THE H-FUNCTIONS AS KERNELS IN CHAIN TRANSFORMS

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In this paper using the idea of the chain-transforms, introduced by Fox (1948), certain K-chains (defined below) in terms of H-function, defined by Fox (1961) have been obtained so as to give rise to T-chains (defined below). The results, recently given by Narain (1965*b*), follow as its particular cases.

§1. Fox (1948), extending the ideas developed by Hardy and Titchmarsh (1933) and Watson (1933), defined chain transforms by the system of equations:

$$\left.\begin{aligned} g_{m+1}(x) &= \int_0^\infty g_m(u)k_m(xu)du(m = 1, \ldots, n-1), \\ g_1(x) &= \int_0^\infty g_n(u)k_n(xu)du(m = n), \end{aligned}\right\} \quad \ldots \quad \ldots \quad (1.1)$$

where n is an even integer.

With $n = 2$, the above system (1.1) reduces to the well-known Fourier transform (Titchmarsh 1948, chapter VIII).

When n is an odd integer the last equation of (1.1), corresponding to $m = n$, is slightly modified and then the chain transform is given by

$$\left.\begin{aligned} g_{m+1}(x) &= \int_0^\infty g_m(u)k_m(xu)du(m = 1, \ldots, n-1) \\ g_1(x) &= \int_0^\infty \frac{1}{u} g_n\left(\frac{1}{u}\right)k_n(xu)du(m = n). \end{aligned}\right\} \quad \ldots \quad \ldots \quad (1.2)$$

Following Fox, the set of eqns. (1.1) or (1.2) will be referred to as a T-chain, the function $k_m(x)$ as kernels and the set of kernels $k_1(x), \ldots, k_n(x)$ as a K-chain.

The object of this paper is to obtain, in terms of H-function introduced by Fox (1961), certain K-chains which may give rise to T-chains of the type (1.1), where n is an even integer.

Adopting the symbolic notation, given by Gupta and Jain (*in press*), the

H-function of Fox will be represented and defined as:

$$H_{p,q}^{m,n}\left[x\left|\begin{matrix}(a_1,\alpha_1),\ldots,(a_p,\alpha_p)\\(b_1,\beta_1),\ldots,(b_q,\beta_q)\end{matrix}\right.\right]$$

$$=\frac{1}{2\Pi i}\int_T \frac{\prod_{j=1}^{m}\Gamma(b_j+\beta_j s)\prod_{j=1}^{n}\Gamma(1-a_j-\alpha_j s)}{\prod_{j=m+1}^{q}\Gamma(1-b_j-\beta_j s)\prod_{j=n+1}^{p}\Gamma(a_j+\alpha_j s)}\cdot x^{-s}\,ds, \quad \ldots \quad (1.3)$$

where the parameters are such that the poles of the integrand are simple and those of $\Gamma(b_j+\beta_j s)(j=1,\ldots,m)$ lie on one side of the contour T and those of $\Gamma(1-a_j-\alpha_j s)(j=1,\ldots,n)$ lie on the other side.

The H-function, defined in (1.3), is further given another symbolic notation in a more compact form as:

$$H_{p,q}^{m,n}\left[x\left|\begin{matrix}\{(a_p,\alpha_p)\}\\\{(b_q,\beta_q)\}\end{matrix}\right.\right],$$

where $\{(a_p,\alpha_p)\}$ stands for the set of parameters $(a_1,\alpha_1),\ldots,(a_p,\alpha_p)$.

