DEFINITE INTEGRALS INVOLVING SELF-RECIPROCAL FUNCTIONS*

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In this paper, some theorems have been given about the functions such that if their $\chi_{\mu,\,k,\,m}$-transform is $F(x)/\sqrt{x}$, their $\chi_{\nu,\,k,\,m}$-transform is $F\left(\frac{1}{x}\right)\Big/\sqrt{x}$. The nature of such functions has been discussed and it has been shown that definite integrals, which involve self-reciprocal functions, satisfy a particular type of relation. With the help of some of these results, self-reciprocal functions have been derived as solutions of integral equations.

§ 1. A generalization of the Hankel transform

$$g(x) = \int_0^\infty \sqrt{xy}\, J_\nu(xy)\, f(y)\, dy \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (1.1)$$

has been introduced by Narain (1956-57, p. 270) in the form

$$g(x) = 2^{-\nu} \int_0^\infty (xy)^{\nu+\frac{1}{2}} \chi_{\nu,\,k,\,m}(x^2y^2/4) f(y)\, dy \qquad \ldots \quad \ldots \quad (1.2)$$

where

$$\chi_{\nu,\,k,\,m}(x) = x^{-\nu} G^{2,\,1}_{2,\,4}\left(x \,\middle|\, \begin{matrix} k-m-\frac{1}{2},\ \nu-k+m+\frac{1}{2} \\ \nu,\ \nu+2m,\ -2m,\ 0 \end{matrix}\right).$$

He calls $g(x)$ in (1.2) to be $\chi_{\nu,\,k,\,m}$-transform of $f(x)$ and has proved in the same paper that this is a transform reciprocal like the Hankel transform. In case $g(x) \equiv f(x)$ we may call it to be self-reciprocal in the $\chi_{\nu,\,k,\,m}$-transform and we shall say that $f(x)$ is $R_\nu\ (k, m)$.

When $k+m = \frac{1}{2}$, (1.2) reduces to (1.1).

Yet another generalization of (1.1) has been given by Narain (1959, p. 298) as:

$$g(x) = \sqrt{2} \int_0^\infty G^{q,\,p}_{2p,\,2q}\left(\frac{x^2y^2}{4} \,\middle|\, \begin{matrix} a_1, \ldots, a_p, \frac{1}{2}-a_1, \ldots, \frac{1}{2}-a_p \\ b_1, \ldots, b_q, \frac{1}{2}-b_1, \ldots, \frac{1}{2}-b_q \end{matrix}\right) f(y)\, dy. \ldots \quad (1.3)$$

On putting $p = 0$, $q = 1$ and $b_1 = \frac{1}{4}+\frac{1}{2}\nu$ (1.3) reduces to (1.1).

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In this paper, we have given some theorems about functions such that if their $\chi_{\mu,k,m}$-transform is $x^{-\frac{1}{2}}F(x)$, their $\chi_{\nu,k,m}$-transform is $x^{-\frac{1}{2}}F\left(\frac{1}{x}\right)$. We have found out the nature of such functions and have also shown that definite integrals, which involve self-reciprocal functions, satisfy a particular type of relation. With the help of some of the results, self-reciprocal functions have been derived as solutions of the integral equation.

§ 2. Theorem 1. If $f(x)$ is $R_\nu(k, m)$, i.e. a solution of the integral equation

$$f(x) = \sqrt{2}\int_0^\infty G_{2,4}^{2,1}\left(\frac{x^2t^2}{4}\,\middle|\,\begin{matrix} k-m-\frac{1}{4}-\frac{\nu}{4}, \frac{\nu}{2}-k+m+\frac{3}{4} \\ \frac{\nu}{2}+\frac{1}{4}, \frac{\nu}{2}+\frac{1}{4}+2m, \frac{1}{4}-\frac{\nu}{2}, \frac{1}{4}-\frac{\nu}{2}-2m \end{matrix}\right) f(t)\,dt$$

and if

$$g(x) = \int_a^{1/a} t^{-\frac{1}{2}}F(t)\,f(xt)\,dt$$

where a is positive or zero and $F(t)$ is any function such that

$$F(t) = F\left(\frac{1}{t}\right),$$

then $g(x)$ is also $R_\nu(k, m)$, i.e. is a solution of the integral equation, provided the integrals involved are absolutely and uniformly convergent.

