EFFECT OF WALL CONDUCTIVITY ON HEAT TRANSFER BY HARTMANN'S FLOW IN THERMAL ENTRANCE REGION

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The problem of temperature distributions and heat transfer in a thermal entrance region, which results when a liquid flows under a constant pressure gradient between parallel walls at rest and are electrically conducting, has been investigated. The effect of internal heat generation by Joule heating has been taken into account while viscous dissipation is neglected. Boundary conditions are prescribed uniform heat flux at wall and uniform wall temperature. The use of orthogonal functions to solve the energy equation in the entrance region leads to an eigen-value problem of Sturm-Liouville type differential equation which has been solved by Ritz variational method. From the solutions and results it can be observed that the temperature distribution and Nusselt number depend on $(\phi_1+\phi_2)$, the sum of the wall conductance ratios rather than on individual values of ϕ_1 and ϕ_2. Considerable effect can be noticed on these due to the wall conductivity.

1. Introduction

As early as 1885, it was demonstrated by Graetz (*see* Jakob 1949) that the temperature solution in the thermal entrance region of a passage could be formulated as an eigen-value problem, but the numerical calculation of the required eigen-values was very tedious and he was able to obtain only the first three. Subsequently this problem has been extended by many authors both when there is no magnetic field (*see* Nusselt 1923; Sellars *et al.* 1956; Brown 1961; Hatton and Quarmby 1962; Hatton and Turton 1962; Sparrow and Siegel 1960; Seban and Shimazaki 1949) and when a magnetic field is present (*see* Singh 1957; Jain and Srinivasan 1964; Michiyoshi and Matsumoto 1964). One important field in which knowledge about the phenomenon is highly useful is the design and operation of heat exchangers equipment for nuclear reactors, where information about heat transfer and temperature distributions in thermal entrance regions is sometimes of great use.

In the present paper, a laminar flow of a uniform conducting incompressible flow is established between two parallel plates, which are electrically conducting, under a uniform magnetic field, perpendicular to the plates; the temperature distribution within the fluid and the local Nusselt number are theoretically investigated for the thermal entrance region, when there exists the internal heat generation within the fluid. The problem reduces to an eigen-value problem of the Sturm-Liouville type, which has been solved by Ritz method. Boundary conditions are considered for the cases of both a prescribed uniform heat flux at wall and of a uniform wall temperature.

2. Basic Equations and Formulation of the Problem

The basic equations governing steady, hydromagnetic flow and heat transfer for viscous incompressible liquids are (in rationalized MKS system of units):

(i) The modified Navier-Stokes equations:

$$\rho\left[\left(\vec{V}\cdot\nabla\right)\vec{V}\right]=-\text{grad}\ p+\rho\vec{f}+\mu\nabla^2\vec{V}+\left(\vec{J}\times\vec{B}\right), \qquad (1)$$

(ii) the equation of continuity

$$\text{div}\ \vec{V}=0, \qquad (2)$$

(iii) the equation of energy (Landau and Lifshitz 1960):

$$\rho c_p\left[\left(\vec{V}\cdot\nabla\right)T\right]=K\nabla^2T+\mu\Phi+\frac{J^2}{\sigma}, \qquad (3)$$

and (iv) the Maxwell's equations

$$\left.\begin{aligned}&\text{div}\ \vec{B}=0,\\&\text{curl}\ \vec{E}=0,\\&\text{curl}\ \vec{H}=\vec{J},\\&\vec{J}=\sigma\left(\vec{E}+\vec{V}\times\vec{B}\right),\end{aligned}\right\} \qquad (4)$$

where $\vec{B}$ and $\vec{H}$ are the magnetic induction and field vectors, $\vec{D}$ and $\vec{E}$ stand for the electric displacement and field vectors, $\vec{J}$ is the current density vector, $\vec{V}$ represents the fluid velocity vector, p, ρ, μ, σ, T, C_p and K are the pressure, the density, the coefficient of viscosity, the electrical conductivity of the fluid, the temperature, the specific heat at constant pressure and the thermal conductivity, $\vec{f}$ and Φ represent the body force per unit mass and the viscous dissipation respectively.

We consider a viscous, incompressible, electrically conducting liquid flowing from left to right in x-direction between two plates, $y=a$ and $y=-a$, at rest with a transverse magnetic field superposed. The motion being considered as two-dimensional, the equations of motion and energy for the fully developed steady flow may be obtained from eqns. (1) to (3) with Maxwell's eqns. (4), by using the following assumptions: (i) the body force, $\vec{f}$, and $\mu\Phi$

have been neglected from the equations; (ii) the flow is established Hartmann's flow with conducting walls; (iii) the heat conduction in x-direction is neglected; (iv) the physical properties of the fluid are independent of temperature and are constant. With the assumptions given above, eqns. (1) to (4) reduce to by setting

$$\vec{V} = u(y)\vec{i},\ \vec{H} = H_x(y)\vec{i}+H_0\vec{j} \text{ and}$$

$$\vec{J} = J_z\vec{K} \text{ where } H_0 \text{ is constant,}$$

$$-\frac{\partial p}{\partial x} = \text{constant},$$

$$\frac{\mu_e}{\rho}\frac{dH_x}{dy}+\frac{\mu}{\rho}\frac{d^2u}{dy^2} = \frac{1}{\rho}\frac{\partial p}{\partial x}, \quad \ldots \quad (5)$$

$$\frac{d^2H_x}{dy^2}+\sigma\mu_e H_0\frac{du}{dy} = 0, \quad \ldots \quad (6)$$

and

$$u\frac{\partial T}{\partial x} = \alpha\frac{\partial^2 T}{\partial y^2}+\frac{1}{\rho c_p}\frac{J_z^2}{\sigma}, \quad \ldots \quad (7)$$

where α is the thermal diffusivity.

