ON THE PETROV CLASSIFICATION OF ORTHOGONAL METRIC-SPACES

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(Communicated by V. V. Narlikar, F.N.I.)

(*Received February 4, 1967*)

The Petrov classification has been used, following the matrix notation, to examine the general metric of the orthogonal type, referred to as Dingle's metric here. The algebraic conditions that have to be satisfied in transition from local anisotropy to local isotropy through the intermediate stages of the Penrose diagram have been obtained. The algebraic conditions are really differential equations, the solution of which presents a series of new problems. The equivalence of neighbourhoods as considered here can be of considerable significance in the exploration of new gravitational fields. The conditions (22) and (23) do not seem to have been studied before. They have been considered with reference to some special metrics of interest.

1. INTRODUCTION

In the general theory of relativity the same physical or purely geometrical situation may be described by a metric tensor field $g_{ab}(x)$ in one coordinate system and by $\bar{g}_{ab}(\bar{x})$ in another coordinate system, g_{ab} and $\bar{g}_{ab}$ being related by

$$\bar{g}_{ab}(\bar{x}) = g_{cd}(x) \frac{\partial x^c}{\partial \bar{x}^a} \cdot \frac{\partial x^d}{\partial \bar{x}^b}. \qquad \text{(1)}$$

The existence of such relations means the existence of functions $x^i = x^i(\bar{x}^j)$ satisfying the differential equations (1). As compared to this equivalence problem the question, whether an event E described by $g_{ab}(x)$ is equivalent to or the same as an event $\bar{E}$ described by $\bar{g}_{ab}(\bar{x})$, is a simple algebraic one. In regular metric fields any two events E, $\bar{E}$ should be equivalent in the sense that

$$[\bar{g}_{ab}(\bar{x})]_{\bar{E}} = \left[g_{cd}(x) \frac{\partial x^c}{\partial \bar{x}^a} \cdot \frac{\partial x^d}{\partial \bar{x}^b}\right]_E.$$

The situation is, however, different when we are concerned not only with E and $\bar{E}$ but with their neighbourhoods also, as far as the second order partial derivatives go. Since we can make all first order partial derivatives vanish at an event, the equivalence of E and $\bar{E}$ remains if the second and higher

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order partial derivatives are ignored. For our purpose it is enough to consider partial derivatives up to the second order, as the geometrical and physical structures of a neighbourhood are given in sufficient detail by the Riemann and Weyl tensors.

If W_{abcd} and $\overline{W}_{abcd}$ are the Weyl tensors at E and $\bar{E}$ then the 16 numbers $\frac{\partial x^a}{\partial \bar{x}^p}$ do not always exist, so that

$$[\overline{W}_{pqrs}]_{\bar{E}} = \left[W_{abcd} \cdot \frac{\partial x^a}{\partial \bar{x}^p} \cdot \frac{\partial x^b}{\partial \bar{x}^q} \cdot \frac{\partial x^c}{\partial \bar{x}^r} \cdot \frac{\partial x^d}{\partial \bar{x}^s}\right]_E. \quad \quad (2)$$

Equations (2) are satisfied—neighbourhoods of events E, $\bar{E}$ are identical—if and only if W_{abcd} and $\overline{W}_{abcd}$ are of the same Petrov class.

The problem of the Petrov classification may be approached in several ways and has led to extensive literature, particularly in connection with the problem of gravitational radiation. A number of known exact solutions of Einstein's field equations have been classified.

It is not without interest to classify and study even those metric fields which have not yet led to exact solutions of the field equations. This study is particularly relevant in view of Synge's argument (1960), that the field equations may be read either from right to left or from left to right. Thus one may start with a metric tensor and see what distribution of sources it represents. In such cases, it may be useful to start with a metric tensor field, belonging to a particular Petrov class.

In this paper we propose to study a four-dimensional Riemannian metric in which the only assumption made is that it is possible to choose orthogonal coordinates everywhere in the region under consideration. Such a metric was studied by Dingle (1933) and it may be written as

$$ds^2 = -A(dx^1)^2 - B(dx^2)^2 - C(dx^3)^2 + D(dx^4)^2, \quad \quad (3)$$

where A, B, C, D are functions of x^1, x^2, x^3, x^4. The physical implications of the limitation

$$g_{ij} = 0 (i \neq j)$$

are not yet fully understood but a necessary and sufficient condition that the congruence of an orthogonal ennuple be normal is $\gamma_{hkl} = 0$, $(h \neq k \neq 1)$ in the usual notation (Eisenhart 1960).

Some spherically symmetric and cylindrically symmetric metrics appear as special cases of (3). Some other special cases are also discussed.

