \text{ and } \chi = 0, \text{ on } \xi = \xi_0 \text{ and } \xi = 1.$$

From the nature of the boundary conditions making use of the same finite transform used in § 3, we get

$$\bar{\rho}\left(1+\frac{C^2_{n,p}}{C^2}\right) = \frac{\tau}{\Lambda}\bar{\chi}(C_{n,p},\theta_0), \qquad (4.3)$$

and

$$\frac{d\bar{\chi}}{d\theta}+\frac{C^2_{n,p}}{C^2}\bar{\chi} = \frac{k}{\tau}\bar{\rho}\delta(\theta), \qquad (4.4)$$

where

$$(\bar{\rho},\bar{\chi}) = \int_1^{\xi_0}\int_{-1}^{1}(\rho,\chi)(\xi^2-\eta^2)R^{(1)}_{0n}(C_{n,p},\xi)S^{(1)}_{0n}(C_{n,p},\eta)\,d\xi\,d\eta,$$

and $C_{n,p}$ is the pth zero of

$$R^{(1)}_{0n}(C_n,\xi_0) = 0.$$

Integration of equation (4.4) gives

$$\bar{\chi} = \frac{\bar{\rho}k}{\tau}\exp\left[-C^2_{n,p}\theta/C^2\right], \qquad (4.5)$$

from (4.3) and (4.5) it follows that

$$\bar{\rho}\left[\left(1+\frac{C^2_{n,p}}{C^2}\right)\exp\left[C^2_{np}\theta/C^2\right]-k\right] = 0.$$

Obviously, $\bar{\rho}(C_{n,p})$ must be zero except possibly for one value of $C_{n,p}$, say, it is $C_{1,1}$ on physical grounds; the lowest, i.e. $\bar{\rho} = 0$ for $C_{n,2}$, $C_{n,3}$ and the critical values of ξ_0 is ξ_c where

$$(1+C^2_{1,1}/C^2)\exp\left[C^2_{1,1}\theta/C^2\right]-k = 0.$$

It is quite evident that if $\xi_0 = \xi_c$ (the critical value of the const $\bar{\rho}$ $(C_1, 1)$ is quite arbitrary, say, A and

$$\rho_c = A \frac{R_{01}^{(1)} S_{01}^{(1)}(C_{1,1}, \eta)}{N_{01} \int_1^{\xi_0} [R_{01}^{(1)}(C_{1,1}, \xi)]^2 [\xi^2 - \Theta_{1,1}]\, d\xi},$$

$$= \bar{A} R_{01}^{(1)}(C_{1,1}, \xi) S_{01}^{(1)}(C_{1,1}, \eta), \qquad (4.6)$$

where

$$\bar{A} = \frac{A}{N_{01} \int_1^{\xi_0} \{R_{01}(C_{1,1}, \xi)\}^2 [\xi^2 - \Theta_{1,1}]\, d\xi}.$$

The value of $\bar{\rho}_c$ can be determined as follows, since the total number of thermal neutrons in the spheroid is

$$N = 2\pi C^3 \int_1^{\xi_0} \int_{-1}^1 \bar{A}(\xi^2 - \eta^2) R_{01}^{(1)}(C_{1,1}, \xi) S_{01}^{(1)}(C_{1,1}, \eta)\, d\xi\, d\eta.$$

Hence, (4.6) may be written in the following form:

$$\rho_c = \frac{N R_{01}^{(1)}(C_{1,1}, \xi) S_{01}^{(1)}(C_{1,1}, \eta)}{2\pi C^3 \int_1^{\xi_0} \int_{-1}^1 (\xi^2 - \eta^2) R_{01}^{(1)}(C_{1,1}, \xi) S_{01}^{(1)}(C_{1,1}, \eta)\, d\xi\, d\eta}, \qquad (4.7)$$

when $C \to 0$ and $\xi \to \infty$ the spheroid degenerates into a sphere and it can be readily shown that the result agrees with that of Sneddon (1951).