The boundary conditions for the velocity and the magnetic fields are

$$\left.\begin{aligned} &u = 0,\ y = \pm a,\\ &\frac{dH_x}{dy}-\frac{H_x}{a\phi_1} = 0,\ y = -a,\\ \text{and}\quad &\frac{dH_x}{dy}+\frac{H_x}{a\phi_2} = 0,\ y = a, \end{aligned}\right\} \quad \ldots \quad (8)$$

where ϕ_1 and ϕ_2 are the electrical conductance ratios, defined as $\phi_1 = (\sigma_{\omega_1}h_1/\sigma_f a)$ and $\phi_2 = (\sigma_{\omega_2}h_2/\sigma_f a)$ where the suffixes ω_1 and ω_2 refer to the plates $y = \mp a$ respectively and f stands for the fluid, and h_1 and h_2 are the thicknesses of the plates $y = \mp a$ respectively.

The solutions of (5) and (6) subject to (8) (*see* Chang and Yen 1962) are

$$\frac{u}{u_m} = \left[\frac{\cosh M-\cosh M\xi}{\cosh M-\frac{1}{M}\sinh M}\right], \quad \ldots \quad (9)$$

and

$$\overline{H}_x = \frac{\bar{P}}{R_m^2}\left[C\sinh M\xi+\Phi_2\left(\frac{1}{M}-\coth M\right)-\xi\right], \quad \ldots \quad (10)$$

where $\bar{u}_m$, the non-dimensional mean velocity of the fluid, is given by

$$\bar{u}_m = \frac{\bar{P}\Phi_1}{M^2R_m}(M\coth M-1), \quad \ldots \quad (11)$$

and

$$P = -\frac{\rho a^3}{\mu^2}\frac{\partial p}{\partial x},\ \xi = \frac{y}{a},$$

$$\Phi_1 = \frac{\phi_1+\phi_2+2}{(\phi_1+\phi_2)M \coth M+2},$$

$$\Phi_2 = \frac{(\phi_2-\phi_1)M}{(\phi_1+\phi_2)M \coth M+2},\ C = \frac{\Phi_1}{\sinh M},$$

$$R_m = \frac{a'\rho a}{\mu}, \text{ the magnetic Reynolds' number,}$$

$$a' = \left(\frac{\mu_e H_0^2}{\rho}\right)^{\frac{1}{2}}, \text{ the Alfven velocity,}$$

$$M = a'a\sqrt{\mu/\rho\mu_e\sigma}, \text{ the Hartmann number.}$$

The thermal entrance region starts from $x = 0$ and in the region $x > 0$. We consider the existence of both wall heat transfer and Joule heating. In the energy eqn. (7) J_z is given by

$$J_z = \frac{H_0 P}{aR_m^2}[1-CM \cosh M\xi]. \qquad (12)$$

3. Solution of the Heat Transfer Equation

In this section we would like to find the solution of the energy equation for the following two situations (i) when there is uniform heat flux at wall and (ii) when there is uniform wall temperature.

(3.1) *Uniform heat flux at wall.*—In view of the linearity of the equation, we can separate the problem into two simpler cases: (*a*) when there is internal heat generation due to Joule heating with the walls being insulated. In this case if we define T_Q as the temperature distribution, then eqn. (7) becomes

$$u\frac{\partial T_Q}{\partial x} = \alpha\frac{\partial^2 T_Q}{\partial y^2}+\frac{1}{\rho c_p}\frac{J_z^2}{\sigma}, \qquad (13)$$

with the boundary conditions

$$\left.\begin{aligned} &T_Q = 0: \text{ at } x = 0,\\ &\frac{\partial T_Q}{\partial y} = 0: \text{ at } y = 0 \text{ and } y = a;\end{aligned}\right\} \qquad (14)$$

(*b*) when there is no Joule heating but uniform heat transfer at wall. Defining the temperature distribution in this case as T_q, we can reduce the problem as

$$u\frac{\partial T_q}{\partial x} = \alpha\frac{\partial^2 T_q}{\partial y^2}, \qquad (15)$$

with the boundary conditions

$$\left.\begin{aligned} T_q &= 0 \quad &: x = 0, \\ \frac{\partial T_q}{\partial y} &= 0 \quad &: y = 0, \\ \frac{\partial T_q}{\partial y} &= -\frac{q}{K} &: y = a. \end{aligned}\right\} \quad \ldots \quad (16)$$