2. Some Essential Formulae

Even for those who avoid the spinor notation and the tensor method of Debever (1959) and Sachs (1961) to stick to the familiar tensors of Riemannian geometry in a matrix method, it is found that there are slight variations in the notation and the preliminaries in the growing literature on the Petrov classification. We, therefore, state the relevant standard results in the notation followed here. The signature of the metric tensor will be taken as $(-1, -1, -1, 1)$. In what follows the Latin suffixes $a, b, c, \ldots i, j, k$, etc., are supposed to run over 1 to 4. The Riemann tensor is

$$R^a_{bcd} = \Gamma^a_{bd,\,c} - \Gamma^a_{bc,\,d} + \Gamma^e_{bd}\Gamma^a_{ec} - \Gamma^e_{bc}\Gamma^a_{ed} \qquad (4)$$

where

$$\Gamma^a_{bc} = \tfrac{1}{2} g^{ae}[g_{ec,\,b} + g_{eb,\,c} - g_{bc,\,e}] \qquad (5)$$

with

$$f,_a \equiv \frac{\partial f}{\partial x^a}.$$

The Ricci tensor R_{ab} is given by

$$R_{ab} = R^h_{abh} \qquad (6)$$

and the scalar curvature R is

$$R = g^{bc} R_{bc}. \qquad (7)$$

The energy tensor T_{ij} is given by the field equations

$$-\frac{8\pi G}{c^4} T_{ij} = R_{ij} - \tfrac{1}{2} g_{ij} R \qquad (8)$$

in the usual notation. The Weyl tensor which is the conformal curvature tensor is given by

$$W^h_{ijk} = R^h_{ijk} + \tfrac{1}{2}[\delta^h_j R_{ik} - \delta^h_k R_{ij} + g_{ik} R^h_j - g_{ij} R^h_k] + \tfrac{1}{6} R[\delta^h_k g_{ij} - \delta^h_j g_{ik}] \qquad (9)$$

where δ^i are the Kronecker symbols

$$W_{lijk} = g_{lh} W^h_{ijk}. \qquad (10)$$

The Weyl tensor satisfies the following identities:

$$W^h_{ijh} = 0, \qquad (11.1)$$

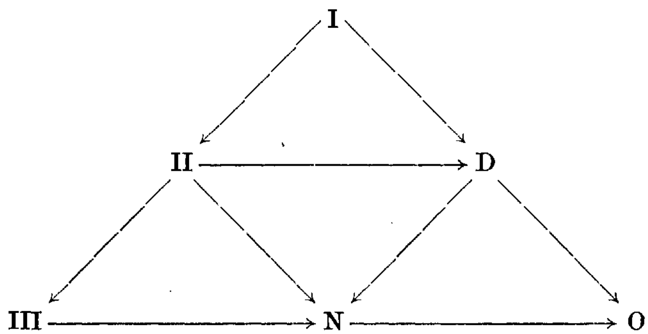
$$W_{lijk} + W_{ljki} + W_{lkij} = 0, \qquad (11.2)$$

$$W_{lijk} = W_{jkli} = -W_{likj} = -W_{iljk}. \qquad (11.3)$$

The classification is made throughout the region of validity of (3). It will be valid for neighbourhoods whether they are empty or occupied by inertial or electromagnetic matter as given by (8). When the vacuum field equations

$$R_{ij} = 0 \qquad (12)$$

are satisfied, the Weyl tensor reduces to the Riemann tensor. For this reason a separate classification of R_{hijk} for the case $T_{ij} = 0$ is not given. In what follows, the classification is carried out according to the Penrose (1960) diagram,



with the arrows pointing in the direction of increasing specialization. From I to O, the march is from anisotropy to isotropy. In the notation of the next section the criteria of classification can be stated as follows: Anticipating equation (21), if x_1, x_2, x_3 are its roots with $x_1+x_2+x_3 = 0$. Class O can be recognized early in the work, as it corresponds to conformally flat space, with the Weyl tensor zero. Table 1 gives the essential structure of the remaining types.

TABLE 1

Petrov classification

Type	*Eigenvalues*	*Eigenvectors*
I	$x_1 \neq x_2 \neq x_3$	3 non-null, mutually orthogonal, with lines uniquely determined
II	$x_1 \neq x_2 = x_3$	1 non-null with line uniquely determined and 1 null orthogonal to non-null and with line uniquely determined
D	$x_1 \neq x_2 = x_3$	1 non-null with line uniquely determined and all vectors in the plane orthogonal to it
III	$x_1 = x_2 = x_3 = 0$	1 null-vector with line uniquely determined and all vectors orthogonal to it
N	$x_1 = x_2 = x_3 = 0$	1 null-vector with line uniquely determined

