GEOMETRODYNAMICS OF THE NULL FIELD

by (Mrs.) Vanaja Iyengar, *Department of Mathematics, Osmania University, Hyderabad*

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For a non-holonomic form of the matrix $((h_{ij}))$, compatible with the real electromagnetic null field, the conditions of existence of two analytic functions ρ and σ satisfying the Einstein-Maxwell equations are investigated.

§ 1. *Introduction:* In a four-dimensional space-time of general relativity we assume a symmetric tensor $h_{\lambda\mu}$ of signature $+++-$ and rank 4. There exists an inverse tensor $h^{\lambda\mu}$ such that

$$h^{\lambda\mu}h_{\lambda\alpha} = \delta^{\mu}_{\alpha} \qquad \text{.. .. .. ..} \quad (1.1)$$

we shall use $h_{\lambda\mu}$ ($h^{\lambda\mu}$) to lower (raise) the indices. Throughout this paper the Latin indices i, j, k are used for the non-holonomic frame and range from 1 to 4 and the Greek indices α, β, γ for the original frame and range from I to IV. The range of Greek indices ξ, η, is I, II and III.

If $k_{\lambda\mu} = -k_{\mu\lambda}$ is a skew-symmetric real electromagnetic tensor field, then the stress tensor $F_{\mu}{}^{\lambda}$ is given by

$$F_{\mu}{}^{\lambda} \overset{\text{def}}{=} k^{\alpha}_{\mu}k_{\alpha}{}^{\lambda} - \tfrac{1}{4}\delta_{\mu}{}^{\lambda}k_{\alpha\beta}k^{\beta\alpha}. \qquad \text{.. .. ..} \quad (1.2)$$

We put

$$^{(p)}k_{\mu}{}^{\alpha} \overset{\text{def}}{=} k_{\mu}{}^{\beta}k_{\beta}{}^{\alpha},\ ^{(0)}k_{\mu}{}^{\alpha} \overset{\text{def}}{=} \delta^{\alpha}_{\mu} (p = 1, 2, \ldots) \qquad \text{.. ..} \quad (1.3a)$$

$$4K \overset{\text{def}}{=} {}^{(2)}k_{\alpha}{}^{\alpha}. \qquad \text{.. .. ..} \quad (1.3b)$$

Then the basic system of field equations of general relativity assumes the form

$$R_{\mu}{}^{\lambda} - \tfrac{1}{2}Rh_{\mu}{}^{\lambda} = \mu u_{\mu}u\lambda - F^{\lambda}_{\mu} + p\delta^{\lambda}_{\mu} \qquad \text{.. .. ..} \quad (1.4a)$$

and the Maxwell equations are

$$\nabla_{[\omega}k_{\mu\lambda]} = 0 \qquad \text{.. .. .. ..} \quad (1.4b)$$

$$\nabla_{\alpha}k^{\alpha\beta} = \epsilon u^{\beta} \qquad \text{.. .. .. ..} \quad (1.4c)$$

where ∇_{α} is the covariant derivative with respect to $h_{\lambda\mu}$

$$R \overset{\text{def}}{=} R_{\alpha}{}^{\alpha} \qquad \text{.. .. .. ..} \quad (1.5)$$

and u^{α} is a unit time-like vector field

$$U_{\alpha}U^{\alpha} = -1 \qquad \text{.. .. .. ..} \quad (1.6)$$

and p is the pressure and ϵ is the energy.

The electromagnetic tensor $k_{\lambda\mu}$ is said to be of the

(a) first class if $k \overset{\text{def}}{=} \det |k_{\alpha\beta}| \neq 0,\ K \neq 0$

(b) second class if $k = 0,\ K \neq 0$

(c) third class if $k = 0,\ K = 0$

If λ be an eigen-value of $k_{\alpha}{}^{\beta}$ and a_{α} be the corresponding eigen-vector, then

$$\left|k_{\alpha}{}^{\beta}-\lambda\delta_{\alpha}{}^{\beta}\right| a_{\beta}=0 \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (1.7)$$

Thus the eigen-values are given by

$$\left|k_{\alpha}{}^{\beta}-\lambda\delta_{\alpha}{}^{\beta}\right|=0 \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (1.8)$$

which reduces to

$$\lambda^{4}+2K\lambda^{2}+k=0. \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (1.9)$$