Then the solution of equation (7) can be written as

$$T = T_Q + T_q. \quad \ldots \quad (17)$$

(3.1*a*) *Joule heating present with insulating walls.*—In this sub-section we would like to find the temperature distribution T_Q by solving eqn. (13) with the boundary conditions (14). In order to solve it, it is further convenient to write T_Q as the sum of the fully developed situation which applies far down the channel and the entrance region solution near the entrance of the channel. Hence we write

$$T_Q = T_{Qd} + T_{Qe}. \quad \ldots \quad (18)$$

In what follows the solution for the fully developed region will be obtained first. In this case the equation is

$$u\frac{\partial T_{Qd}}{\partial x} = \alpha \frac{\partial^2 T_{Qd}}{\partial y^2} + \frac{1}{\rho c_p}\frac{J_z^2}{\sigma}. \quad \ldots \quad (19)$$

The mixed mean temperature T_m is defined as

$$T_m = \int_{-a}^{a} Tu\, dy \Big/ \int_{-a}^{a} u\, dy. \quad \ldots \quad (20)$$

Then with the help of (20), we can obtain from (19) an equation

$$\frac{\partial T_{Qd}}{\partial x} = \frac{\partial T_m}{\partial x} = \frac{1}{au_m c_p \rho}\left[\int_0^a \frac{J_z^2}{\sigma} dy\right]. \quad \ldots \quad (21)$$

Substituting the value of J_z given by eqn. (12), (21) becomes after integration

$$T_{Qd} = \frac{\bar{p}^2 x\nu^3}{a^4 c_p u_m}\left[\frac{1}{M^2} + \frac{C^2}{4}\left(2 + \frac{\sinh 2M}{2}\right) - \frac{2C}{M^2}\sinh M\right] + F(y). \quad \ldots \quad (22)$$

Where $F(y)$ is a function of y alone, and it is given by

$$F(\xi) = \frac{(P_r R_e)^2 \nu\alpha}{a^2 c_p}\left\{\frac{1}{C^2}\left(\frac{1}{\cosh M - \frac{1}{M}\sinh M}\right)^2\left[A\left(\frac{\xi^2}{2}\cosh M - \frac{\cosh M\xi}{M^2}\right) - \left\{\frac{\xi^2}{2} - \frac{2C}{M}\cosh M\xi + \frac{C^2 M^2}{2}\left(\frac{\xi^2}{2} + \frac{\cosh 2M\xi}{4M^2}\right)\right\}\right] + C_0\right\} \quad \ldots \quad (23)$$

where

$$A = \left[\frac{1+\frac{CM^2}{4}\left(2+\frac{\sinh 2M}{2}\right)-2C\sinh M}{\left(\cosh M-\frac{1}{M}\sinh M\right)}\right]$$

$R_e = \frac{u_m a}{\nu}$, the Reynolds' number,

$P_r = \frac{\nu}{\alpha}$, the Prandtl number,

and C_0, the constant of integration to be determined later. We shall now determine T_{Qe}. The equation satisfied by T_{Qe} is

$$u\,\frac{\partial T_{Qe}}{\partial x} = \alpha\,\frac{\partial^2 T_{Qe}}{\partial y^2}, \quad \ldots \quad (24)$$

with the boundary conditions

$$\left.\begin{aligned} T_{Qe} &= 0 : x = 0; \\ \frac{\partial T_{Qe}}{\partial y} &= 0 : y = 0 \text{ and } y = a. \end{aligned}\right\} \quad \ldots \quad (25)$$

Following Michiyoshi and Matsumoto (1964), the solution of (24) can be written as

$$T_{Qe} = \frac{(R_e P_r)^2 \nu\alpha}{a^2 c_p} \sum_{n=1}^{\infty} C_n Y_n \exp\left[-\beta_n \frac{u_m}{u_{\max}}\,\xi'\right], \quad \ldots \quad (26)$$

where

$$\xi' = \frac{1}{R_e P_r}\left(\frac{x}{a}\right), \quad \ldots \quad (27)$$

and β_n and y_n are eigen-values and eigen-functions respectively of the following Sturm-Liouville type differential equation,

$$\frac{d^2 Y_n}{d\xi^2} + \beta_n \frac{u}{u_{\max}} Y_n = 0, \quad \ldots \quad (28)$$

with boundary conditions:

$$\left.\begin{aligned} Y_n' &= \frac{dY_n}{d\xi} = 0 \; : \; \xi = 0; \\ Y_n' &= 0 \qquad\quad\;\, : \; \xi = 1, \end{aligned}\right\} \quad \ldots \quad (29)$$

in which

$$\frac{u}{u_{\max}} = \left[\frac{\cosh M - \cosh M\xi}{\cosh M - 1}\right], \quad \ldots \quad (30)$$

$u_{\max}$ is the velocity at the centre of the axis, $y = 0$.