According to Braaksma (1963)

$$H_{p,q}^{m,n}\left[x\left|\begin{matrix}\{(a_p,\alpha_p)\}\\\{(b_q,\beta_q)\}\end{matrix}\right.\right]=0(|x|^{\alpha})\text{ for small } x,$$

where

$$\sum_{1}^{p}\alpha_j-\sum_{1}^{q}\beta_j\leqslant 0 \quad\text{and}\quad \alpha=\text{Re}\,(b_h/\beta_h)(h=1,\ldots,m),$$

and

$$H_{p,q}^{m,n}\left[x\left|\begin{matrix}\{(a_p,\alpha_p)\}\\\{(b_q,\beta_q)\}\end{matrix}\right.\right]=0(|x|^{\beta})\text{ for large } x,$$

where

$$\sum_{1}^{p}\alpha_j-\sum_{1}^{q}\beta_j<0,\ \sum_{1}^{n}\alpha_j-\sum_{n+1}^{p}\alpha_j+\sum_{1}^{m}\beta_j-\sum_{m+1}^{q}\beta_j\equiv\lambda>0,$$

$$|\arg x|<\tfrac{1}{2}\lambda\Pi \quad\text{and}\quad \beta=\text{Re}\,[(a_i-1)/\alpha_i](i=1,\ldots,n).$$

Also, some of the known properties of the H-function (1.3), which will be useful in our present work, are mentioned here:

$$H_{p,q}^{m,n}\left[x\left|\begin{matrix}\{(a_p,1)\}\\\{(b_q,1)\}\end{matrix}\right.\right]\equiv G_{p,q}^{m,n}\left[x\left|\begin{matrix}a_1,\ldots,a_p\\b_1,\ldots,b_q\end{matrix}\right.\right], \quad \ldots \quad \ldots \quad (1.4)$$

where the right-hand function is the Meijer's G-function (Erdélyi *et al.* 1953), which is a sum of hypergeometric functions, each of which is usually an entire function.

$$x^{\sigma}H_{p,q}^{m,n}\left[x\left|\begin{matrix}\{(a_p,\alpha_p)\}\\\{(b_q,\beta_q)\}\end{matrix}\right.\right]\equiv H_{p,q}^{m,n}\left[x\left|\begin{matrix}\{(a_p+\sigma\alpha_p,\alpha_p)\}\\\{(b_q+\sigma\beta_q,\beta_q)\}\end{matrix}\right.\right]. \quad \ldots \quad (1.5)$$

In what follows $K_m(s)$ will mean the Mellin transform of $k_m(x)(m = 1, \ldots n)$, i.e.

$$K_m(s) = \int_0^\infty k_m(x)\, x^{s-1}\, ds\ (m = 1, \ldots, n),$$

and $\mathbf{f}(x)$ will be taken to be related to $f(x)$ by the equation

$$\mathbf{f}(x) = \int_0^x f(u)\, du. \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (1.6)$$

Any pair of such functions, *viz.* $\mathbf{h}(x)$ and $h(x)$, etc., will be obeying the relation of the type (1.6).

§2. In our discussion, we shall require the following result, due to Fox (1948).

In case when n is an even integer, if

(i) the Mellin transforms $K_m(s)(m = 1, \ldots, n)$ of K-chain satisfy

$$K_1(s)K_2(1-s)K_3(s)K_4(1-s) \ldots K_{n-1}(s)K_n(1-s) = 1, \quad \ldots \quad (2.1)$$

(ii) $K_m(s)(m = 1, \ldots, n)$ be bounded on the line $\frac{1}{2}+it$ for all values of t, and

(iii) $g_1(x)$ belongs to $L^2(0, \infty)$, then

$$\left.\begin{aligned} g_{m+1}(x) &= \frac{d}{dx}\int_0^\infty g_m(u)\mathbf{k}_m(xu)\,\frac{du}{u}\ (m = 1, \ldots, n-1), \\ g_1(x) &= \frac{d}{dx}\int_0^\infty g_n(u)\mathbf{k}_n(xu)\,\frac{du}{u}\ (m = n). \end{aligned}\right\} \quad \ldots \quad (2.2)$$

When differentiation through the integral signs, with respect to x, is permissible, (2.2) reduce to (1.1).