Hlavaty (1961) has shown that when $k_{\alpha\beta}$ is of the first and second class a set of four mutually independent eigen-vectors $\overset{i}{a}_{\beta}$ can always be found, which, together with its inverse set $\underset{i}{a}^{\beta}$, form a frame which is called the non-holonomic frame of the first two classes. For the third class which includes the null electromagnetic field, there is only one basic scalar $\lambda = 0$, and therefore a non-holonomic frame cannot be found out of the basic vectors. However, Hlavaty (1961) using line geometry has shown that for the third class also a set of two complex conjugate null vectors $\underset{1}{a}^{\lambda}$, $\underset{2}{a}^{\lambda}$ and two real null vectors $\underset{3}{a}^{\lambda}$, $\underset{4}{a}^{\lambda}$, all linearly independent can be found such that

$$\left.\begin{aligned} \underset{1}{a}^{\lambda}k_{\lambda}{}^{\mu} &= \frac{1}{\sqrt{2}}\underset{4}{a}^{\mu}e^{+i\sigma} \\ \underset{3}{a}^{\lambda}k_{\lambda}{}^{\mu} &= \frac{1}{\sqrt{2}}\left(e^{+i\sigma}\underset{1}{a}^{\mu}+e^{+i\sigma}\underset{2}{a}^{\mu}\right) \\ \underset{4}{a}^{\lambda}k_{\lambda}{}^{\mu} &= 0 \end{aligned}\right\} \qquad \ldots \quad \ldots \quad \ldots \quad (1.10)$$

Mishra (1963) has obtained this result without using line geometry, with the help of the property

$$^{(3)}k_{\alpha}{}^{\beta}=0.$$

The non-holonomic form for the canonical matrix $((h_{ij}))$ compatible with real magnetic null field was given by Mishra (1963) as

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (1.11)$$

§ 2. *Determination of $k_{\lambda\mu}$ and u_λ:* We take a generalization of form (1.1) as

$$((h_{ij})) = \begin{pmatrix} \alpha & 0 & 0 & \rho \\ 0 & \beta & 0 & 0 \\ 0 & 0 & 0 & -\alpha \\ \sigma & 0 & -\alpha & \sigma \end{pmatrix} \quad \ldots \quad (2.1)$$

$$\det ((h_{\lambda\mu})) = -\alpha^3\beta$$

$$h_{\mu v} \overset{\text{def}}{=} h_{ij} \overset{i}{a}_\mu \overset{j}{a}_\lambda \quad \ldots \quad (2.2)$$

where α, β, ρ, σ are four parameters of which α and β are to be determined by Maxwell equations. We then have

$$((h^{ij})) = \begin{pmatrix} \frac{1}{\alpha} & 0 & \frac{\rho}{\alpha^2} & 0 \\ 0 & \frac{1}{\beta} & 0 & 0 \\ \frac{\rho}{\alpha^2} & 0 & \frac{\alpha\rho^2-\sigma}{\alpha^3} & -\frac{1}{\alpha} \\ 0 & 0 & -\frac{1}{\alpha} & 0 \end{pmatrix} \quad \ldots \quad (2.3)$$

Since the field is null and k_{ij} is skew-symmetric, using conditions (Mishra 1963)

$$k_1{}^4 k_4{}^2 = k_1{}^2 k_2{}^4 = k_4{}^1 k_2{}^4 = k_2{}^1 k_1{}^4 = k_4{}^2 k_2{}^1 = k_4{}^1 k_1{}^2 = 0 \quad \ldots \quad (2.4a)$$

and

$$k_4{}^1 k_1{}^3 + k_4{}^2 k_2{}^3 = 1. \quad \ldots \quad (2.4b)$$

We obtain for the non-holonomic frame (2.1)

$$k_1{}^3 = k_4{}^1 = k_4{}^2 = \alpha,\ k_2{}^3 = \beta \quad \ldots \quad (2.5a)$$

$$k_3{}^2 = k_1{}^4 = k_3{}^4 = k_3{}^1 = k_2{}^4 = k_2{}^1 = k_1{}^2 = 0 \quad \ldots \quad (2.5b)$$

$$\alpha(\alpha+\beta) = 1 \quad \ldots \quad (2.5c)$$

We get

$$((k_{ij})) = \begin{pmatrix} 0 & 0 & 0 & -\alpha^2 \\ 0 & 0 & 0 & -\alpha\beta \\ 0 & 0 & 0 & 0 \\ \alpha^2 & \alpha\beta & 0 & 0 \end{pmatrix} \quad \ldots \quad (2.6)$$

consequently

$$^{(2)}k_{\lambda\mu} \overset{\text{def}}{=} {}^{(2)}k_{ij} \overset{i}{a}_\lambda \overset{j}{a}_\mu$$