From the characteristics of Sturm-Liouville differential equation and the conditions $T_{Qe} = 0$ at $x = 0$, the coefficients of C_n can be obtained from the following relations:

$$C_n = -\frac{\int_0^1 u\left[\frac{F(\xi)}{(R_eP_r)^2\nu\alpha/(a^2c_p)} - c_0\right] Y_n \, d\xi}{\int_0^1 u Y_n^2 \, d\xi},$$

$$C_0 = -\frac{\int_0^1 u\left[\frac{F(\xi)}{(R_eP_r)^2\nu\alpha/(a^2c_p)} - c_0\right] d\xi}{\int_0^1 u \, d\xi}. \quad \ldots \quad (31)$$

Eqn. (28) subject to the conditions (29) has been solved by the Ritz variational method. Sparrow and Siegel (1960) have successfully demonstrated this method for the problems confronting in thermal entrance region of ducts. Taking

$$F_1\left(\xi, Y_n, \frac{dY_n}{d\xi}\right) = \left[\left(\frac{dY_n}{d\xi}\right)^2 - \beta_n \frac{u}{u_{\max}} Y_n^2\right], \quad \ldots \quad (32)$$

eqn. (28) can be rewritten as

$$\frac{d}{d\xi}\left(\frac{\partial F_1}{\partial Y_n'}\right) - \frac{\partial F_1}{\partial Y_n} = 0, \quad \left(Y_n' = \frac{dY_n}{d\xi}\right) \quad \ldots \quad (33)$$

which is Euler's equation for

$$I = \int_0^1 F_1(\xi, Y_n, Y_n')\, d\xi \quad \ldots \quad (34)$$

to be minimum.

To satisfy the conditions (29), we take

$$Y_n = A_0 + \sum_{r=1}^{5} A_r \cos r\pi\xi, \quad \ldots \quad (35)$$

involving six unknown constants, $A_0, A_1, \ldots, A_5$.

The values of these constants are determined by substituting (35) in (34) and expressing the conditions for the integral to be minimum. Further we would like to impose the condition that $Y_n(o) = 1$ following Sparrow and Siegel (1960). The simultaneous equations involving the eigen-values β_n and constants $A_0, A_1, \ldots, A_5$ have been solved on the digital computer and for $M = 4$, the first five positive eigen-values have been evaluated and compared with those obtained by Michiyoshi and Matsumoto (1964). Surprisingly our values are in good agreement with those values (*see* Table I).

(3.1*b*) *No Joule heating but uniform wall heat flux.*—As before let us take

$$T_q = T_{qd} + T_{qe}, \quad \ldots \quad (36)$$

TABLE I

Eigen-values and eigen-functions at wall

	Author's values		Michiyoshi and Matsumoto's values	
n	β_n	$y_n(1)$	β_n	$y_n(1)$
1	14·8356	−1·1654	14·8752	−1·1528
2	56·6952	1·2522	56·8958	1·2579
3	25·0123	−1·3403	125·7058	−1·3304
4	220·2161	1·4020	221·3816	1·3868
5	341·3246	−1·4368	341·3523	−1·4285

where T_{qd} and T_{qe} are the contributions of T_q due to fully developed situation and the entrance region.

In the case of fully developed region, eqn. (15) will be

$$u\frac{\partial T_{qd}}{\partial x} = \alpha\frac{\partial^2 T_{qd}}{\partial y^2}. \qquad (37)$$

From a heat balance on the fluid for a length dx of the channel, we have

$$-q\,dx = \rho c_p u_m a T_{qd}, \qquad (38)$$

i.e.

$$\frac{\partial T_{qd}}{\partial x} = \frac{\partial T_m}{\partial x} = -\frac{q}{a}\frac{1}{\rho c_p u_m} \qquad (39)$$

($q > 0$ for heat release from the fluid),

which on integration gives

$$T_{qd} = -\frac{q}{a}\frac{x}{\rho c_p u_m} + G(y). \qquad (40)$$

Substituting in (37), integrating and using the boundary conditions, we obtain

$$G(\xi) = -\frac{qa}{k}\left[\frac{\frac{\xi^2}{2}\cosh M - \frac{1}{M^2}\cosh M\xi}{\left(\cosh M - \frac{1}{M}\sinh M\right)} - D_0\right], \qquad (41)$$

where D_0 is a constant of integration.

The equation satisfied by T_{qe}, the temperature contribution in the thermal entrance region, is

$$u\frac{\partial T_{qe}}{\partial x} = \alpha\frac{\partial^2 T_{qe}}{\partial y^2}, \qquad (42)$$

with the boundary conditions

$$\left.\begin{aligned} x = 0 &: \quad T_{qe} = 0, \\ y = 0,\ y = a &: \quad \frac{\partial T_{qe}}{\partial y} = 0. \end{aligned}\right\} \qquad (43)$$