3. General Classification

Using the notation of collective indices, $23 \to 1$, $31 \to 2$, $12 \to 3$, $14 \to 4$, $24 \to 5$, $34 \to 6$, the Weyl tensor can be written as a 6×6 matrix. Referred to the tetrad of vectors $(\sqrt{A}, 0, 0, 0)$, $(0, \sqrt{B}, 0, 0)$, $(0, 0, \sqrt{C}, 0)$ and

$(0, 0, 0, \sqrt{D})$ at $E(x^1, x^2, x^3, x^4)$ the array of the Weyl tensor components assumes the form

$$W = \begin{pmatrix} M & N \\ N & -M \end{pmatrix} \qquad (13)$$

where

$$M = \begin{pmatrix} \alpha_1 & \lambda & \mu \\ \lambda & \alpha_2 & \nu \\ \mu & \nu & \alpha_3 \end{pmatrix} \qquad (14)$$

and

$$N = \begin{pmatrix} 0 & \lambda' & \mu' \\ \lambda' & 0 & \nu' \\ \mu' & \nu' & 0 \end{pmatrix} \qquad (15)$$

are symmetric matrices of zero trace, since

$$\alpha_1+\alpha_2+\alpha_3 = 0. \qquad (16)$$

The elements of matrices M and N are given in (17) and (18) in the Appendix. We construct the complex matrix K as

$$K = M+\sqrt{-1}N. \qquad (19)$$

The characteristic equation for the complex eigenvalues x of this matrix is

$$\text{Det } (K-xI) = 0, \qquad (20)$$

where I is a unit 3×3 matrix. Using (16) we can write (20) as

$$\begin{aligned} -x^3+[(\lambda+i\lambda')^2+(\mu+i\mu')^2+(\nu+i\nu')^2-\alpha_1\alpha_2-\alpha_2\alpha_3-\alpha_3\alpha_1]x \\ +\alpha_1\alpha_2\alpha_3+2(\lambda+i\lambda')(\mu+i\mu')(\nu+i\nu') \\ -\alpha_1(\nu+i\nu')^2-\alpha_2(\mu+i\mu')^2-\alpha_3(\lambda+i\lambda')^2 = 0, \end{aligned} \qquad (21)$$

where

$$i \equiv \sqrt{-1}.$$

x_1, x_2, x_3 are the roots of this cubic already referred to, at the end of the last section.

Equation (21) has three distinct roots when no restrictions are imposed on the coefficients of the various powers of x. Therefore, in general, the metric (3) is of type I of the Penrose diagram.

The metric (3) will be type II or type D if (21) has a pair of equal roots, the condition for which is

$$\begin{aligned} 27[\alpha_1\alpha_2\alpha_3+2(\lambda+i\lambda')(\mu+i\mu')(\nu+i\nu')-\alpha_1(\nu+i\nu')^2-\alpha_2(\mu+i\mu')^2-\alpha_3(\lambda+i\lambda')^2]^2 \\ -4[(\lambda+i\lambda')^2+(\mu+i\mu')^2+(\nu+i\nu')^2-\alpha_1\alpha_2-\alpha_2\alpha_3-\alpha_3\alpha_1]^3 = 0. \end{aligned} \qquad (22)$$

However, when (22) is satisfied further analysis is required to separate types II and D. This depends on the knowledge of α_1, α_2, ... and, therefore, the analysis can be conveniently performed only with reference to special cases.

Conditions for eqn. (21) to have all three roots equal (and then necessarily zero), are

$$\lambda\lambda'+\mu\mu'+\nu\nu' = 0, \quad \ldots \quad (23.1)$$

$$\alpha_1\alpha_2+\alpha_2\alpha_3+\alpha_3\alpha_1-\lambda^2-\mu^2-\nu^2+\lambda'^2+\mu'^2+\nu'^2 = 0, \quad \ldots \quad (23.2)$$

$$\alpha_1\alpha_2\alpha_3-\alpha_1\nu^2-\alpha_2\mu^2-\alpha_3\lambda^2+\alpha_1\nu'^2+\alpha_2\mu'^2+\alpha_3\lambda'^2$$
$$+2(\lambda\mu\nu-\lambda\mu'\nu'-\lambda'\mu\nu'-\lambda'\mu'\nu) = 0, \quad \ldots \quad (23.3)$$

and

$$\lambda\mu\nu'+\lambda\mu'\nu+\lambda'\mu\nu-\lambda'\mu'\nu'+\alpha_1\nu\nu'-\alpha_2\mu\mu'-\alpha_3\lambda\lambda' = 0. \quad \ldots \quad (23.4)$$

Again, excluding the trivial case of the matrix W being zero, which happens when the space is conformally flat, the separation into types III and N can be conveniently effected with respect to only special cases.