$$= -\alpha \overset{4}{a}_\lambda \overset{4}{a}_\mu \quad \ldots \quad (2.7)$$

Since

$$\underset{i}{a}_\lambda = h_{ij} \overset{j}{a}_\lambda$$

we have

$$\left.\begin{aligned} \underset{1}{a}_\lambda &= \alpha \overset{1}{a}_\lambda + \rho \overset{4}{a}_\lambda \\ \underset{2}{a}_\lambda &= \beta^2 \overset{2}{a}_\lambda \\ \underset{3}{a}_\lambda &= -\alpha \overset{4}{a}_\lambda \\ \underset{4}{a}_\lambda &= \rho \overset{1}{a}_\lambda - \alpha \overset{3}{a}_\lambda + \sigma \overset{4}{a}_\lambda \end{aligned}\right\} \quad \ldots \quad (2.8)$$

and

$$\left.\begin{aligned} \overset{1}{a}_\lambda &= \frac{1}{\alpha} \underset{1}{a}_\lambda + \frac{\rho}{\alpha} \underset{3}{a}_\lambda \\ \overset{2}{a}_\lambda &= \frac{1}{\beta} \underset{2}{a}_\lambda \\ \overset{3}{a}_\lambda &= \frac{\rho}{\alpha} \underset{1}{a}_\lambda + \frac{\rho^2 - \sigma}{\alpha^2} \underset{3}{a}_\lambda - \frac{1}{\alpha} \underset{4}{a}_\lambda \\ \overset{4}{a}_\lambda &= -\frac{1}{\alpha} \underset{3}{a}_\lambda \end{aligned}\right\} \quad \ldots \quad (2.9)$$

From (2.8) and (2.9) we get,

$$h^{ij} = h^{\lambda\mu} \overset{i}{a}_\lambda \overset{j}{a}_\mu \quad \ldots \quad (2.10)$$

From (2.2)

$$h_{\mu\nu} = \alpha \overset{1}{a}_\mu \overset{1}{a}_\nu + 2\rho \overset{(1}{a}_\mu \overset{4)}{a}_\nu + \beta \overset{2}{a}_\mu \overset{2}{a}_\nu + 2\alpha \overset{(4}{a}_\mu \overset{1)}{a}_\nu + \sigma \overset{4}{a}_\mu \overset{4}{a}_\nu \quad \ldots \quad (2.11)$$

Using (2.7), (1.4a) can now be written as

$$R_{\mu\lambda} - (\tfrac{1}{2}R + p) h_{\mu\lambda} = \mu U_\mu U_\lambda - \alpha \overset{4}{a}_\mu \overset{4}{a}_\lambda \quad \ldots \quad (2.12)$$

The right-hand side of (2.12) is of rank two, therefore the left-hand side must be of second rank. Therefore we can take

$$U_\lambda = A \overset{3}{a}_\lambda + B \overset{4}{a}_\lambda \quad \ldots \quad (2.13)$$

($\overset{1}{a}_\lambda$ and $\overset{2}{a}_\lambda$ are excluded since they are complex conjugate) where A and B have to be determined. We have

$$U_\lambda U^\lambda = -1 = \frac{A(\rho^2 - \alpha\sigma)}{\alpha^3} - \frac{2AB}{\alpha} \quad \ldots \quad (2.14)$$

If we put $AB = 1$, then

$$U_\lambda = A \overset{3}{a}_\lambda + \frac{1}{A} \overset{4}{a}_\lambda \quad \ldots \quad (2.15)$$

where

$$A = \pm \alpha \sqrt{\frac{2 - \alpha}{\rho^2 - \alpha\sigma}}$$

From the Maxwell equations (1.4b) and (1.4c) we have

$$\nabla_{[i} k_{jk]} = 0 \quad \ldots \quad (2.16)$$

$$\nabla_i k_i^{\,i} = 0 \quad \ldots \quad (2.17)$$

Substituting from (2.6) in (2.16) we get

$$\nabla_3\alpha = 0 \qquad \ldots \quad (2.18a)$$

$$\nabla_3\beta = 0 \qquad \ldots \quad (2.18b)$$

Since α and β are scalars it follows that α and β are constant along the X^3 axis. From (2.17), substituting from (2.5), we get

$$\left.\begin{aligned} \nabla_3\alpha &= \epsilon u_1 = 0 \\ \nabla_3\beta &= \epsilon u_2 = 0 \\ \nabla_1\alpha &= \epsilon u_4 \neq 0 \end{aligned}\right\} \qquad \ldots \quad (2.19)$$

Therefore, if there is energy, then the first and second components of u_j are zero while the fourth component is non-zero.