The differential equation (42) with the boundary conditions (43) is exactly the same equation with the boundary conditions which we have obtained in the previous section and if we write the solution in the form

$$T_{qe} = \frac{qa}{k} \sum_{n=1}^{\infty} \left[D_n Y_n \exp\left[-\frac{u_m}{u_{\max}} \beta_n \xi' \right] \right], \qquad (44)$$

the eigen-values and eigen-functions will be the same as we have obtained earlier. From the condition of $T_{qe} = 0$ at $x = 0$ in eqn. (42), the coefficients D_n are

$$D_n = -\frac{\int_0^1 u \left[\frac{G(\xi)}{(qa/k)} - D_0 \right] Y_n \, d\xi}{\int_0^1 u Y_n^2 \, d\xi},$$

$$D_0 = -\frac{\int_0^1 u \left[\frac{G(\xi)}{(qa/k)} - D_0 \right] d\xi}{\int_0^1 u \, d\xi}. \qquad (45)$$

(3.1*c*) *Wall heat transfer* (*uniform wall heat flux*) *with Joule heating.*—The resultant temperature T, given by eqn. (17) is nothing but the superposition of all the results obtained till now and it can be seen as

$$T = \left[\frac{P^2 \nu^3}{a^4 c_p u_m} \left\{ \frac{1}{M^2} + \frac{C^2}{4} \left(2 + \frac{\sinh 2M}{2} \right) - \frac{2C}{M^2} \sinh M \right\} - \frac{q}{a \rho c_p u_m} \right] x + F(\xi) + G(\xi)$$
$$+ \frac{(R_e P_r)^2 \nu \alpha}{a^2 c_p} \left[\sum_{n=1}^{\infty} C_n Y_n \exp\left[-\frac{u_m}{u_{\max}} \beta_n \xi' \right] \right]$$
$$+ \frac{qa}{K} \left[\sum_{n=1}^{\infty} D_n Y_n \exp\left[-\frac{u_m}{u_{\max}} \beta_n \xi' \right] \right]. \qquad (46)$$

Once we have got the temperature distribution, we consider the following heat transfer coefficient h:

$$h = \frac{q}{T_m - T_w}, \qquad (47)$$

where T_w is the wall temperature and T_m represents the mixed mean temperature.

The local Nusselt number Nu is defined as

$$Nu = \frac{2a \cdot h}{K} = \frac{2a}{K} \frac{q}{(T_m - T_w)} = -2a \frac{\left(\frac{\partial T}{\partial y} \right)_{y=a}}{(T_m - T_w)}, \qquad (48)$$

in which $(T_m - T_w)$ can be obtained by the following expression:

$$T_m - T_w = -\left[F(1)+G(1)+\frac{(R_eP_r)^2\nu\alpha}{a^2c_p}\sum_{n=1}^{\infty}\left\{C_nY_n(1)\exp\left[-\frac{u_m}{u_{\max}}\beta_n\xi'\right]\right\}\right.$$
$$\left.+\frac{qa}{K}\sum_{n=1}^{\infty}\left\{D_nY_n(1)\exp\left[-\frac{u_m}{u_{\max}}\beta_n\xi'\right]\right\}\right]. \quad \ldots \quad (49)$$

Using this, the Nusselt number, Nu_1, will come out to be

$$Nu_1 = -(2qa/K)\Big/\left[F(1)+G(1)+\frac{(R_eP_r)^2\alpha\nu}{a^2c_p}\sum_{n=1}^{\infty}\left\{C_nY_n(1)\exp\left[-\frac{u_m}{u_{\max}}\beta_n\xi'\right]\right\}\right.$$
$$\left.+\frac{qa}{K}\sum_{n=1}^{\infty}\left\{D_nY_n(1)\exp\left[-\frac{u_m}{u_{\max}}\beta_n\xi'\right]\right\}\right]. \quad \ldots \quad (50)$$

The temperature distributions have been evaluated (T_Q and T) for $M = 4$, $\xi' = 0{\cdot}03$, various values of $(\phi_1+\phi_2)$, the parameter associated with the electrical conductance ratios, $M = [(R_eP_r)^2\nu\alpha k/(a^3c_pq)]$, and have been illustrated in Figs. (1) and (2). From these figures we can conclude that the

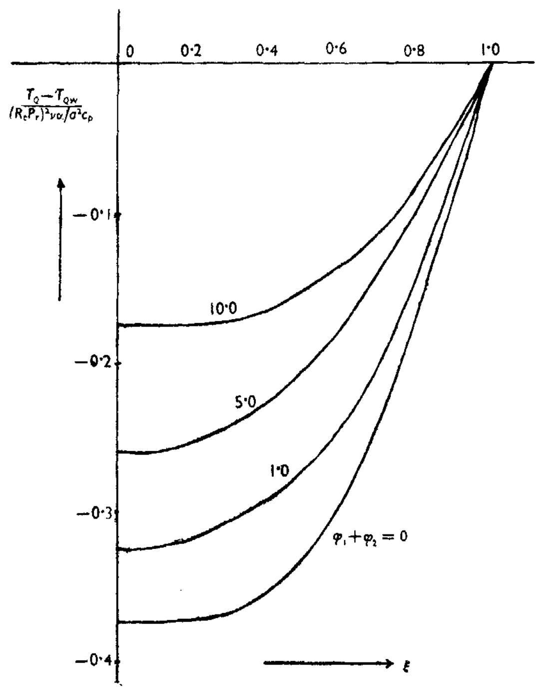

Fig. 1. Effect of wall conductance on the fluid temperature with Joule heating and no heat flow through the wall ($M = 4$, $\xi' = 0{\cdot}03$).

increase in $(\phi_1+\phi_2)$ results in the increase of both T_Q and T, i.e. the temperature rises sharply due to the introduction of wall conductivity.