Proof: We have

$$\sqrt{2}\int_0^\infty G_{2,4}^{2,1}\left(\frac{x^2t^2}{4}\,\middle|\,\begin{matrix} k-m-\frac{1}{4}-\frac{\nu}{2}, \frac{\nu}{2}-k+m+\frac{3}{4} \\ \frac{\nu}{2}+\frac{1}{4}, \frac{\nu}{2}+\frac{1}{4}+2m, \frac{1}{4}-\frac{\nu}{2}, \frac{1}{4}-\frac{\nu}{2}-2m \end{matrix}\right) g(t)\,dt$$

$$= \sqrt{2}\int_0^\infty G_{2,4}^{2,1}\left(\frac{x^2t^2}{4}\,\middle|\,\begin{matrix} k-m-\frac{1}{4}-\frac{\nu}{2}, \frac{\nu}{2}-k+m+\frac{3}{4} \\ \frac{\nu}{2}+\frac{1}{4}, \frac{\nu}{2}+\frac{1}{4}+2m, \frac{1}{4}-\frac{\nu}{2}, \frac{1}{4}-\frac{\nu}{2}-2m \end{matrix}\right) dt \int_a^{1/a} u^{-\frac{1}{2}}F(u)\,f(tu)\,du$$

$$= \sqrt{2}\int_a^{1/a} u^{-\frac{1}{2}}F(u)\,du \int_0^\infty G_{2,4}^{2,1}\left(\frac{x^2t^2}{4}\,\middle|\,\begin{matrix} k-m-\frac{1}{4}-\frac{\nu}{2}, \frac{\nu}{2}-k+m+\frac{3}{4} \\ \frac{\nu}{2}+\frac{1}{4}, \frac{\nu}{2}+\frac{1}{4}+2m, \frac{1}{4}-\frac{\nu}{2}, \frac{1}{4}-\frac{\nu}{2}-2m \end{matrix}\right) f(tu)\,dt$$

$$= \sqrt{2}\int_a^{1/a} u^{-\frac{1}{2}}F(u)\,du \int_0^\infty G_{2,4}^{2,1}\left(\frac{x^2t^2}{4u^2}\,\middle|\,\begin{matrix} k-m-\frac{1}{4}-\frac{\nu}{2}, \frac{\nu}{2}-k+m+\frac{3}{4} \\ \frac{\nu}{2}+\frac{1}{4}, \frac{\nu}{2}+\frac{1}{4}+2m, \frac{1}{4}-\frac{\nu}{2}, \frac{1}{4}-\frac{\nu}{2}-2m \end{matrix}\right) f(t)\,dt$$

$$= \int_a^{1/a} u^{-\frac{1}{2}}F(u)\,f\left(\frac{x}{u}\right)du = \int_a^{1/a} t^{-\frac{1}{2}}F\left(\frac{1}{t}\right)f(xt)\,dt$$

$$= g(x),$$

since $F(t) = F\left(\frac{1}{t}\right)$.

When $a = 0$, the theorem can be expressed in the form that

$$g(x) = \int_0^\infty (xu)^{-\frac{1}{2}} F(u/x) f(u)\, du$$

is self-reciprocal in $\chi_{\nu, k, m}$-transform.

The most obvious example is obtained by taking

$$F(t) = (t^{\alpha}+t^{-\alpha})^{-\lambda},$$

in which case we get the self-reciprocal function in $\chi_{\nu, k, m}$-transform

$$\int_0^\infty \frac{(xu)^{\alpha\lambda-\frac{1}{2}}}{(u^{2\alpha}+x^{2\alpha})^{\lambda}} f(u)\, du.$$

The argument can also be generalized when $a = 0$. The corresponding result is then:

THEOREM 2. If $f(x)$ is $R_\nu(k, m)$, i.e. a solution of the integral equation

$$f(x) = \sqrt{2} \int_0^\infty G_{2,4}^{2,1}\left(\frac{x^2t^2}{4}\middle|\begin{matrix} k-m-\frac{1}{4}-\frac{\nu}{2}, \frac{\nu}{2}-k+m+\frac{3}{4} \\ \frac{\nu}{2}+\frac{1}{4}, \frac{\nu}{2}+\frac{1}{4}+2m, \frac{1}{4}-\frac{\nu}{2}, \frac{1}{4}-\frac{\nu}{2}-2m \end{matrix}\right) f(t)\, dt$$

and $G(x)$ is such that

$$x^{-\frac{1}{2}}F(x) = \sqrt{2} \int_0^\infty G_{2,4}^{2,1}\left(\frac{x^2t^2}{4}\middle|\begin{matrix} k-m-\frac{1}{4}-\frac{\nu}{2}, \frac{\nu}{2}-k+m+\frac{3}{4} \\ \frac{\nu}{2}+\frac{1}{4}, \frac{\nu}{2}+\frac{1}{4}+2m, \frac{1}{4}-\frac{\nu}{2}, \frac{1}{4}-\frac{\nu}{2}-2m \end{matrix}\right) G(t)\, dt,$$

$$x^{-\frac{1}{2}}F\left(\frac{1}{x}\right) = \sqrt{2} \int_0^\infty G_{2,4}^{2,1}\left(\frac{x^2t^2}{4}\middle|\begin{matrix} k-m-\frac{1}{4}-\frac{\mu}{2}, \frac{\mu}{2}-k+m+\frac{3}{4} \\ \frac{\mu}{2}+\frac{1}{4}, \frac{\mu}{2}+\frac{1}{4}+2m, \frac{1}{4}-\frac{\mu}{2}, \frac{1}{4}-\frac{\mu}{2}-2m \end{matrix}\right) G(t)\, dt,$$