Conditions (23) are four partial differential equations in four unknowns A, B, C, D and if their solution exists with suitable initial conditions then the metric (3) is completely determined. Thus in this case the condition that (3) is of type III or N is strong enough to determine (3) completely.

4. Some Special Cases

We consider now the following special cases:

Case (i):

A, B, C, D are functions of only three variables, say, x^1, x^2, x^4. In this case

$$\mu = \nu = 0 = \lambda'. \quad \ldots \quad (24)$$

Equation (22) becomes

$$27[\alpha_1\alpha_2\alpha_3-2\lambda\mu'\nu'+\alpha_1\nu'^2+\alpha_2\mu'^2-\alpha_3\lambda^2]^2$$
$$-4[\lambda^2-\mu'^2-\nu'^2-\alpha_1\alpha_2-\alpha_2\alpha_3-\alpha_3\alpha_1]^3 = 0. \quad \ldots \quad (25)$$

Equation (23) becomes

$$\alpha_1\alpha_2+\alpha_2\alpha_3+\alpha_3\alpha_1-\lambda^2+\mu'^2+\nu'^2 = 0, \quad \ldots \quad (26.1)$$

and

$$\alpha_1\alpha_2\alpha_3-\alpha_3\lambda^2+\alpha_1\nu'^2+\alpha_2\mu'^2-2\lambda\mu'\nu' = 0. \quad \ldots \quad (26.2)$$

We observe that eqn. (26) is satisfied by

$$\alpha_1 = 0,\ \alpha_2 = \nu',\ \lambda = \mu'. \quad \ldots \quad (27)$$

This gives

$$K = \begin{pmatrix} 0 & \lambda & i\lambda \\ \lambda & \alpha_2 & i\alpha_2 \\ i\lambda & i\alpha_2 & -\alpha_2 \end{pmatrix}. \quad \ldots \quad (28)$$

Equation (21) reduces to $x^3 = 0$.

If $\lambda = 0$, then the metric (3) is of type III. Eigenvectors consist of one null-vector with line uniquely determined and all vectors orthogonal to it and hence forming a plane through it (in the terminology of the complex Euclidean 3-space).

If $\lambda \neq 0$, then the matrix K can be transformed by means of an orthogonal matrix T into matrix K',

$$K' = TKT^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & i \\ 0 & i & 0 \end{pmatrix} \quad \ldots \quad \ldots \quad \ldots \quad (29)$$

where

$$T = \begin{pmatrix} -\frac{1}{2}\alpha_2\lambda^{-2} & \frac{1}{2}\lambda\left(1+\lambda^{-2}-\frac{1}{4}\alpha_2^2\lambda^{-4}\right) & \frac{i\lambda}{2}\left(1+\lambda^{-2}-\frac{1}{4}\alpha_2^2\lambda^{-4}\right) \\ 1 & \frac{1}{2}\alpha_2\lambda^{-1} & \frac{i}{2}\alpha_2\lambda^{-1} \\ -\frac{i}{2}\alpha_2\lambda^{-2} & \frac{1}{2}\lambda\left(-1+\lambda^{-2}-\frac{1}{4}\alpha_2^2\lambda^{-4}\right) & \frac{1}{2}\lambda\left(1+\lambda^{-2}+\frac{1}{4}\alpha_2^2\lambda^{-4}\right) \end{pmatrix}. \quad (30)$$

For a non-singular transformation from K to K' the characteristic roots remain invariant. The characteristic equation for K' also is $x^3 = 0$. The matrix (29) shows that in this case the metric (3) is of type N.

There is only one eigenvector in this case and it is a null-vector with line uniquely determined.

The well-known spherically symmetric metric is a particular form of Case (i), when it is expressed as follows:

$$ds^2 = -e^{\alpha}(dx^1)^2 - e^{\beta}[(dx^2)^2 + \sin^2 x^2 (dx^3)^2] + e^{\nu}(dx^4)^2. \quad \ldots \quad (31)$$

Here α, β, ν are functions of x^1 and x^4 only.
In this case, the Weyl tensor is already in the canonical form, with

$$M = \begin{pmatrix} \alpha_1 & 0 & 0 \\ 0 & \alpha_2 & 0 \\ 0 & 0 & \alpha_3 \end{pmatrix}, \; N = 0, \; -\tfrac{1}{2}\alpha_1 = \alpha_2 = \alpha_3 = S. \quad \ldots \quad (32)$$