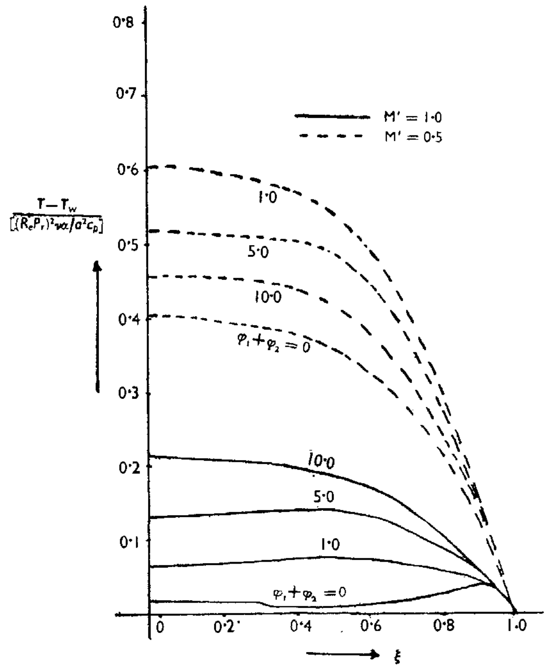

FIG. 2. Effect of wall conductance on fluid temperature distribution in the thermal entrance region under uniform heat flux at wall ($\xi' = 0{\cdot}03$, $M = 4{\cdot}0$).

Local Nusselt number given in Table II has been evaluated for $M = 4$, $\xi' = 1{\cdot}5$, 10, for different values of M' and $(\phi_1+\phi_2)$. The local Nusselt

TABLE II

Effect of wall conductance on local Nusselt number under uniform heat flux, for $M = 4$, $\xi' = 1{\cdot}5$, $10{\cdot}0$ *and various values of* M'

$\phi_1+\phi_2$ \ M'	0·25	0·50	0·75	1·00	
0	9·12565	10·45652	19·25678	132·2343	$\xi' = 1{\cdot}5$
1·0	13·23467	15·32416	21·35627	133·4963	
5·0	20·20347	24·36412	30·31232	134·4865	
10·0	21·36482	25·13432	31·01232	135·00112	
0	6·01234	8·82534	16·54367	147·3123	$\xi' = 10{\cdot}0$
1·0	8·91245	10·23437	19·13237	148·1234	
5·0	9·63247	11·69327	20·69347	148·6937	
10·0	10·12345	12·12348	21·29456	149·0316	

number can be seen to be increasing considerably with the imposition of conductivity conditions at the walls or with increase of the parameter M'. In all these results if we consider $(\phi_1+\phi_2) = 0$, then the results are in good agreement with the results obtained by Michiyoshi and Matsumoto (1964).

(3.2) *Uniform wall temperature.*—As we take the inlet temperature at $x = 0$ as T_0 and the wall temperature being zero, the problem can be divided into two cases (*a*) in which the internal heat generation by Joule heating and the inlet temperature are zero and (*b*) in which the Joule heating has been neglected, but an arbitrary inlet temperature T_0 has been taken. In these cases the fluid temperature is taken as the excess temperature over the value at the wall. If we take each of these solutions for (*a*) and (*b*) as T_H and T_h, then the general solution of the problem is

$$T = T_H + T_h. \qquad (51)$$

The two separate problems for these cases (*a*) and (*b*) will be

$$u\frac{\partial T_H}{\partial x} = \alpha\frac{\partial^2 T_H}{\partial y^2} + \frac{J_z^2}{\sigma c_p \rho}, \qquad (52)$$

with boundary conditions,

$$(a)\quad \left.\begin{aligned} x = 0: &\quad T_H = 0, \\ y = 0: &\quad \frac{\partial T_H}{\partial y} = 0, \\ y = a: &\quad T_H = 0. \end{aligned}\right\} \qquad (53)$$

$$u\frac{\partial T_h}{\partial x} = \alpha\frac{\partial^2 T_h}{\partial y^2}, \qquad (54)$$

with boundary conditions,

$$(b)\quad \left.\begin{aligned} x = 0: &\quad T_h = T_0, \\ y = 0: &\quad \frac{\partial T_h}{\partial y} = 0, \\ y = a: &\quad T_h = 0. \end{aligned}\right\} \qquad (55)$$

