A PROPERTY OF SELF-RECIPROCAL FUNCTIONS

by V. V. L. N. Rao, *No. 6-4-429, Bholakpur, Secunderabad 3 (A.P.)*

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In this paper a certain relation concerning self-reciprocal functions and their kernels has been found with the help of confluent hypergeometric function; so that when a kernel is known the corresponding self-reciprocal function can be obtained by the mere change of the variable. Examples of self-reciprocal functions, thus obtained from various kernels, are found to agree with the previously obtained results.

§ 1. Following the notation of Hardy and Titchmarsh (1930) we denote a function $f(x)$ as $R\mu$, if it is self-reciprocal for Hankel transforms, of order μ, so that it is given by the formula,

$$f(x) = \int_0^\infty J\mu(xy) f(y) \sqrt{xy}\, dy, \qquad \ldots \quad \ldots \quad \ldots \quad (1.1)$$

where $f(x)$ is a Bessel function of order μ. For $\mu = \frac{1}{2}$ and $-\frac{1}{2}$, $f(x)$ is denoted by Rs and Rc respectively. Formulae for self-reciprocal functions and their kernels have been established separately by Hardy and Titchmarsh (1930) and Brij Mohan (1939). In this paper we proceed to establish a property between self-reciprocal functions and their kernels which may be stated as follows:

'If a kernel $P(x)$ transforms $R(\mu-1)$ into $R\mu$ then $P\frac{(x^2)}{2}$ becomes Rc.'

We proceed to establish this property by means of an example. For this, we start with the integral given by Slater (1960), viz.

$$\int_0^\infty t^{(s-1)} {}_1F_1(a\,;\, b\,;\, -t)\, dt = \frac{\Gamma(b)\Gamma(s)\Gamma(a-s)}{\Gamma(a)\Gamma(b-s)}, \qquad \ldots \quad \ldots \quad (1.2)$$

where

$$0 < R(s) < R(a).$$

Writing $s+\frac{a}{2}-\frac{1}{4}$ for s, we obtain that

$$\int_0^\infty t^{(s-1)} t^{\frac{a}{2}-\frac{1}{4}} {}_1F_1(a\,;\, b\,;\, -t)\, dt = \frac{\Gamma(b)\Gamma\left(s+\frac{a}{2}-\frac{1}{4}\right)\Gamma\left(\frac{a}{2}+\frac{1}{4}-s\right)}{\Gamma(a)\Gamma(b-s)}, \quad \ldots \quad (1.3)$$

where

$$\frac{1}{4}-R\left(\frac{a}{2}\right) < R(s) < R\left(\frac{a}{2}\right)+\frac{1}{4}.$$

Further, putting $b=\frac{a}{2}+\frac{1}{4}$ and applying duplication formula for gamma functions, we obtain that

$$\int_0^\infty t^{(s-1)} t^{\frac{a}{2}-\frac{1}{4}} {}_1F_1\left(a;\ \frac{a}{2}+\frac{1}{4};\ -t\right) dt$$

$$=\Gamma\left(\frac{a}{2}+\frac{1}{4}\right)2^{\frac{a}{2}-\frac{1}{4}}\cdot 2^s\Gamma\left(\frac{s}{2}+\frac{2a-3}{2}\cdot\frac{1}{2}+\frac{1}{4}\right)\times\Gamma\left(\frac{s}{2}+\frac{2a-1}{2}\cdot\frac{1}{2}+\frac{1}{4}\right). \quad .. \quad (1.4)$$

On applying Mellin's inversion formula (Hardy 1918) to (1.4) we obtain that

$$t^{\frac{a}{2}-\frac{1}{4}} {}_1F_1\left(a;\ \frac{a}{2}+\frac{1}{4};\ -t\right)$$

$$=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} 2^{\frac{a}{2}-\frac{1}{4}}\Gamma\left(\frac{a}{2}+\frac{1}{4}\right)2^s\Gamma\left(\frac{1}{4}+\frac{2a-3}{2}\cdot\frac{1}{2}+\frac{s}{2}\right)\times\Gamma\left(\frac{1}{4}+\frac{2a+1}{2}\cdot\frac{1}{2}+\frac{s}{2}\right)t^{-s}\,ds, \quad .. \quad (1.5)$$

where

$$0 < C < 1.$$

Brij Mohan (1939) has shown that, if $f(x)$ is $R\mu$ and

$$P(x)=\frac{1}{2\pi i}\int_{k-i\infty}^{k+i\infty} 2^s\Gamma\left(\frac{1}{4}+\frac{\mu}{2}+\frac{s}{2}\right)\Gamma\left(\frac{1}{4}+\frac{\nu}{2}+\frac{s}{2}\right)\times\psi(s)x^{-s}\,ds, \quad .. \quad (1.6)$$

where

$$0 < K < 1,$$

and

$$\psi(s)=\psi(1-s),$$

then

$$G(x)=\int_0^\infty f(y)P(xy)dy \quad .. \quad .. \quad .. \quad (1.7)$$

is $R\nu$. Hence, from (1.5) and (1.6), we conclude that the kernel

$$P(x)=x^{\frac{a}{2}-\frac{1}{4}} {}_1F_1\left(a;\ \frac{a}{2}+\frac{1}{4};\ -x\right) \quad .. \quad .. \quad .. \quad (1.8)$$

transforms $R_{\frac{2a-3}{4}}$ into $R_{\frac{2a+1}{4}}$ if $\frac{1}{2}\leq a\leq 3/2$.