then

$$g(x) = \int_0^\infty t^{-\frac{1}{2}} F(t) f(xt)\, dt$$

is $R_\mu(k, m)$, i.e. a solution of the integral equation

$$g(x) = \sqrt{2} \int_0^\infty G_{2,4}^{2,1}\left(\frac{x^2t^2}{4}\middle|\begin{matrix} k-m-\frac{1}{4}-\frac{\mu}{2}, \frac{\mu}{2}-k+m+\frac{3}{4} \\ \frac{\mu}{2}+\frac{1}{4}, \frac{\mu}{2}+\frac{1}{4}+2m, \frac{1}{4}-\frac{\mu}{2}, \frac{1}{4}-\frac{\mu}{2}-2m \end{matrix}\right) g(t)\, dt,$$

where $G(x)$ is a continuous function in $(0, \infty)$ is $O(x^{\lambda})$ for small x and $O(x^{-\delta})$ as $x \to \infty$, $f(x)$ belongs to $A(\alpha, a)$, Re $(\delta+1+2m-2k) > 0$, Re $(2\nu+\lambda+2+2m \pm 2m) > 0$, Re $(2\mu+\lambda+2+2m\pm 2m) > 0$ and $2m$ is not an integer or zero.

PROOF: We have

$$g(x) = \sqrt{2}\int_0^\infty f(xt)\,dt \int_0^\infty G^{2,1}_{2,4}\left(\frac{u^2t^2}{4}\middle|\begin{matrix} k-m-\frac{1}{4}-\frac{\nu}{2}, \frac{\nu}{2}-k+m+\frac{3}{4} \\ \frac{\nu}{2}+\frac{1}{4}, \frac{\nu}{2}+\frac{1}{4}+2m, \frac{1}{4}-\frac{\nu}{2}, \frac{1}{4}-\frac{\nu}{2}-2m \end{matrix}\right) G(u)\,du$$

$$= \sqrt{2}\int_0^\infty G(u)\,du \int_0^\infty G^{2,1}_{2,4}\left(\frac{u^2t^2}{4}\middle|\begin{matrix} k-m-\frac{1}{4}-\frac{\nu}{2}, \frac{\nu}{2}-k+m+\frac{3}{4} \\ \frac{\nu}{2}+\frac{1}{4}, \frac{\nu}{2}+\frac{1}{4}+2m, \frac{1}{4}-\frac{\nu}{2}, \frac{1}{4}-\frac{\nu}{2}-2m \end{matrix}\right) f(xt)\,dt$$

$$= \frac{\sqrt{2}}{x}\int_0^\infty G(u)\,du \int_0^\infty G^{2,1}_{2,4}\left(\frac{u^2t^2}{4x^2}\middle|\begin{matrix} k-m-\frac{1}{4}-\frac{\nu}{2}, \frac{\nu}{2}-k+m+\frac{3}{4} \\ \frac{\nu}{2}+\frac{1}{4}, \frac{\nu}{2}+\frac{1}{4}+2m, \frac{1}{4}-\frac{\nu}{2}, \frac{1}{4}-\frac{\nu}{2}-2m \end{matrix}\right) f(t)\,dt$$

$$= x^{-1}\int_0^\infty f\left(\frac{u}{x}\right) G(u)\,du$$

$$= \int_0^\infty f(u)G(xu)\,du. \qquad (2.1)$$

Hence

$$\sqrt{2}\int_0^\infty G^{2,1}_{2,4}\left(\frac{x^2t^2}{4}\middle|\begin{matrix} k-m-\frac{1}{4}-\frac{\mu}{2}, \frac{\mu}{2}-k+m+\frac{3}{4} \\ \frac{\mu}{4}+\frac{1}{4}, \frac{\mu}{4}+\frac{1}{4}+2m, \frac{1}{4}-\frac{\mu}{2}, \frac{1}{4}-\frac{\mu}{2}-2m \end{matrix}\right) g(t)\,dt$$

$$= \sqrt{2}\int_0^\infty G^{2,1}_{2,4}\left(\frac{x^2t^2}{4}\middle|\begin{matrix} k-m-\frac{1}{4}-\frac{\mu}{2}, \frac{\mu}{2}-k+m+\frac{3}{4} \\ \frac{\mu}{2}+\frac{1}{4}, \frac{\mu}{2}+\frac{1}{4}+2m, \frac{1}{4}-\frac{\mu}{2}, \frac{1}{4}-\frac{\mu}{2}-2m \end{matrix}\right) dt$$

$$\times \int_0^\infty f(u)\,G(tu)\,du$$

$$= \sqrt{2}\int_0^\infty f(u)\,du \int_0^\infty G^{2,1}_{2,4}\left(\frac{x^2t^2}{4}\middle|\begin{matrix} k-m-\frac{1}{4}-\frac{\mu}{2}, \frac{\mu}{2}-k+m+\frac{3}{4} \\ \frac{\mu}{2}+\frac{1}{4}, \frac{\mu}{2}+\frac{1}{4}+2m, \frac{1}{4}-\frac{\mu}{2}, \frac{1}{4}-\frac{\mu}{2}-2m \end{matrix}\right) G(ut)\,dt$$