(3.2*a*) *Joule heating with inlet temperature being zero.*—Once u is known, eqn. (52) is linear in T_H and can be further written as before as the sum of two, T_{Hd} and T_{He}, i.e. fully developed solution and the entrance region solution near the entrance of the channel and then the complete solution can be written as

$$T_H = T_{Hd} + T_{He}. \qquad (56)$$

We shall now establish first the solution for T_{Hd}. For this case, eqn. (52) becomes

$$\alpha\frac{\partial^2 T_{Hd}}{\partial y^2} + \frac{J_z^2}{c_p \rho \sigma} = 0, \qquad (57)$$

which on integration twice, on using eqn. (12) for the boundary conditions (53), reduces to

$$T_{Hd} = -\frac{(R_eP_r)^2\nu\alpha}{a^2c_p}\left[\frac{1}{C^2}\left(\frac{1}{\cosh M-\frac{1}{M}\sinh M}\right)^2\right]\left[\frac{\xi^2}{2}+\frac{C^2M^2}{2}\left(\frac{\xi^2}{2}+\frac{\cosh 2M\xi}{4M^2}\right)\right.$$
$$\left.-\frac{2C}{M}\cosh M\xi-\tfrac{1}{2}+\frac{2C}{M}\cosh M-\frac{C^2M^2}{2}\left(\tfrac{1}{2}+\frac{\cosh 2M}{4M^2}\right)\right]. \quad \ldots \quad (58)$$

In order to establish the thermal entrance region, we get an equation in terms of T_{He} with the help of (56) and (52) which we should solve. The equation in T_{He} can be seen as

$$u\frac{\partial T_{He}}{\partial x} = \alpha\frac{\partial^2 T_{He}}{\partial y^2}, \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (59)$$

with the boundary conditions

$$\left.\begin{aligned} x = 0 &: T_{He} = 0, \\ y = 0 &: \frac{\partial T_{He}}{\partial y} = 0, \\ y = a &: T_{He} = 0. \end{aligned}\right\} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (60)$$

The solution of this equation can be expressed as follows:

$$T_{He} = \frac{(R_eP_r)^2\,\nu\alpha}{a^2c_p}\sum_{n=1}^{\infty} E_n\psi_n(\xi)\left\{\exp\left[-\frac{u_m}{u_{\max}}\gamma_n\,\xi'\right]\right\} \quad \ldots \quad \ldots \quad (61)$$

where γ_n and ψ_n are the eigen-values and eigen-functions of the Sturm-Liouville type differential equation

$$\frac{d^2\psi_n}{d\xi^2}+\gamma_n\frac{u}{u_{\max}}\psi_n = 0, \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (62)$$

with boundary conditions

$$\left.\begin{aligned} \xi = 0 &: \frac{d\psi_n}{d\xi} = 0, \\ \xi = 1 &: \psi_n = 0. \end{aligned}\right\} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (63)$$

And the coefficients E_n are given by considering the boundary condition, $T_{He} = 0$ at $x = 0$,

$$E_n = -\frac{\int_0^1 u\,\frac{T_{Hd}}{[(R_eP_r)^2\nu\alpha/a^2c_p]}\,\psi_n\,d\xi}{\int_0^1 u\psi_n^2\,d\xi}. \quad \ldots \quad \ldots \quad \ldots \quad (64)$$

Equation (62), subject to the boundary conditions (63), has been solved by Ritz method as before by taking the function which satisfies the conditions taken as

$$\psi_n(\xi) = \sum_{r=0}^{5} A_r\cos\frac{(2r+1)\pi}{2}\xi. \quad \ldots \quad \ldots \quad \ldots \quad (65)$$

The first five eigen-values have been tabulated and compared with the numerical results given by Michiyoshi and Matsumoto (1964) for $M = 4$. They are in good agreement with those eigen-values (*see* Table III).

TABLE III

Eigen-values

n	Author's values	Michiyoshi and Matsumoto's values
1	2·6603	2·6390
2	27·3269	27·1269
3	78·5843	78·3049
4	157·0345	156·2148
5	261·6935	261·0073

(3.2*b*) *Joule heating but arbitrary inlet temperature.*—As before, the fluid temperature will have two distributions T_{ha} and T_{he}, the temperature in the fully developed region and entrance region respectively. In this case T_{hd} will be zero, since the fluid flows with wall heat transfer under the condition of $T_w = 0$.

Then the fundamental equation (54) becomes

$$u \frac{\partial T_{he}}{\partial x} = \alpha \frac{\partial^2 T_{he}}{\partial y^2}, \quad \ldots \quad (66)$$

with boundary conditions,

$$\left. \begin{aligned} y = 0 &: \frac{\partial T_{he}}{\partial y} = 0, \\ y = a &: \quad T_{he} = 0. \end{aligned} \right\} \quad \ldots \quad (67)$$

Since this equation has the same formula and the same boundary conditions as equation (59), T_{he} can be expressed as the infinite series by using the above-mentioned eigen-values γ_n and eigen-functions ψ_n. Hence, T_{he} can be written as

$$\frac{T_{he}}{[(R_e P_r)^2 \nu\alpha / a^2 c_p]} = \frac{T_0}{[(R_e P_r)^2 \nu\alpha / a^2 c_p]} \sum_{n=1}^{\infty} \left\{ F_n \psi_n \exp\left[-\frac{u_m}{u_{\max}} \gamma_n \xi' \right] \right\} \quad \ldots \quad (68)$$

and the coefficients F_n are expressed as follows by the characteristics of Sturm-Liouville equation and the condition of $T_{he} = 0$ at $x = 0$:

$$F_n = \frac{-\int_0^1 u\psi_n \, d\xi}{\int^1 u\psi_n^2 \, d\xi}. \quad \ldots \quad (69)$$