S is given by

$$S = -\frac{1}{24}e^{-\alpha}l - \frac{1}{24}e^{-\nu}m + \frac{1}{6}e^{-\beta} \quad \ldots \quad \ldots \quad \ldots \quad (33)$$

where

$$l = 2\nu_{11} + \nu_1^2 - 2\beta_{11} + \alpha_1\beta_1 - \alpha_1\nu_1 - \beta_1\nu_1$$

and

$$m = 2\beta_{44} - 2\alpha_{44} - \alpha_4^2 + \alpha_4\beta_4 + \alpha_4\nu_4 + \beta_4\nu_4.$$

Although there are only two distinct eigenvalues here the system of eigenvectors consists of one non-null vector with line uniquely determined and all vectors lying in the plane orthogonal to it. Hence the spherically symmetric metric is of type D, provided $S \neq 0$ (Krishna Rao 1966). $S = 0$ corresponds to $W = 0$ and the metric is then of type O. In fact $S = 0$ is the condition for (31) to be conformally flat.

Case (ii):

A, B, C, D are functions of only two variables, say, x^1 and x^4. In this case

$$\lambda = \mu = \nu = 0 = \lambda' = \mu', \qquad \ldots \quad (34)$$

and in the absence of further conditions (3) is of type I. Equation (22) becomes in this case

$$27\alpha_1^2(\alpha_2\alpha_3+\nu'^2)^2+4(\nu'^2+\alpha_2\alpha_3+\alpha_3\alpha_1+\alpha_1\alpha_2)^3 = 0. \qquad \ldots \quad (35)$$

(35) is satisfied by

$$\alpha_2-\alpha_3 = \pm 2\nu', \qquad \ldots \quad (36)$$

and K becomes

$$K = \begin{pmatrix} \alpha_1 & 0 & 0 \\ 0 & \alpha\pm\nu' & i\nu' \\ 0 & i\nu' & \alpha\pm\nu' \end{pmatrix} \qquad \ldots \quad (37)$$

where

$$\alpha_2 = \alpha\pm\nu'.$$

From (37) it follows that (3) is of type II provided $\nu' \neq 0$. If $\nu' = 0$ then (3) goes over from type II to type D. If $B = C$, then we have for K

$$K = \begin{pmatrix} -2\alpha & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \alpha \end{pmatrix} \qquad \ldots \quad (38)$$

with

$$\alpha = \frac{A_{44}}{12AD}-\frac{A_4^2}{24A^2D}-\frac{D_{11}}{12AD}+\frac{D_1^2}{24AD^2}-\frac{B_1^2}{24AB^2}+\frac{B_{11}}{12AB}-\frac{B_{44}}{12BD}+\frac{B_4^2}{24B^2D}$$
$$+\frac{A_1D_1}{24A^2D}-\frac{A_1B_1}{24A^2B}+\frac{B_1D_1}{24ABD}-\frac{A_4B_4}{24ABD}+\frac{B_4D_4}{24BD^2}-\frac{A_4D_4}{24AD^2}.$$

Hence (3) is of type D, provided $\alpha \neq 0$.
$\alpha = 0$ gives conformally flat space.
Equation (23) becomes in this case

$$\alpha_1\alpha_2+\alpha_2\alpha_3+\alpha_3\alpha_1+\nu'^2 = 0 \qquad \ldots \quad (39.1)$$

and

$$\alpha_1\alpha_2\alpha_3+\alpha_1\nu'^2 = 0. \qquad \ldots \quad (39.2)$$

Equation (39) imply

$$\alpha_1 = 0, \quad \alpha_2\alpha_3+\nu'^2 = 0 \qquad \ldots \quad (40)$$

giving

$$K = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \alpha_2 & \pm i\alpha_2 \\ 0 & \pm i\alpha_2 & -\alpha_2 \end{pmatrix} \qquad \ldots \quad (41)$$

(41) shows that the metric (3) is now of type III provided $\alpha_2 \neq 0$.

The metric considered by Einstein and Rosen (1937), viz.

$$ds^2 = -A(dx^1)^2-B(dx^2)^2-C(dx^3)^2+A(dx^4)^2 \qquad \ldots \quad (42)$$

where A, B, C are functions of x^1 and x^4 only, belongs to Case (ii). It is of type I in general. If (35) is satisfied it is of type D and if (40) is satisfied it is of type III.