§ 5. Unsteady Flow between two Confocal Spheroids

In this section we have studied the rotational flow of the liquid contained between two confocal spheroids. Initially both the spheroids and the liquid are at rest. Suddenly the outer spheroid is started rotating impulsively with a uniform velocity Ω while the inner one is kept fixed. Here in this case the equation of motion in spheroidal coordinates is given by

$$\frac{\partial}{\partial \xi}(\xi^2 - 1)\frac{\partial v}{\partial \xi} + \frac{\partial}{\partial \eta}(1 - \eta^2)\frac{\partial v}{\partial \eta} + \frac{v}{\xi^2 - 1} - \frac{v}{1 - \eta^2} - \frac{C^2}{\nu}(\xi^2 - \eta^2)\frac{\partial v}{\partial t} = 0, \qquad (5.1)$$

with the following boundary conditions

(1) $v(\xi, \eta, t) = 0$ if $t = +0$ $\xi_0 \leqslant \xi \leqslant \xi_1$, $-1 \leqslant \eta \leqslant 1$;

(2) $v(\xi_0, \eta, t) = 0$ $t \geqslant 0$;

(3) $v(\xi_1, \eta, 0) = C\sqrt{\xi_1^2 - 1}\sqrt{1 - \eta^2}\,\Omega$, $-1 \leqslant \eta \leqslant 1$.

Now we assume that $C_{n,p}$ is a root of the equation

$$R_{1n}^{(1)}(C_{n,p}, \xi_1) R_{n1}^{(2)}(C_{n,p}, \xi_0) - R_{n1}^{(1)}(C_{n,p}, \xi_0) R_{n1}^{(2)}(C_{n,p}, \xi_1) = 0.$$

Multiplying (5.1) by $B_{n_1}(C_{n,p}, \xi) S_{n1}^{(1)}(C_{n,p}, \eta)$ where

$$B_n(C_{n,p}, \xi) = R_{n1}^{(1)}(C_{n,p}, \xi) R_{n1}^{(2)}(C_{n,p}, \xi_1) - R_{n1}^{(1)}(C_{n,p}, \xi_1) R_{n1}^{(2)}(C_{n,p}, \xi),$$

and proceeding exactly as in case II eqn. (2.6), we get

$$\frac{C\sqrt{\xi_1^2-1}}{\nu}\,\Omega \int_{-1}^{1} S_{n1}^{(1)}(C_{n,p}, \eta)\sqrt{1-\eta^2}\,d\eta + C_{n,p}^2 \bar{v} + \frac{C^2}{\nu}\,\frac{\partial \bar{v}}{dt} = 0,$$

or

$$\frac{d\bar{v}}{dt} + \frac{\nu C_{n,p}^2}{C^2}\,\bar{v} = \frac{4}{3C}\,\Omega\sqrt{\xi_1^2-1}\; d_0^{1n}(C_{n,p}), \qquad \ldots \quad \ldots \quad (5.2)$$

with the initial conditions

$$\bar{v} = 0, \quad \text{when } t = 0.$$

Hence, the appropriate solution of (5.2) is

$$\bar{v} = \frac{4\Omega\nu\sqrt{\xi_1^2-1}}{3C\;C_{n,p}^2}\cdot d_0^{1n}(C_{n,p}).\left\{1-\exp\left[\frac{-\nu C_{n,p}^2 t}{C^2}\right]\right\}, \qquad \ldots \quad (5.3)$$

which by the inversion formula (1.14) is given by

$$v = \sum_{n=1}^{\infty}\sum_{p=1}^{\infty} \frac{4\nu\Omega}{C}\cdot\frac{\sqrt{\xi_1^2-1}\,d_0^{1n}(C_{n,p})}{C_{n,p}^2}\cdot\frac{\left\{1-\exp\left[\frac{-\nu C_{n,p}^2 t}{C^2}\right]\right\} B_n(C_{n,p}, \xi) S_{n1}^{(1)}(C_{n,p}, \eta)}{N_{n1}^{(1)}\int_{\xi_0}^{\xi_1} B_{np}^2(\xi^2-\Theta_{n,p})\,d\xi}. \qquad \ldots \quad (5.4)$$

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