$$= \sqrt{2}\int_0^\infty u^{-1}f(u)\,du \int_0^\infty G^{2,1}_{2,4}\left(\frac{x^2t^2}{4u^2}\middle|\begin{matrix} k-m-\frac{1}{4}-\frac{\mu}{2}, \frac{\mu}{2}-k+m+\frac{3}{4} \\ \frac{\mu}{2}+\frac{1}{4}, \frac{\mu}{2}+\frac{1}{4}+2m, \frac{1}{4}-\frac{\mu}{2}, \frac{1}{4}-\frac{\mu}{2}-2m \end{matrix}\right) G(t)\,dt$$

$$= \int_0^\infty (xu)^{-\frac{1}{2}} f(u)F(u/x)\,du$$

$$= \int_0^\infty t^{-\frac{1}{2}} f(xt)F(t)\,dt = g(x). \qquad (2.2)$$

To justify the changes in the order of integration in (2.1) and (2.2), we see that u-integral is absolutely and uniformly convergent if Re $(2\nu+\lambda+2+2m\pm 2m)>0$, Re $(\delta+1+2m-2k)>0$, Re $(2\mu+\lambda+2+2m\pm 2m)>0$, where $G(u)=O(u^{\lambda})$ for small u and $O(x^{-\delta})$ as $u\to\infty$ and t-integral is absolutely and uniformly convergent if $f(t)$ belongs to $A(\alpha, a)$ and one of the repeated integrals is absolutely convergent if $f(t)$ belongs to $A(\alpha, a)$. Hence by de la Valle Poussin's theorem (Bromwich 1959, p. 504) the inversion of order of integration is justified. This proves the theorem. The argument also gives:

Theorem 3. If $G(t)$ satisfies the same condition as in Theorem 2 then

$$g(x)=\int_0^\infty G(xu)f(u)\,du$$

is $R_\mu(k, m)$.

Theorem 4. If $f(x)$ is $R_\mu(k, m)$ and $k(x)$ is $R_\nu(k, m)$, then the function

$$x^{-\frac{1}{2}}F(x)=\int_0^\infty k(y)f(xy)\,dy \qquad \ldots \quad \ldots \quad \ldots \quad (2.3)$$

is such that the $\chi_{\mu, k, m}$-transform of $x^{-\frac{1}{2}}F(x)$ [$Q(x)$, say] is equal to $\chi_{\nu, k, m}$-transform of $x^{-\frac{1}{2}}F\left(\frac{1}{x}\right)$.

Proof: As $k(x)$ is $R_\nu(k, m)$, by known theorem (Narain 1956-57, p. 284)

$$x^{-\frac{1}{2}}F(x)=\frac{1}{2\pi i}\int_0^\infty f(xy)\,dy\int_{c-i\infty}^{c+i\infty} 2^{s/2}\,\frac{\Gamma_*\left(\frac{\nu}{2}+\frac{s}{2}+\frac{1}{4}+m\pm m\right)}{\Gamma\left(\frac{\nu}{2}+\frac{s}{2}+\frac{3}{4}+m-k\right)}\,\psi(s)y^{-s}\,ds,$$

where

$$\psi(s)=\psi(1-s).$$

Hence

$$x^{-\frac{1}{2}}F(x)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} 2^{s/2}\,\frac{\Gamma_*\left(\frac{\nu}{2}+\frac{s}{2}+\frac{1}{4}+m\pm m\right)}{\Gamma\left(\frac{\nu}{2}+\frac{s}{2}+\frac{3}{4}+m-k\right)}\,\psi(s)\,ds\int_0^\infty f(xy)y^{-s}\,dy.$$

The inversion of order of integration will be justified if $f(x)$ and $k(x)$ both belong to $A(\alpha, a)$.

Hence

$$x^{-\frac{1}{2}}F(x)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} 2^{s/2}\,\frac{\Gamma_*\left(\frac{\nu}{2}+\frac{s}{2}+\frac{1}{4}+m\pm m\right)}{\Gamma\left(\frac{\nu}{2}+\frac{s}{2}+\frac{3}{4}+m-k\right)}\,\psi(s)x^{s-1}\,ds\int_0^\infty u^{-s}f(u)\,du.$$