§3. Let us consider the function

$$k(x) = 2\beta\gamma x^{\gamma-\frac{1}{2}} H^{q,\,p}_{2p,\,2q}\left[\beta^2 x^{2\gamma}\left|\begin{matrix}\{(a_p, \alpha_p)\}, \{(1-a_p-\alpha_p, \alpha_p)\} \\ \{(b_q, \beta_q)\}, \{(1-b_q-\beta_q, \beta_q)\}\end{matrix}\right.\right], \quad \ldots \quad (3.1)$$

where β and γ are real constants, $\sum_1^q \beta_j - \sum_1^p \alpha_j > 0$, Re $[(1-a_j)/\alpha_j] > \frac{1}{2}(j = 1, \ldots, p)$ and Re $(b_j/\beta_j) > -\frac{1}{2}(j = 1, \ldots, q)$.

By using (1.5) and the definition (1.3) of the H-function and making obvious replacements, (3.1) can be thrown into the form:

$$k(x) = \frac{1}{2\Pi i}\int_T \beta^{\frac{1-2s}{2\gamma}} \cdot \frac{\prod_{j=1}^q \Gamma\left(b_j+\frac{2\gamma-1}{4\gamma}\beta_j+\frac{s}{2\gamma}\beta_j\right)\prod_{j=1}^p \Gamma\left(1-a_j-\frac{2\gamma-1}{4\gamma}\alpha_j-\frac{s}{2\gamma}\alpha_j\right)}{\prod_{j=1}^q \Gamma\left(b_j+\frac{2\gamma+1}{4\gamma}\beta_j-\frac{s}{2\gamma}\beta_j\right)\prod_{j=1}^p \Gamma\left(1-a_j-\frac{2\gamma+1}{4\gamma}\alpha_j+\frac{s}{2\gamma}\alpha_j\right)} \cdot x^{-s} ds, \quad (3.2)$$

which is clearly of the same form, as given by Fox (1961) for H-function to be a symmetrical Fourier kernel. Therefore $k(x)$ is a symmetrical Fourier kernel.

If $K(s)$ denotes the coefficient of x^{-s} in the integrand of (3.2) then evidently $K(s)$ is the Mellin transform of $k(s)$ and satisfies the condition

$$K(s)K(1-s) = 1.$$

Now on applying Gauss' multiplication formula (Erdélyi *et al.* 1953) for Gamma-function:

$$\Gamma(nz) = n^{nz-\frac{1}{2}}(2\Pi)^{\frac{1}{2}(1-n)} \prod_{r=1}^{n} \left(z+\frac{r-1}{n}\right) (n = 2, 3, 4, \ldots),$$

and writing, for the sake of brevity, D instead of $2\left(\sum_1^q \beta_j - \sum_1^p \alpha_j\right)$, $K(s)$ reduces to

$$K(s) = \prod_{r=1}^{n} K_r(s), \qquad \ldots \quad (3.3)$$

where

$$K_r(s) = (n^D)^{\frac{1}{2\gamma n}(s-\frac{1}{2})} \beta^{\frac{1-2s}{2\gamma n}}$$

$$\times \frac{\prod_{j=1}^{q} \Gamma\left[\left(b_j+\frac{2\gamma-1}{4\gamma}\beta_j+\frac{s}{2\gamma}\beta_j+r-1\right)\Big/n\right] \prod_{j=1}^{p} \Gamma\left[\left(-a_j-\frac{2\gamma-1}{4\gamma}\alpha_j-\frac{s}{2\gamma}\alpha_j+r\right)\Big/n\right]}{\prod_{j=1}^{q} \Gamma\left[\left(b_j+\frac{2\gamma+1}{4\gamma}\beta_j-\frac{s}{2\gamma}\beta_j+r-1\right)\Big/n\right] \prod_{j=1}^{p} \Gamma\left[\left(-a_j-\frac{2\gamma+1}{4\gamma}\alpha_j+\frac{s}{2\gamma}\alpha_j+r\right)\Big/n\right]}, \quad \ldots \quad (3.4)$$