The cylindrically symmetric metric, in the standard notation, may be written as

$$ds^2 = -e^{2(\nu-\psi)}(dx^1)^2 - e^{-2\psi}(x^1)^2(dx^2)^2 - e^{2\psi}(dx^3)^2 + e^{2(\nu-\psi)}(dx^4)^2 \quad .. \quad (43)$$

and is a particular case of (42). The metric (43) will be of type II if any one of the conditions (36) is satisfied. These may be written as

$$\psi_{11}+\psi_{44}+2\psi_{14}-2(\nu_1-\psi_1)(\psi_1+\psi_4)$$
$$-2(\nu_4-\psi_4)(\psi_1+\psi_4)+\frac{1}{x^1}(\nu_1+\nu_4) = 0, \quad .. \quad .. \quad (44)$$

$$\psi_{11}+\psi_{44}-2\psi_{14}+2(\nu_1-\psi_1)(\psi_4-\psi_1)$$
$$-2(\nu_4-\psi_4)(\psi_4-\psi_1)+\frac{1}{x^1}(\nu_1-\nu_4) = 0. \quad .. \quad .. \quad .. \quad (45)$$

Pirani (1957) observes that (43) is of type II in general. Our considerations show that this observation is incorrect.*

Case (iii):

A, B, C, D are functions of only one variable, say, x^4.
In this case

$$\lambda = \mu = \nu = 0 = \lambda' = \mu' = \nu'. \quad .. \quad .. \quad .. \quad (46)$$

The metric is in general of type I with the Weyl tensor already in the canonical form as may be expected from the inequality of A, B, C. If for example $B = C$, the metric becomes of type D.

The metric considered by Narlikar and Karmarkar (1946) illustrates our point. The metric is given by

$$ds^2 = -(1+kt)^p dx^2 - (1+kt)^q dy^2 - (1+kt)^r dz^2 + dt^2 \quad .. \quad .. \quad (47)$$

where

$$p+q+r = 2 \text{ and } pq+qr+rp = 0.$$

(47) is of type I, but goes over to type D, if any two of p, q, r are equal.

Conditions (23) imply in this case $\alpha_1 = \alpha_2 = \alpha_3 = 0$, and hence a conformally flat space.

We have, therefore, a general result that an orthogonal metric, which is not conformally flat and whose metric tensor is a function of only one variable, is necessarily of type I or type D.

* (This fact was independently brought to my notice by Shri J. Krishna Rao of Vallabh Vidyanagar, in the course of a discussion).

5. Concluding Remarks

A number of critics like A. N. Whitehead and V. Fock have raised serious objections to the formulation of general relativity because of the heterogeneity of space-time. When the space-time is heterogeneous the pattern of curvature changes from point to point. By Schur's theorem if the Riemannian curvature at each point is the same for every orientation it does not vary from point to point. But in general relativity we do not always deal with space-times of constant curvature. The problem of equivalence of neighbourhoods assumes, therefore, considerable importance.

From a conformally flat space-time to the heterogeneous space-time of general relativity is a far cry. This gap is described through the various types of local anisotropy classified by Penrose. The study of these types, in various notations and diverse methods, first arose with reference to R_{hijk} in connection with the problems of gravitational radiation.

Similar studies with reference to the Weyl tensor W_{hijk} have also been carried out. The author has benefited considerably from the lucid exposition of the subject by Synge (1964). The particular cases considered here do not seem to have been discussed elsewhere. It is one thing to lay down the algebraic conditions like (22) or (23) for transition from type to type and quite a different thing to examine the analytical implications of these conditions by studying the partial differential equations involved. As is to be expected the equations do not generally constitute a sufficient set signifying a multiplicity of metrical possibilities. There seem to be a sufficient set of conditions to determine A, B, C, D in the general case but no conclusions can be drawn from a superficial survey of the set of equations. Even a plausible reasoning supported by simple particular solutions can throw much light on the local evolution of a space-time pattern from anisotropy to isotropy. We have not found any such non-trivial solutions.

Acknowledgement

The author's thanks are due to Professor V. V. Narlikar, Lokamanya Tilak Professor of Applied Mathematics, University of Poona, for many useful discussions, during the preparation of this paper.

Appendix

The elements of matrices M and N are given below. A suffix i attached to A, B, C, D denotes a partial derivative with respect to x^i.

$$BC\alpha_1 = -\frac{BC}{12A}\left(\frac{A_{22}}{B}+\frac{A_{33}}{C}+\frac{2A_{44}}{D}\right)-\frac{C}{12}\left(\frac{B_{11}}{A}-\frac{2B_{33}}{C}-\frac{B_{44}}{D}\right)$$
$$-\frac{B}{12}\left(\frac{C_{11}}{A}-\frac{2C_{22}}{B}-\frac{C_{44}}{D}\right)-\frac{BC}{12D}\left(-\frac{2D_{11}}{A}+\frac{D_{22}}{B}+\frac{D_{33}}{C}\right)$$

$$+\frac{BC}{24A^2}\left(\frac{A_2^2}{B}+\frac{A_3^2}{C}+\frac{2A_4^2}{D}\right)+\frac{C}{24B}\left(\frac{B_1^2}{A}-\frac{2B_3^2}{C}-\frac{B_4^2}{D}\right)$$