Now, as $f(x)$ is $R_\mu(k, m)$,

$$f(u) = \frac{1}{2\pi i}\int_{c'-i\infty}^{c'+i\infty} 2^{s/2} \frac{\Gamma_*\left(\frac{\mu}{2}+\frac{s}{2}+\frac{1}{4}+m\pm m\right)}{\Gamma\left(\frac{\mu}{2}+\frac{s}{2}+\frac{3}{4}+m-k\right)} \psi'(s)u^{-s}\,ds,$$

where

$$\psi'(s) = \psi'(1-s).$$

Hence, by Mellin's inversion formula,

$$\int_0^\infty u^{s-1}f(u)\,du = 2^{s/2} \frac{\Gamma_*\left(\frac{\mu}{2}+\frac{s}{2}+\frac{1}{4}+m\pm m\right)}{\Gamma\left(\frac{\mu}{2}+\frac{s}{2}+\frac{3}{4}+m-k\right)} \psi'(s),$$

changing s into $(1-s)$, we get

$$\int_0^\infty u^{-s}f(u)\,du = 2^{\frac{1}{2}-\frac{1}{2}s} \frac{\Gamma_*\left(\frac{\mu}{2}+\frac{3}{4}-\frac{s}{2}+m\pm m\right)}{\Gamma\left(\frac{\mu}{2}+\frac{5}{4}-\frac{s}{2}+m-k\right)} \psi'(s).$$

Therefore

$$x^{-\frac{1}{2}}F(x) = \frac{\sqrt{2}}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{\Gamma_*\left(\frac{\nu}{2}+\frac{1}{4}+\frac{s}{2}+m\pm m\right)\Gamma_*\left(\frac{\mu}{2}+\frac{3}{4}-\frac{s}{2}+m\pm m\right)}{\Gamma\left(\frac{\nu}{2}+\frac{s}{2}+\frac{3}{4}+m-k\right)\Gamma\left(\frac{\mu}{2}+\frac{5}{4}-\frac{s}{2}+m-k\right)}$$

$$\times\psi(s)\psi'(s)x^{s-1}\,ds,$$

$$= \frac{1}{2\pi i}\int_{1-c-i\infty}^{1-c+i\infty} \frac{\Gamma_*\left(\frac{\nu}{2}+\frac{3}{4}-\frac{s}{2}+m\pm m\right)\Gamma_*\left(\frac{\mu}{2}+\frac{1}{4}+\frac{s}{2}+m\pm m\right)}{\Gamma\left(\frac{\nu}{2}+\frac{5}{4}-\frac{s}{2}+m-k\right)\Gamma\left(\frac{\mu}{2}+\frac{3}{4}+\frac{s}{2}+m-k\right)}$$

$$\times\chi(s)x^{-s}\,ds, \quad \ldots \quad (2.4)$$

where

$$\chi(s) = \chi(1-s).$$

The form of this integral shows that $x^{-\frac{1}{2}}F(x)$ is a function such that $\chi_{\mu,k,m}$-transform of $x^{-\frac{1}{2}}F(x)$ is equal to $\chi_{\nu,k,m}$-transform of $x^{-\frac{1}{2}}F\left(\frac{1}{x}\right)$.

Putting $k+m = \frac{1}{2}$ in the above Theorem 4, we get a result in Hankel transform (Brij Mohan 1932).

THEOREM 5. If

$$Q(x) = \int_0^\infty \frac{1}{y}k(y)f\left(\frac{x}{y}\right)dy = \int_0^\infty \frac{1}{y}k\left(\frac{x}{y}\right)f(y)\,dy,$$

where $f(x)$ is $R_\mu(k, m)$ and $k(x)$ is $R_\nu(k, m)$ and both belong to $A(\alpha, a)$, then $Q(x)$ is of the form

$$Q(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} 2^{s/2}\frac{\Gamma_*\left(\frac{\mu}{2}+\frac{1}{4}+\frac{s}{2}+m\pm m\right)\Gamma_*\left(\frac{\nu}{2}+\frac{1}{4}+\frac{s}{2}+m\pm m\right)}{\Gamma\left(\frac{\mu}{2}+\frac{3}{4}+\frac{s}{2}+m-k\right)\Gamma\left(\frac{\nu}{2}+\frac{3}{4}+\frac{s}{2}+m-k\right)} \times \chi(s)x^{-s}\,ds, \quad \ldots \quad (2.5)$$

where

$$\chi(s) = \chi(1-s).$$

The theorem follows if we proceed in the same way as in Theorem 4.

As can be easily verified, $Q(x)$ is such that if $\chi_{\mu,k,m}$-transform of $Q(x)$ is $x^{-\frac{1}{2}}F(x)$, then $\chi_{\nu,k,m}$-transform of $Q(x)$ is $x^{-\frac{1}{2}}F\left(\frac{1}{x}\right)$.

On taking $k+m = \frac{1}{2}$ the above theorem will give a known result (Brij Mohan 1932).

§ 2.1. If, in the result of Theorem 4, we put $\mu = \nu$, we get by (2.4)

$$x^{-\frac{1}{2}}F(x) = \frac{1}{2\pi i}\int_{1-c-i\infty}^{1-c+i\infty} \lambda(s)x^{-s}\,ds, \quad \ldots \quad (2.6)$$

where

$$\lambda(s) = \lambda(1-s).$$

The symmetry of the result shows that we shall arrive at the same result, if we put

$$x^{-\frac{1}{2}}F(x) = \int_0^\infty k(xy)f(y)\,dy.$$

So we get a verification of the following theorem, which is a particular case of Parseval formula.