which is obviously the Mellin transform of

$$k_r(x) = 2\beta\gamma n^{1-D/2} x^{\gamma n-\frac{1}{2}}$$

$$\times H^{q,p}_{2p,2q}\left[\frac{\beta^2 x^{2\gamma n}}{n^D} \left| \begin{matrix} \left\{\left(1-\frac{r}{n}+\frac{a_p}{n}+\frac{1-n}{2n}\alpha_p, \alpha_p\right)\right\}, \left\{\left(\frac{r}{n}-\frac{a_p}{n}-\frac{1+n}{2n}\alpha_p, \alpha_p\right)\right\} \\ \left\{\left(\frac{r}{n}-\frac{1}{n}+\frac{b_q}{n}+\frac{1-n}{2n}\beta_q, \beta_q\right)\right\}, \left\{\left(1-\frac{r}{n}+\frac{1}{n}-\frac{b_q}{n}-\frac{1+n}{2n}\beta_q, \beta_q\right)\right\} \end{matrix} \right.\right]. \quad \ldots \quad (3.5)$$

For large s, the asymptotic expansion (Erdélyi *et al.* 1953) of Gamma-function is given by

$$\log \Gamma(s+a) = (s+a-\tfrac{1}{2}) \log (s) - s + \tfrac{1}{2} \log (2\Pi) + O\left(\frac{1}{|s|}\right), \qquad \ldots \quad (3.6)$$

where $|\arg s| < \Pi$. Let $s = \sigma+it$. Using (3.6), and the relation

$$\Gamma(z)\Gamma(1-z) = \Pi \operatorname{cosec} (\Pi z),$$

in case of such Gamma-functions in (3.4) where s occurs with negative sign, we obtain on the line σ = constant,

$$K_r(s) = |t|^{\mu(\sigma-\frac{1}{2})} \exp [it(\mu \log |t|+B)]\left[Q+O\left(\frac{1}{|t|}\right)\right], \qquad \ldots \quad (3.7)$$

for large $|t|$, where $\mu = D/(2\gamma n)$, B is a constant and Q is also a constant having one value for large positive t and another for large negative t. It is evident from (3.7) that $K_r(s)$ are all bounded on the line $\sigma = \frac{1}{2}$.

Now if we consider the set of $2n$-functions

$$k_1(x),\ k_1(x),\ k_2(x),\ k_2(x),\ \ldots,\ k_n(x),\ k_n(x), \qquad \ldots \quad \ldots \quad (3.8)$$

given by (3.5), we see that their Mellin transforms

$$K_1(s),\ K_1(s),\ K_2(s),\ K_2(s),\ \ldots,\ K_n(s),\ K_n(s)$$

satisfy the functional relation (2.1) and also they are bounded on the line $s = \frac{1}{2}+it$.

Hence the functions (3.8) form a K-chain of order $2n$.

Since each $k_r(x)$ is itself a Fourier kernel, we get a trivial T-chain if we take the functions in (3.8) in the same order as they are written. For at every alternate step in the transformations (1.1), we will get the functions, with which we started, *viz.* $g_1(x)$. It is only when the order of the functions (3.8) is rearranged that we get non-trivial T-chains. Note that the validity of (2.1) remains unaffected by rearrangement among themselves of either the odd or the even factors. Making corresponding changes in the order of the kernels (3.8) we obtain a number of T-chains.

§4. Following Narain (1965*a*) a pair of unsymmetrical Fourier kernels can be put in the form as:

$$k(x) = 2\beta\gamma x^{\gamma-\frac{1}{2}} H^{p',p}_{p+q,\,p'+q'}\left[\beta^2 x^{2\gamma}\,\middle|\, \begin{matrix}\{(a_p,\alpha_p)\},\ \{(b_q,\beta_q)\}\\ \{(c_{p'},\gamma_{p'})\},\ \{(d_{q'},\delta_{q'})\}\end{matrix}\right],$$

$$h(x) = 2\beta\gamma x^{\gamma-\frac{1}{2}} H^{q',q}_{p+q,\,p'+q'}\left[\beta^2 x^{2\gamma}\,\middle|\, \begin{matrix}\{(1-b_q-\beta_q,\beta_q)\},\ \{(1-a_p-\alpha_p,\alpha_p)\}\\ \{(1-d_{q'}-\delta_{q'},\delta_{q'})\},\ \{(1-c_{p'}-\gamma_{p'},\gamma_{p'})\}\end{matrix}\right], \qquad \ldots \quad (4.1)$$