$$+\frac{B}{24C}\left(\frac{C_1^2}{A}-\frac{2C_2^2}{B}-\frac{C_4^2}{D}\right)-\frac{BC}{24D^2}\left(\frac{2D_1^2}{A}-\frac{D_2^2}{B}-\frac{D_3^2}{C}\right)$$

$$+\frac{BCA_1}{24A^2}\left(\frac{B_1}{B}+\frac{C_1}{C}-\frac{2D_1}{D}\right)-\frac{BC}{24A}\left(\frac{B_1D_1}{BD}+\frac{C_1D_1}{CD}-\frac{2B_1C_1}{BC}\right)$$

$$+\frac{CA_2}{24A}\left(\frac{B_2}{B}-\frac{C_2}{C}+\frac{2D_2}{D}\right)-\frac{C}{24}\left(\frac{2B_2C_2}{BC}-\frac{B_2D_2}{BD}+\frac{C_2D_2}{CD}\right)$$

$$+\frac{BA_3}{24A}\left(-\frac{B_3}{B}+\frac{C_3}{C}+\frac{2D_3}{D}\right)-\frac{B}{24}\left(\frac{2B_3C_3}{BC}+\frac{B_3D_3}{BD}-\frac{C_3D_3}{CD}\right)$$

$$+\frac{BCA_4}{24AD}\left(\frac{B_4}{B}+\frac{C_4}{C}+\frac{2D_4}{D}\right)-\frac{BC}{24D}\left(\frac{2B_4C_4}{BC}+\frac{B_4D_4}{BD}+\frac{C_4D_4}{CD}\right),$$

$$CA\alpha_2=-\frac{CA}{12B}\left(\frac{B_{33}}{C}+\frac{B_{11}}{A}+\frac{2B_{44}}{D}\right)-\frac{A}{12}\left(\frac{C_{22}}{B}-\frac{2C_{11}}{A}-\frac{C_{44}}{D}\right)$$

$$-\frac{C}{12}\left(\frac{A_{22}}{B}-\frac{2A_{33}}{C}-\frac{A_{44}}{D}\right)-\frac{CA}{12D}\left(-\frac{2D_{22}}{B}+\frac{D_{33}}{C}+\frac{D_{11}}{A}\right)$$

$$+\frac{CA}{24B^2}\left(\frac{B_3^2}{C}+\frac{B_1^2}{A}+\frac{2B_4^2}{D}\right)+\frac{A}{24C}\left(\frac{C_2^2}{B}-\frac{2C_1^2}{A}-\frac{C_4^2}{D}\right)$$

$$+\frac{C}{24A}\left(\frac{A_2^2}{B}-\frac{2A_3^2}{C}-\frac{A_4^2}{D}\right)-\frac{CA}{24D^2}\left(\frac{2D_2^2}{B}-\frac{D_3^2}{C}-\frac{D_1^2}{A}\right)$$

$$+\frac{CAB_2}{24B^2}\left(\frac{C_2}{C}+\frac{A_2}{A}-\frac{2D_2}{D}\right)-\frac{CA}{24B}\left(\frac{C_2D_2}{CD}+\frac{A_2D_2}{AD}-\frac{2C_2A_2}{CA}\right)$$

$$+\frac{AB_3}{24B}\left(\frac{C_3}{C}-\frac{A_3}{A}+\frac{2D_3}{D}\right)-\frac{A}{24}\left(\frac{2C_3A_3}{CA}-\frac{C_3D_3}{CD}+\frac{A_3D_3}{AD}\right)$$

$$+\frac{CB_1}{24B}\left(-\frac{C_1}{C}+\frac{A_1}{A}-\frac{2D_1}{D}\right)-\frac{C}{24}\left(\frac{2C_1A_1}{CA}+\frac{C_1D_1}{CD}-\frac{A_1D_1}{AD}\right)$$

$$+\frac{CAB_4}{24BD}\left(\frac{C_4}{C}+\frac{A_4}{A}+\frac{2D_4}{D}\right)-\frac{CA}{24D}\left(\frac{2C_4A_4}{CA}+\frac{C_4D_4}{CD}+\frac{A_4D_4}{AD}\right),$$

$$AB\alpha_3=-\frac{AB}{12C}\left(\frac{C_{11}}{A}+\frac{C_{22}}{B}+\frac{2C_{44}}{D}\right)-\frac{B}{12}\left(\frac{A_{33}}{C}-\frac{2A_{22}}{B}-\frac{A_{44}}{D}\right)$$

$$-\frac{A}{12}\left(\frac{B_{33}}{C}-\frac{2B_{11}}{A}-\frac{B_{44}}{D}\right)-\frac{AB}{12D}\left(-\frac{2D_{33}}{C}+\frac{D_{11}}{A}+\frac{D_{22}}{B}\right)$$