§ 3. Using (2.7) and (2.11) eqn. (1.4) can be written as

$$R_{\mu\nu} - \left(\frac{R}{2}+p\right)\left(\alpha \overset{1}{a}_\mu \overset{1}{a}_\nu + 2\rho \overset{(1}{a}_\mu \overset{4)}{a}_\nu + \beta \overset{2}{a}_\mu \overset{2}{a}_\nu + 2\alpha \overset{(4}{a}_\mu \overset{1)}{a}_\nu\right)$$

$$= \left(\frac{\mu}{A^2} - \sigma - \alpha\right) \overset{4}{a}_\mu \overset{4}{a}_\gamma + \mu A^2 \overset{3}{a}_\mu \overset{3}{a}_\nu + \mu\left(\overset{3}{a}_\nu \overset{4}{a}_\mu + \overset{4}{a}_\nu \overset{4}{a}_\mu\right) \qquad \ldots \quad (3.1)$$

In order to simplify eqn. (3.1) we write

$$h_{ij} = \tilde{h}_{ij} + X_{ij} \qquad \ldots \quad (3.2)$$

where

$$((\tilde{h}_{ij})) = \begin{pmatrix} \alpha & 0 & 0 & 0 \\ 0 & \beta & 0 & 0 \\ 0 & 0 & 0 & -\alpha \\ 0 & 0 & -\alpha & 0 \end{pmatrix} \qquad \ldots \quad (3.3)$$

$$\det\left|\tilde{h}_{ij}\right| = \alpha^3\beta$$

and

$$((X_{ij})) = \begin{pmatrix} 0 & 0 & 0 & \rho \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \rho & 0 & 0 & \sigma \end{pmatrix} \qquad \ldots \quad (3.4)$$

$$\tilde{h}_{\lambda\mu} \overset{\text{def}}{=} \tilde{h}_{ij} \overset{i}{a}_\lambda \overset{j}{a}_\mu \qquad \ldots \quad (3.5a)$$

$$X_{\lambda\mu} \overset{\text{def}}{=} X_{ij} \overset{i}{a}_\lambda \overset{j}{a}_\mu \qquad \ldots \quad (3.5b)$$

If $\Gamma^\nu_{\lambda\mu}$ are the connections formed with $h_{\mu\nu}$ and $\tilde{\Gamma}^\nu_{\lambda\mu}$ are the connections formed with $\tilde{h}_{\mu\nu}$ and $\tilde{\nabla}_\mu$ stands for the covariant derivative with respect to $\tilde{h}_{\mu\nu}$ then we make use of the following results (Schmidt 1966):

$$T_{\lambda\mu|\nu} \overset{\text{def}}{=} \tfrac{1}{2}(\tilde{\nabla}_\lambda T_{\mu\nu} + \tilde{\nabla}_\mu T_{\lambda\nu} - \tilde{\nabla}_\nu T_{\lambda\mu}) \qquad \ldots \quad (3.6a)$$

for any tensor $T_{\lambda\mu}$

$$T_{\lambda\mu|}{}^{\nu} \overset{\text{def}}{=} T_{\lambda\mu|\alpha} h^{\alpha\nu} \quad \ldots \quad (3.6b)$$

$$h_{\lambda\mu|}{}^{\nu} = X_{\lambda\mu|}{}^{\nu} \quad \ldots \quad (3.6c)$$

$$\Gamma^{\nu}_{\lambda\mu} = \tilde{\Gamma}^{\nu}_{\lambda\mu} + X_{\lambda\mu|}{}^{\nu} \quad \ldots \quad (3.6d)$$

$$X_{\lambda\mu|}{}^{\nu} = 0 \quad \ldots \quad (3.6e)$$

The equations (3.1) become on using (3.6)

$$\tilde{R}_{\mu\nu} - \tilde{\nabla}_{\alpha} X_{\mu\nu|}{}^{\alpha} + X_{\alpha\mu|}{}^{\beta} - \left(\frac{R}{2}+p\right)(\tilde{h}_{\mu\nu} + X_{\mu\nu})$$

$$= \left(\frac{\mu}{A^2} - \alpha\right) \overset{4}{a}_{\mu}\overset{4}{a}_{\nu} + \mu A^2 \overset{3}{a}_{\mu}\overset{3}{a}_{\nu} + \mu\left(\overset{3}{a}_{\nu}\overset{4}{a}_{\mu} + \overset{4}{a}_{\nu}\overset{3}{a}_{\mu}\right) \quad \ldots \quad (3.7)$$