If $f(x)$ and $k(x)$ are $R_\nu(k, m)$, then

$$\int_0^\infty k(xy)\,f(y)\,dy = \int_0^\infty k(y)\,f(xy)\,dy. \quad \ldots \quad (2.7)$$

§ 2.2. If $f(x)$ and $k(x)$ are both $R_\nu(k, m)$ and both belong to $A(\alpha, a)$, we have by (2.6), (2.3) and (2.7)

$$x^{-\frac{1}{2}}F(x) = \int_0^\infty k(xy)\,f(y)\,dy$$

$$= \frac{1}{2\pi i}\int_{1-c-i\infty}^{1-c+i\infty} \lambda(s)\,x^{-s}\,ds,$$

where

$$\lambda(s) = \lambda(1-s).$$

Hence

$$F(x) = \frac{1}{2\pi i}\int_{1-c-i\infty}^{1-c+i\infty} \lambda(s)\, x^{\frac{1}{2}-s}\, ds$$

$$= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \lambda(1-s) x^{s-\frac{1}{2}}\, ds$$

$$= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \lambda(s) x^{s-\frac{1}{2}}\, ds = F\left(\frac{1}{x}\right).$$

§ 3. *Examples.*—Using the results of §2.1, we can derive some examples of self-reciprocal function, i.e. solution of the integral equation.

$$f(x) = \sqrt{2}\int_0^\infty G_{2,4}^{2,1}\left(\frac{x^2y^2}{4}\,\middle|\, \begin{matrix} k-m-\frac{1}{4}-\frac{\nu}{2},\ \frac{\nu}{2}-k+m+\frac{3}{4} \\ \frac{\nu}{2}+\frac{1}{4},\ \frac{\nu}{2}+\frac{1}{4}+2m,\ \frac{1}{4}-\frac{\nu}{2},\ \frac{1}{4}-\frac{\nu}{2}-2m \end{matrix}\right) f(y)\, dy.$$

(i) Let us take (Narain 1956-57)

$$k(x) = G_{1,2}^{2,0}\left(\frac{x^2}{2}\,\middle|\, \begin{matrix} \frac{\nu}{2}+\frac{3}{4}+m-k \\ \frac{\nu}{2}+\frac{1}{4},\ \frac{\nu}{2}+\frac{1}{4}+2m \end{matrix}\right)$$

$$= x^{\nu+2m-\frac{1}{2}} e^{-\frac{1}{4}x^2} W_{k,m}\left(\frac{x^2}{2}\right),$$

then

$$F(x) = \sqrt{x}\int_0^\infty G_{1,2}^{2,0}\left(\frac{x^2}{2}\,\middle|\, \begin{matrix} \frac{\nu}{2}+\frac{3}{4}+m-k \\ \frac{\nu}{2}+\frac{1}{4},\ \frac{\nu}{2}+\frac{1}{4}+2m \end{matrix}\right) f(y)\, dy.$$

Inverting this integral by Inversion formula of Meijer-Laplace transform (Bhise 1959) we get a second solution.

(ii) Let us have (Saxena *unpublished*)

$$k(x) = G_{2,4}^{2,1}\left(\frac{x^4}{16}\,\middle|\, \begin{matrix} -\frac{\nu}{4}-\frac{1}{8}-\frac{m}{2}+\frac{k}{2},\ \frac{\nu}{4}+\frac{1}{8}+\frac{m}{2}-\frac{k}{2} \\ \frac{\nu}{4}+\frac{1}{8},\ \frac{\nu}{4}+\frac{1}{8}+m,\ \frac{1}{8}-\frac{\nu}{4},\ \frac{1}{8}-\frac{\nu}{4}-m \end{matrix}\right),$$

then

$$F(x) = \sqrt{x}\int_0^\infty G_{2,4}^{2,1}\left(\frac{x^4y^4}{16}\,\middle|\, \begin{matrix} -\frac{\nu}{4}-\frac{1}{8}-\frac{m}{2}+\frac{k}{2},\ \frac{\nu}{4}+\frac{1}{8}+\frac{m}{2}-\frac{k}{2} \\ \frac{\nu}{4}+\frac{1}{8},\ \frac{\nu}{4}+\frac{1}{8}+m,\ \frac{1}{8}-\frac{\nu}{4},\ \frac{1}{8}-\frac{\nu}{4}-m \end{matrix}\right) f(y)\, dy.$$

Now using the inversion formula (Fox 1961) for the transform (1.3), we can get the third solution.