$$+\frac{AB}{24C^2}\left(\frac{C_1^2}{A}+\frac{C_2^2}{B}+\frac{2C_4^2}{D}\right)+\frac{B}{24A}\left(\frac{A_3^2}{C}-\frac{2A_2^2}{B}-\frac{A_4^2}{D}\right)$$

$$+\frac{A}{24B}\left(\frac{B_3^2}{C}-\frac{2B_1^2}{A}-\frac{B_4^2}{D}\right)-\frac{AB}{24D^2}\left(\frac{2D_3^2}{C}-\frac{D_1^2}{A}-\frac{D_2^2}{B}\right)$$

$$+\frac{ABC_3}{24C^2}\left(\frac{A_3}{A}+\frac{B_3}{B}-\frac{2D_3}{D}\right)-\frac{AB}{24C}\left(\frac{A_3D_3}{AD}+\frac{B_3D_3}{BD}-\frac{2A_3B_3}{AB}\right)$$

$$+\frac{BC_1}{24C}\left(\frac{A_1}{A}-\frac{B_1}{B}+\frac{2D_1}{D}\right)-\frac{B}{24}\left(\frac{2A_1B_1}{AB}-\frac{A_1D_1}{AD}+\frac{B_1D_1}{BD}\right)$$

$$+\frac{AC_2}{24C}\left(-\frac{A_2}{A}+\frac{B_2}{B}+\frac{2D_2}{D}\right)-\frac{A}{24}\left(\frac{2A_2B_2}{AB}+\frac{A_2D_2}{AD}-\frac{B_2D_2}{BD}\right)$$

$$+\frac{ABC_4}{24CD}\left(\frac{A_4}{A}+\frac{B_4}{B}+\frac{2D_4}{D}\right)-\frac{AB}{24D}\left(\frac{2A_4B_4}{AB}+\frac{A_4D_4}{AD}+\frac{B_4D_4}{BD}\right),$$

$$\sqrt{A}\sqrt{B}\lambda=\frac{1}{4}\left[-\frac{C_{12}}{C}+\frac{D_{12}}{D}+\frac{C_1C_2}{2C^2}+\frac{B_1C_2}{2BC}+\frac{C_1A_2}{2AC}-\frac{B_1D_2}{2BD}-\frac{A_2D_1}{2AD}-\frac{D_1D_2}{2D^2}\right],$$

$$\sqrt{A}\sqrt{C}\mu=\frac{1}{4}\left[-\frac{B_{31}}{B}+\frac{D_{31}}{D}+\frac{B_3B_1}{2B^2}+\frac{A_3B_1}{2AB}+\frac{B_3C_1}{2BC}-\frac{A_3D_1}{2AD}-\frac{C_1D_3}{2CD}-\frac{D_3D_1}{2D^2}\right],$$

$$\sqrt{B}\sqrt{C}\nu=\frac{1}{4}\left[-\frac{A_{23}}{A}+\frac{D_{23}}{D}+\frac{A_2A_3}{2A^2}+\frac{C_2A_3}{2CA}+\frac{A_2B_3}{2AB}-\frac{C_2D_3}{2CD}-\frac{B_3D_2}{2BD}-\frac{D_2D_3}{2D^2}\right]; \qquad \ldots (17)$$

and

$$\sqrt{C}\sqrt{D}\lambda'=\frac{1}{4}\left[\frac{B_{34}}{B}-\frac{A_{34}}{A}+\frac{A_3A_4}{2A^2}+\frac{C_4A_3}{2AC}-\frac{B_3B_4}{2B^2}-\frac{B_3C_4}{2BC}-\frac{B_4D_3}{2BD}+\frac{D_3A_4}{2AD}\right],$$

$$\sqrt{B}\sqrt{D}\mu'=\frac{1}{4}\left[\frac{A_{24}}{A}-\frac{C_{24}}{C}+\frac{C_2C_4}{2C^2}+\frac{B_4C_2}{2BC}-\frac{A_2A_4}{2A^2}-\frac{A_2B_4}{2AB}-\frac{A_4D_2}{2AD}+\frac{D_2C_4}{2CD}\right],$$

$$\sqrt{A}\sqrt{D}\nu'=\frac{1}{4}\left[\frac{C_{14}}{C}-\frac{B_{14}}{B}+\frac{B_1B_4}{2B^2}+\frac{A_4B_1}{2AB}-\frac{C_1C_4}{2C^2}-\frac{C_1A_4}{2AC}-\frac{C_4D_1}{2CD}+\frac{D_1B_4}{2BD}\right]. \qquad (18)$$

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