where β and γ are real constants; $\sum_1^{p'}\gamma_j - \sum_1^{q}\beta_j = \sum_1^{q'}\delta_j - \sum_1^{p}\alpha_j > 0$,

$$\text{Re}\,[(1-a_j)/\alpha_j] > \tfrac{1}{2}(j = 1,\ldots,p),\ \text{Re}\,(b_j/\beta_j) > -\tfrac{1}{2}(j = 1,\ldots,q),$$

$$\text{Re}\,[(1-d_j)/\delta_j] > \tfrac{1}{2}(j = 1,\ldots,q'),\ \text{Re}\,(c_j/\gamma_j) > -\tfrac{1}{2}(j = 1,\ldots,p')$$

and

$$\sum_1^{p} a_j + \sum_1^{q} b_j = \sum_1^{p'} c_j + \sum_1^{q'} d_j.$$

Starting with these functions (4.1) and proceeding on the similar lines as those of §3 it can be shown that the functions

$$k_1(x),\ h_1(x),\ k_2(x),\ h_2(x),\ \ldots,\ k_n(x),\ h_n(x)$$

form a K-chain of order $2n$, where these functions are given by formulae:

$$k_r(x) = 2\beta\gamma n^{1-D/2} x^{\gamma n - \frac{1}{2}}$$

$$\times H^{p',p}_{p+q,\,p'+q'}\left[\frac{\beta^2 x^{2\gamma n}}{n^D}\left|\begin{matrix}\left\{\left(1-\frac{r}{n}+\frac{a_p}{n}+\frac{1-n}{2n}\alpha_p,\, \alpha_p\right)\right\},\ \left\{\left(\frac{r}{n}-\frac{1}{n}+\frac{b_q}{n}+\frac{1-n}{2n}\beta_q,\, \beta_q\right)\right\}\\ \left\{\left(\frac{r}{n}-\frac{1}{n}+\frac{c_{p'}}{n}+\frac{1-n}{2n}\gamma_{p'},\, \gamma_{p'}\right)\right\},\ \left\{\left(1-\frac{r}{n}+\frac{d_{q'}}{n}+\frac{1-n}{2n}\delta_{q'},\, \delta_{q'}\right)\right\}\end{matrix}\right.\right],$$

$$h_r(x) = 2\beta\gamma n^{1-D/2} x^{\gamma n - \frac{1}{2}}$$

$$\times H^{q',q}_{p+q,\,p'+q'}\left[\frac{\beta^2 x^{2\gamma n}}{n^D}\left|\begin{matrix}\left\{\left(1-\frac{r}{n}+\frac{1}{n}-\frac{b_q}{n}-\frac{1+n}{2n}\beta_q,\, \beta_q\right)\right\},\ \left\{\left(\frac{r}{n}-\frac{a_p}{n}-\frac{1+n}{2n}\alpha_p,\, \alpha_p\right)\right\}\\ \left\{\left(\frac{r}{n}-\frac{d_{q'}}{n}-\frac{1+n}{2n}\delta_{q'},\, \delta_{q'}\right)\right\},\ \left\{\left(1-\frac{r}{n}+\frac{1}{n}-\frac{c_{p'}}{n}-\frac{1+n}{2n}\gamma_{p'},\, \gamma_{p'}\right)\right\}\end{matrix}\right.\right]$$

where

$$\tfrac{1}{2}D \equiv \left(\sum_1^{p'}\gamma_j - \sum_1^{q}\beta_j\right) = \left(\sum_1^{q'}\delta_j - \sum_1^{p}\alpha_j\right).$$

By using (1.4) and setting the parameters suitably in our results of this paper, we get the corresponding results in Meijer's G-functions, recently obtained by Narain (1965*b*).

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