§ 4. In order to apply the Cauchy existence theorem we separate in (3.7) the second derivatives from the rest (Schmidt 1966). We get

$$h^{\alpha\beta}(\partial_{\mu\alpha} X_{\nu\beta} + \partial_{\nu\alpha} X_{\mu\beta} - \partial_{\alpha\beta} X_{\mu\nu} - \partial_{\mu\nu} X_{\alpha\beta}) = Q_{\mu\nu} \quad \ldots \quad (4.1)$$

where $Q_{\mu\nu}$ denote the rest of the equations. Now separating the derivatives $\partial_{\text{IV IV}}$ from the rest and letting the indices ξ, η run through I, II and IH only, we get

$$h^{\text{IV IV}} \partial_{\text{IV IV}} X_{\xi\eta} = M_{\xi\eta} \quad \text{for } \mu, \nu = \xi, \eta \quad \ldots \quad (4.2)$$

$$h^{\text{IV}\,\eta} \partial_{\text{IV IV}} X_{\xi\eta} = M_{\xi\,\text{IV}} \quad \text{for } \mu, \nu = \text{IV}, \xi \quad \ldots \quad (4.3)$$

$$h^{\xi\eta} \partial_{\text{IV IV}} X_{\xi\eta} = M_{\text{IV IV}} \quad \text{for } \mu, \nu = \text{IV}, \text{IV} \quad \ldots \quad (4.4)$$

where $M_{\mu\nu}$ stands for the rest.
From (3.4) and (3.5*b*) the above equations become on substitution

$$(\partial_{\text{IV IV}}\sigma) \overset{4}{a}_{\xi}\overset{4}{a}_{\eta} + 2(\partial_{\text{IV IV}}\rho) \overset{(1}{a}_{\xi}\overset{4)}{a}_{\eta} = N_{\xi\eta} \quad \ldots \quad (4.5)$$

$$h^{\text{IV}\,\eta}\left[(\partial_{\text{IV III}}\sigma) \overset{4}{a}_{\xi}\overset{4}{a}_{\eta} + 2(\partial_{\text{IV IV}}\rho) \overset{(1}{a}_{\xi}\overset{4)}{a}_{\eta} = N_{\text{IV}\xi}\right. \quad \ldots \quad (4.6)$$

$$h^{\xi\eta}\left[(\partial_{\text{IV IV}}\sigma) \overset{4}{a}_{\xi}\overset{4}{a}_{\eta} + 2(\partial_{\text{IV IV}}\rho) \overset{(1}{a}\,\overset{4)}{a}_{\eta}\right] = N_{\text{IV IV}} \quad \ldots \quad (4.7)$$

where $N_{\mu\nu}$ stands for the rest.

Now if we denote the dual vectors to $\overset{r}{a}_{\xi}$ by $\underset{r}{a}^{\xi}$ then

$$\overset{r}{a}_{\xi} \underset{s}{a}^{\xi} = \delta^{r}_{s}$$

$$\overset{r}{a}_{\xi} \underset{r}{a}^{\eta} = \delta^{\eta}_{\xi} \quad \ldots \quad (4.8)$$

where $r = 1, 2, 3$.

Equations (4.5), (4.6) and (4.7) become

$$\left.\begin{aligned} \partial_{\mathrm{IV\,IV}}\sigma &= \underset{4}{a^{\xi}}\underset{4}{a^{\eta}}N_{\xi\eta} \\ \partial_{\mathrm{IV\,IV}}\rho &= \underset{1}{a^{\xi}}\underset{4}{a^{\eta}}N_{\xi\eta} \end{aligned}\right\} \quad \ldots \quad (4.9)$$

and

$$\left.\begin{aligned} N_{\xi\eta}\underset{r}{a^{\xi}}\underset{s}{a^{\eta}} &= 0 \\ h^{\mathrm{IV}\,\eta}N_{\xi\eta} &= N_{\xi\,\mathrm{IV}} \\ h^{\xi\eta}N_{\xi\eta} &= N_{\mathrm{IV\,IV}} \end{aligned}\right\} \quad \ldots \quad (4.10)$$

for $(r, s) = (1, 1), (1, 2), (1, 3), (2, 2), (2, 3)$ and $(3, 3)$.

A unique solution (ρ, σ) can be obtained by prescribing boundary values for ρ, σ, $\partial_{\mathrm{VI}}\rho$, $\partial_{\mathrm{IV}}\sigma$ along the hypersurface $X^{\mathrm{IV}} = 0$ such that the boundary values satisfy (4.10) (Schmidt 1966).

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