Also,

$$P\left(\frac{x^2}{2}\right)=x^{a-\frac{1}{2}} {}_1F_1\left(a;\ \frac{a}{2}+\frac{1}{4};\ -\frac{x^2}{2}\right) \quad .. \quad .. \quad (1.9)$$

Dineschandra (1952) has shown that the function

$$x^{\alpha-\frac{1}{2}} {}_1F_1\left(\alpha;\ \frac{\alpha+\nu+1}{2};\ -\frac{x^2}{2}\right) \quad .. \quad .. \quad .. \quad (1.10)$$

is R_ν. Hence, identifying (1.9) with (1.10), we conclude that

$$x^{a-\frac{1}{2}} {}_1F_1\left(a;\ \frac{a}{2}+\frac{1}{4};\ -\frac{x^2}{2}\right) \qquad \ldots \quad \ldots \quad \ldots \ (1.11)$$

is Rc, establishing the property stated.

In particular, putting $a = \frac{1}{2}$ we find that the kernel given in (1.8) reduces to

$$e^{-x}, \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \ (1.12)$$

given by Brij Mohan (1942*a*); while (1.11) reduces to a familiar R_c function

$$e^{-x^2/2}, \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \ (1.13)$$

also given by Brij Mohan (1951).

§ 2. Applying Kummer's formula (Rao 1958*a*) to (1.8) we obtain that

$$P(x) = e^{-x} x^{\frac{a}{2}-\frac{1}{4}} {}_1F_1\left(\frac{1}{4}-\frac{a}{2};\ \frac{1}{4}+\frac{a}{2};\ x\right). \qquad \ldots \quad \ldots \ (2.1)$$

This result is capable of giving several interesting particular cases.

(1) If n is a positive integer

$${}_1F_1(-n;\ \alpha+1;\ x) = \frac{n!\,(\alpha+1)!}{(\alpha+n+1)!} L_n^\alpha(x), \qquad \ldots \quad \ldots \ (2.2)$$

$L_n^\alpha(x)$ being generalized Laguerore polynomial; so that from (2.1) and (2.2) we find that the kernel

$$x^{\frac{2(\alpha+n)+1}{4}} e^{-x} L_n^\alpha(x) \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \ (2.3)$$

transforms $R_{\frac{\alpha+n-2}{4}}$ into $R_{\frac{\alpha+n+2}{4}}$ if $\frac{1}{2} \leq \alpha+n+1 \leq 3/2$.

The corresponding R_c function in this case becomes

$$x^{\alpha+n+1/2} e^{-x^2/2} L_n^\alpha\left(\frac{x^2}{2}\right) \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \ (2.4)$$

(2) If n is a positive integer

$${}_1F_1(-n;\ \beta;\ x) = \frac{\Gamma(1-n-\beta)}{\Gamma(1-\beta)} e^{x/2} x^{-\beta/2} \mathcal{W}_{n+\beta/2,\ \frac{1}{2}-\beta}(x), \qquad \ldots \ (2.5)$$

where $\mathcal{W}_{d_1 m}(x)$ is Whittaker's function. Hence, from (2.1) and (2.5), we also conclude that the kernel

$$e^{-x/2} x^{\frac{n}{2}-\frac{1}{4}} \mathcal{W}_{n+\beta/2,\ \frac{1}{2}-\beta}(x) \qquad \ldots \quad \ldots \quad \ldots \ (2.6)$$

transforms $R_{\frac{2(\beta+n)-3}{4}}$ into $R_{\frac{2(\beta+n)+1}{4}}$ if $\frac{1}{2} \leq n+\beta \leq 3/2$;

while the corresponding R_c function becomes

$$e^{-x^2/4} x^{n-\frac{1}{2}} \mathcal{W}_{n+\beta/2,\ \frac{1}{2}-\beta}\left(\frac{x^2}{2}\right). \qquad \ldots \quad \ldots \quad \ldots \ (2.7)$$

(3) If n is a positive integer we further have

$$ {}_1F_1(-n\,;\ \beta\,;\ x) = (-1)^n n!\ \Gamma(\beta) T^n_{(\beta-1)}(x), \quad \ldots \quad \ldots \quad (2.8)$$

where $T^n_{(\beta-1)}(x)$ is a Sonine polynomial order n. Hence, we further conclude from (2.1) and (2.8) that the kernel

$$ e^{-x} x^{\frac{\beta+n}{2}-\frac{1}{4}} T^n_{(\beta-1)}(x), \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (2.9)$$

transforms $R_{\frac{2(n+\beta)-3}{4}}$ into $R_{\frac{2(n+\beta+1)}{4}}$ if $\frac{1}{2} \leq \beta+n \leq 3/2$.