§ 3.1. We work out the last solution in a slightly different form, thus: Let us take

$$\psi(\beta) = \beta^{\frac{1}{4}} \int_0^\infty G_{2,4}^{2,1}\left(\frac{x^4\beta^2}{16}\middle| \begin{matrix} -\frac{\nu}{4}-\frac{1}{8}-\frac{m}{2}+\frac{k}{2},\ \frac{\nu}{4}+\frac{3}{8}+\frac{m}{2}-\frac{k}{2} \\ \frac{\nu}{4}+\frac{1}{8},\ \frac{\nu}{4}+\frac{1}{8}+m,\ \frac{1}{8}-\frac{\nu}{4},\ \frac{1}{8}-\frac{\nu}{4}-m \end{matrix}\right) f(x)\,dx.$$

If $f(x)$ is $R_\nu\ (k, m)$, then

$$f(x) = \sqrt{2} \int_0^\infty G_{2,4}^{2,1}\left(\frac{x^2y^2}{4}\middle| \begin{matrix} k-m-\frac{1}{4}-\frac{\nu}{2},\ \frac{\nu}{2}+\frac{3}{4}-k+m \\ \frac{\nu}{2}+\frac{1}{4},\ \frac{\nu}{2}+\frac{1}{4}+2m,\ \frac{1}{4}-\frac{\nu}{2},\ \frac{1}{4}-\frac{\nu}{2}-2m \end{matrix}\right) f(y)\,dy.$$

Multiplying both the sides by

$$\beta^{\frac{1}{4}} G_{2,4}^{2,1}\left(\frac{x^4\beta^2}{16}\middle| \begin{matrix} -\frac{\nu}{4}-\frac{1}{8}-\frac{m}{2}+\frac{k}{2},\ \frac{\nu}{4}+\frac{3}{8}+\frac{m}{2}-\frac{k}{2} \\ \frac{\nu}{4}+\frac{1}{8},\ \frac{\nu}{4}+\frac{1}{8}+m,\ \frac{1}{8}-\frac{\nu}{4},\ \frac{1}{8}-\frac{\nu}{4}-m \end{matrix}\right)$$

and integrating, we get

$$\psi(\beta) = \sqrt{2}\beta^{\frac{1}{4}} \int_0^\infty G_{2,4}^{2,1}\left(\frac{x^4\beta^2}{16}\middle| \begin{matrix} -\frac{\nu}{4}-\frac{1}{8}-\frac{m}{2}+\frac{k}{2},\ \frac{\nu}{4}+\frac{3}{8}+\frac{m}{2}-\frac{k}{2} \\ \frac{\nu}{4}+\frac{1}{8},\ \frac{\nu}{4}+\frac{1}{8}+m,\ \frac{1}{8}-\frac{\nu}{4},\ \frac{1}{8}-\frac{\nu}{4}-m \end{matrix}\right) dx$$

$$\times \int_0^\infty G_{2,4}^{2,1}\left(\frac{x^2y^2}{4}\middle| \begin{matrix} k-m-\frac{1}{4}-\frac{\nu}{2},\ \frac{\nu}{2}+\frac{3}{4}-k+m \\ \frac{\nu}{2}+\frac{1}{4},\ \frac{\nu}{2}+\frac{1}{4}+2m,\ \frac{1}{4}-\frac{\nu}{2},\ \frac{1}{4}-\frac{\nu}{2}-2m \end{matrix}\right) f(y)\,dy$$

$$= \sqrt{2}\beta^{\frac{1}{4}} \int_0^\infty f(y)dy \int_0^\infty G_{2,4}^{2,1}\left(\frac{x^2y^2}{4}\middle| \begin{matrix} k-m-\frac{1}{4}-\frac{\nu}{2},\ \frac{\nu}{2}+\frac{3}{4}-k+m \\ \frac{\nu}{2}+\frac{1}{4},\ \frac{\nu}{2}+\frac{1}{4}+2m,\ \frac{1}{4}-\frac{\nu}{2},\ \frac{1}{4}-\frac{\nu}{2}-2m \end{matrix}\right)$$

$$\times G_{2,4}^{2,1}\left(\frac{x^4\beta^2}{16}\middle| \begin{matrix} -\frac{\nu}{4}-\frac{1}{8}-\frac{m}{2}+\frac{k}{2},\ \frac{\nu}{4}+\frac{3}{8}+\frac{m}{2}-\frac{k}{2} \\ \frac{\nu}{4}+\frac{1}{8},\ \frac{\nu}{4}+\frac{1}{8}+m,\ \frac{1}{8}-\frac{\nu}{4},\ \frac{1}{8}-\frac{\nu}{4}-m \end{matrix}\right) dx,$$

replacing x^2 by x and using (Saxena 1960), (Erdélyi *et al.* 1953) we have

$$\psi(\beta) = \beta^{-\frac{1}{4}} \int_0^\infty G_{2,4}^{2,1}\left(\frac{y^4}{16\beta^2}\middle| \begin{matrix} -\frac{\nu}{4}-\frac{1}{8}+\frac{k}{2}-\frac{m}{2},\ \frac{\nu}{4}+\frac{1}{8}+\frac{m}{2}-\frac{k}{2} \\ \frac{\nu}{4}+\frac{1}{8},\ \frac{\nu}{4}+\frac{1}{8}+m,\ \frac{1}{8}-\frac{\nu}{4},\ \frac{1}{8}-\frac{\nu}{4}-m \end{matrix}\right) f(y)\,dy$$