(3.2c) *Joule heating with wall heat transfer (uniform wall temperature).*— The resultant temperature has been obtained by substituting these results in T_H and T_h in eqn. (51). It is

$$T = -\frac{(R_eP_r)^2\nu\alpha}{a^2c_p}\left[\frac{1}{C^2}\left(\frac{1}{\cosh M - \frac{1}{M}\sinh M}\right)^2\right]\left[\frac{\xi^2}{2}+\frac{C^2M^2}{2}\left(\frac{\xi^2}{2}+\frac{\cosh 2M\xi}{4M^2}\right)\right.$$
$$\left. -\frac{2C}{M}\cosh M\xi - \tfrac{1}{2} + \frac{2C}{M}\cosh M - \frac{C^2M^2}{2}\left(\tfrac{1}{2}+\frac{\cosh 2M}{4M^2}\right)\right]$$
$$+\frac{(R_eP_r)^2\nu\alpha}{a^2c_p}\sum_{n=1}^{\infty}\left\{E_n\psi_n \exp\left[-\frac{u_m}{u_{\max}}\gamma_m\xi'\right]\right\}$$
$$+T_0\sum_{n=1}^{\infty}\left\{F_n\psi_n \exp\left[-\frac{u_m}{u_{\max}}\gamma_n\xi'\right]\right\}. \qquad (70)$$

The numerical calculations of the temperature distributions, T_H and T, have been recorded in Figs. (3) and (4). It could be observed that the temperature increases sharply due to the wall conductance ratios (ϕ_1 and ϕ_2). The

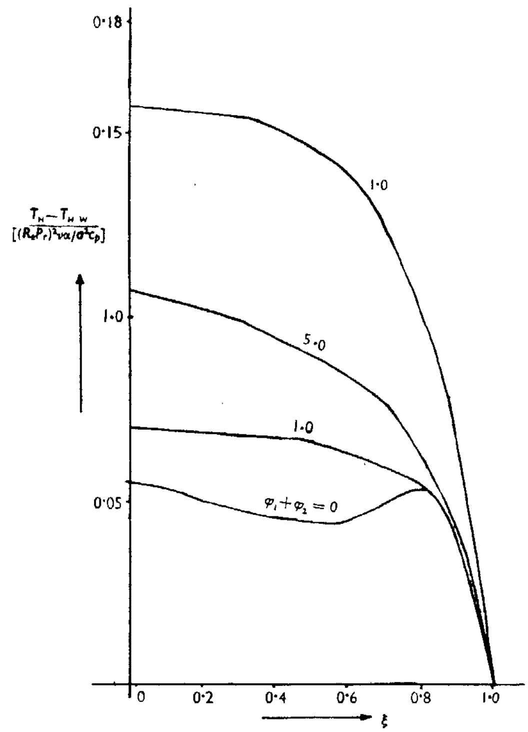

Fig. 3. Effect of wall conductance on fluid temperature distribution with Joule heating and the same inlet temperature as the wall ($M = 4$, $\xi' = 0{\cdot}03$).

temperature T increases with the increase in $T_0' = T_0/[(R_eP_r)^2\nu\alpha/a^2c_p]$, a parameter associated with the arbitrary inlet temperature.

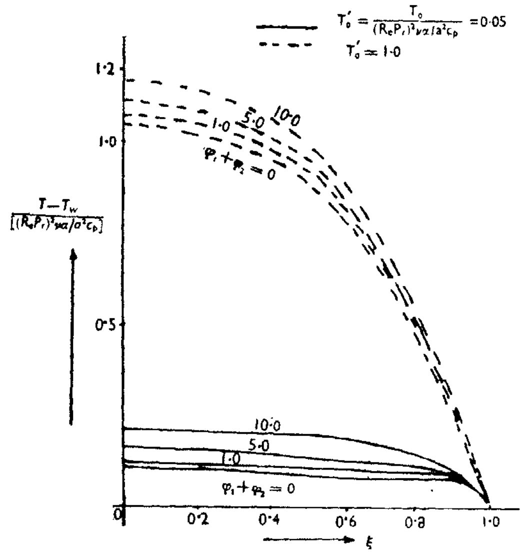

Fig. 4. Effect of wall conductance on fluid temperature distribution in thermal entrance region under uniform wall temperature ($M = 4$, $\xi' = 0{\cdot}03$).

(3.3) Conclusion

The results obtained in this paper will be applicable to the convective heat transfer in thermal entrance region even though, in order to determine the heat transfer in thermal entrance region, the eigen-values and eigen-functions taken here are not sufficient. From the results we can conclude that the local Nusselt number and temperatures are seen to be increased considerably with the imposition of conductivity conditions at the walls. For a given value of M, M', T_0', we see that the thermal entry lengths diminish with the increase in $(\phi_1+\phi_2)$.

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