Hence, the corresponding R_c function becomes

$$ e^{-x^2/2} x^{\beta+n-\frac{1}{2}} T^n_{(\beta-1)}\left(\frac{x^2}{2}\right). \quad \ldots \quad \ldots \quad \ldots \quad (2.10)$$

(4) Again, using the formula

$$ {}_1F_1(\alpha\,;\ 2\alpha\,;\ x) = 2^{2\alpha-1}\Gamma(\alpha+\tfrac{1}{2}) e^{x/2} x^{1/2-\alpha} I_{\alpha-\frac{1}{2}}(x), \quad \ldots \quad \ldots \quad (2.11)$$

where $I_\alpha(x)$ is a Bessel function with imaginary argument, we also conclude that the kernel

$$ x^{\frac{1}{4}-\frac{\alpha}{2}} e^{-x/2} I_{\alpha-\frac{1}{2}}\left(\frac{x}{2}\right) \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (2.12)$$

transforms $R_{\frac{2\alpha-3}{4}}$ into $R_{\frac{2\alpha+1}{4}}$ if $\frac{1}{2} \leq \alpha \leq 3/2$.

Hence, the corresponding R_c function becomes

$$ x^{1/2-\alpha} e^{-x^2/4} I_{\alpha-\frac{1}{2}}\left(\frac{x^2}{4}\right), \quad \ldots \quad \ldots \quad \ldots \quad (2.13)$$

putting $\alpha = \frac{1}{2}$ we find that (2.13) reduces to

$$ e^{-x^2/4} I_0\left(\frac{x^2}{4}\right) \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (2.14)$$

given by the author in a previous paper (Rao 1958*b*).

(5) Also, using the formula

$$ {}_1F_1(-n;\ \tfrac{1}{2};\ x) = \frac{2^{-n}}{\sqrt{\pi}} \Gamma(\tfrac{1}{2}-n) e^{x/2} D_{2n}(\sqrt{2x}), \quad \ldots \quad \ldots \quad (2.15)$$

where $D_n(x)$ is a parabolic cylinder function of order n, we further obtain from (2.1) that the kernel

$$ e^{-x/2} x^{n/2} D_{2n}(\sqrt{2x}) \quad \ldots \quad \ldots \quad \ldots \quad (2.16)$$

transforms $R_{\frac{n-1}{2}}$ into $R_{\frac{n+1}{2}}$ while the corresponding R_c function becomes

$$ e^{-x^2/4} x^n D_{2n}(x). \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (2.17)$$

As a particular case, putting $n = 0$ (2.17) reduces to

$$e^{-x^2/4}D_0(x), \quad \ldots \quad (2.18)$$

given by the author in a previous paper (Rao 1961).

(6) Further, using also the formula

$${}_1F_1(-n\,;\ 3/2\,;\ x) = \frac{\Gamma(-\frac{1}{2}-n)}{2^{n+1/2}\Gamma(\frac{1}{2})} x^{-\frac{1}{2}} e^{x/2} D_{2n+1}(\sqrt{2x}), \quad \ldots \quad (2.19)$$

we conclude that the kernel

$$x^{n/2}e^{-x/2}D_{2n+1}(\sqrt{2x}) \quad \ldots \quad (2.20)$$

transforms $R_{n/2}$ into $R_{n/2}+1$ giving the corresponding R_c function to be

$$x^n e^{-x^2/4}D_{(2n+1)}(x). \quad \ldots \quad (2.21)$$

(7) Finally, using the formula

$${}_1F_1(q\,;\,2\,;\,x) = (-1)^{-q}\frac{e^{x/2}}{x} k_{2-2q}(x/2), \quad \ldots \quad (2.22)$$

where $k_n(x)$ is a Bateman polynomial of order n, we conclude that the kernel

$$\frac{e^{-x/2}k_{2-2q}\left(\frac{x}{2}\right)}{x^{1/4+q/2}} \quad \ldots \quad (2.23)$$

transforms $R_{\frac{1-2q}{4}}$ into $R_{\frac{5-2q}{4}}$ giving the R_c function to be

$$\frac{e^{-x^2/4}k_{2-2q}\left(\frac{x^2}{4}\right)}{x^{q+\frac{1}{2}}}. \quad \ldots \quad (2.24)$$

This becomes a particular case of the $R_{(3-n)}$ function

$$x^{n-5/2}e^{-x^2/4}k_{2n-2}\left(\frac{x^2}{4}\right), \quad \ldots \quad (2.25)$$

given by the author in a previous paper (Rao 1959).

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