$$= \psi\left(\frac{1}{\beta}\right).$$

Now

$$\psi(\beta) = \beta^{\frac{1}{2}} \int_0^\infty G_{2,4}^{2,1}\left(\frac{x^4\beta^2}{16} \middle| \begin{matrix} -\frac{\nu}{4} - \frac{1}{8} + \frac{k}{2} - \frac{m}{2}, \frac{\nu}{4} + \frac{3}{8} + \frac{m}{2} - \frac{k}{2} \\ \frac{\nu}{4} + \frac{1}{8}, \frac{\nu}{4} + \frac{1}{8} + m, \frac{1}{8} - \frac{\nu}{4}, \frac{1}{8} - \frac{\nu}{4} - m \end{matrix} \right) f(x)\, dx$$

$$= \beta^{\frac{1}{2}} \int_0^\infty G_{2,4}^{2,1}\left(\frac{\xi^2\beta^2}{4} \middle| \begin{matrix} -\frac{\nu}{4} - \frac{1}{8} + \frac{k}{2} - \frac{m}{2}, \frac{\nu}{4} + \frac{3}{8} + \frac{m}{2} - \frac{k}{2} \\ \frac{\nu}{4} + \frac{1}{8}, \frac{\nu}{4} + \frac{1}{8} + m, \frac{1}{8} - \frac{\nu}{4}, \frac{1}{8} - \frac{\nu}{4} - m \end{matrix} \right) \frac{f(2\sqrt{\xi})}{\sqrt{\xi}}\, d\xi.$$

Therefore by inversion formula (Fox 1961), we get

$$\xi^{-\frac{1}{2}} f(2\sqrt{\xi}) = \int_0^\infty G_{2,4}^{2,1}\left(\frac{\xi^2 t^2}{4} \middle| \begin{matrix} -\frac{\nu}{4} - \frac{1}{8} + \frac{k}{2} - \frac{m}{2}, \frac{\nu}{4} + \frac{3}{8} + \frac{m}{2} - \frac{k}{2} \\ \frac{\nu}{4} + \frac{1}{8}, \frac{\nu}{4} + \frac{1}{8} + m, \frac{1}{8} - \frac{\nu}{4}, \frac{1}{8} - \frac{\nu}{4} - m \end{matrix} \right) \psi(t)\, dt.$$

Thus

$$\frac{2}{x} f(x) = \int_0^\infty G_{2,4}^{2,1}\left(\frac{x^4 t^2}{16} \middle| \begin{matrix} -\frac{\nu}{4} - \frac{1}{8} + \frac{k}{2} - \frac{m}{2}, \frac{\nu}{4} + \frac{3}{8} + \frac{m}{2} - \frac{k}{2} \\ \frac{\nu}{4} + \frac{1}{8}, \frac{\nu}{4} + \frac{1}{8} + m, \frac{1}{8} - \frac{\nu}{4}, \frac{1}{8} - \frac{\nu}{4} - m \end{matrix} \right) \psi(t)\, dt.$$

Therefore, we obtain

$$f(x) = \frac{x}{2} \int_0^\infty G_{2,4}^{2,1}\left(\frac{x^4 t^2}{16} \middle| \begin{matrix} -\frac{\nu}{4} - \frac{1}{8} + \frac{k}{2} - \frac{m}{2}, \frac{\nu}{4} + \frac{3}{8} + \frac{m}{2} - \frac{k}{2} \\ \frac{\nu}{4} + \frac{1}{8}, \frac{\nu}{4} + \frac{1}{8} + m, \frac{1}{8} - \frac{\nu}{4}, \frac{1}{8} - \frac{\nu}{4} - m \end{matrix} \right) \psi(t)\, dt$$

where

$$\psi(t) = \psi\left(\frac{1}{t}\right).$$

This is the third solution.

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References

Bhise, V. M. (1959). Inversion formula for a generalized Laplace integral. *J. Vikram Univ.*, **3** (3), 57–63.

Brij Mohan (1932). Definite integrals involving self-reciprocal functions. *Bull. Calcutta math. Soc.*, **24**, 163–176.

Bromwich, T. J. I'a (1959). An Introduction to the Theory of Infinite Series. Macmillan and Co. Ltd., London.

Erdélyi, A., *et al.* (1953). Higher Transcendental Functions, Vol. I. McGraw-Hill Book Co. Inc., New York.

Fox, C. (1961). The *G* and *H* functions as symmetrical Fourier kernels. *Trans. Am. math. Soc.*, **98**, 395–429.

Narain, R. (1956-57). On a Generalization of Hankel transform and self-reciprocal functions. *Rc. Semin. mat. Univ. Torino*, **16**, 269–300.

——— (1959). A Fourier kernel. *Math. Z.*, **70**, 297–299.

Saxena, R. K. (1960). Some theorems on generalized Laplace transform-I. *Proc. natn. Inst. Sci. India*, A **26**, 400–413.

Saxena, R. K. (Present author, different from the author of previous reference). On self-reciprocal functions in $\chi_{\nu, k, m}$-transform. *Ganita* (*Communicated*).