NON-NULL ELECTROMAGNETIC FIELDS IN CERTAIN RIEMANNIAN FOURFOLDS

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Non-null electromagnetic fields are considered in the non-static cylindrically symmetric space-time represented by Einstein-Rosen metric. The field equations considered here are those of Rainich's 'already unified theory'. The algebraic relations of the field theory give rise to three possible cases of electromagnetic fields. Several exact solutions of the Rainich equations are obtained which represent cylindrically symmetric non-null electromagnetic fields *in vacuo*. It is found that the electromagnetic fields correspond to electromagnetic wrenches, in the terminology of flat space-time, along one of the spatial coordinated lines. Solutions corresponding to purely electric and magnetic fields are obtained from them as special cases. A product space ($V_3 \times V_1$) is considered. It is shown that the corresponding Riemannian fourfold does not admit non-null electromagnetic fields. $R_{ij} = 0$ in this case is a necessary and sufficient condition for the fourfold to be flat. Finally, the Liouville form of the line-element is discussed. It is found to be incompatible with non-null electromagnetic field. The only field of electromagnetic significance which it represents is the well-known plane symmetric null electromagnetic field.

1. INTRODUCTION

The Einstein-Maxwell equations of the electromagnetic field theory appear to be only partially geometrical in character as the antisymmetric electromagnetic field tensor appearing in these equations are extraneous quantities not derivable from purely geometrical considerations. There is, however, a different approach—first made by Rainich (1925) which has of late attracted considerable attention due to the recent investigations by Misner and Wheeler (1957), Rosen (1959), Witten (1959), Bertotti (1959)—which show that in source-free regions, where electromagnetism is the only contributor to the stress energy tensor, the entire content of the combined Einstein-Maxwell theory can be replaced by a purely geometrical theory. This theory is referred to as Rainich's 'already unified theory'. In this paper, we make use of the Rainich equations to investigate the electromagnetic significance of the non-static cylindrically symmetric metric of Einstein and Rosen. This metric has extensively been investigated by Einstein and Rosen (1937), Rosen (1954, 1955, 1958), Weber and Wheeler (1957), Bonnor (1957) and others in connection with gravitational radiation in empty space-time. Misra and Radhakrishna (1962) have given several exact solutions of the Einstein-Maxwell field equations *in vacuo* corresponding to this metric. An entirely different approach is made in this paper to obtain non-null electromagnetic fields of cylindrical symmetry *in vacuo*. The Rainich algebraic relations give rise to three possible cases of electromagnetic fields, reference to which has already been made in a recent paper by Singh *et al.* (1965). Several exact solutions of Rainich equations are obtained representing non-null electromagnetic fields *in vacuo*. The electromagnetic fields correspond to electromagnetic wrenches, in the terminology of flat space-time, along one of the spatial coordinate lines. The fields corresponding to purely electric and purely magnetic fields are obtained by equating certain constants to zero.

A Riemannian V_4 may admit only of two types of product spaces, viz. $(V_2 \times V_2)$ and $(V_3 \times V_1)$. In a recent paper (Singh and Sharan 1965), we investigated the significance of product of two surfaces $(V_2 \times V_2)$ in general relativity. It has been shown there that only non-null uniform electromagnetic fields are possible in such a Riemannian fourfold. We consider in § 4 of this paper the electromagnetic significance of the other type of product space $(V_3 \times V_1)$. It is shown that such a fourfold is incompatible with non-null electromagnetic field. Also in this case $R_{ij} = 0$ is a necessary and sufficient condition that the Riemannian fourfold be flat.

It is shown in the last section of this paper that the Liouville form of the line-element in the space-time of general relativity does not admit non-null electromagnetic field. The Liouville form on being subjected to the field equations for electromagnetic distribution leads to the well-known null electromagnetic field of plane symmetry.

2. The Rainich Equations

The necessary and sufficient conditions given by Rainich for the existence of a non-null electromagnetic field *in vacuo* are (Adler *et al.* 1965)

$$R = 0, \qquad (2.1)$$

$$R^j_i R^k_j = \tfrac{1}{4}\delta^k_i R^b_a R^a_b, \qquad (2.2)$$

$$R^4_4 < 0, \qquad (2.3)$$

$$\alpha_{i;\,j} = \alpha_{j;\,i}, \qquad (2.4)$$

where

$$\alpha_i \equiv \frac{g_{ij}\epsilon^{jklm} R^p_k R_{pl;\,m}}{\sqrt{-g}R^b_a R^a_b}. \qquad (2.5)$$

Here R^j_i are the components of the Ricci tensor, R the scalar curvature, α_i the complexion vector, ϵ^{jklm} the Levi-Civita symbol skew-symmetric in all pairs of indices with $\epsilon^{1234} = 1$. A semicolon denotes covariant differentiation. Equations (2.1) to (2.4) together are completely equivalent to the Einstein-Maxwell field equations for electromagnetic fields *in vacuo* provided the field is non-null. For null fields, however, eqn. (2.4) falls off and Mishra (1964) has demonstrated the theoretical possibility of unification in such a case. The inequality (2.3) is necessary to ensure that the energy density of an electromagnetic field is positive definite.

We consider the non-static cylindrically symmetric metric of Einstein and Rosen, viz.

$$ds^2 = e^{\lambda-\mu}(dt^2-d\rho^2)-\rho^2 e^{-\mu}d\phi^2-e^{\mu}\,dz^2, \qquad \dots \quad (2.6)$$

λ, μ being functions of ρ and t only. The variables ρ, ϕ, z and t correspond to x^1, x^2, x^3 and x^4 respectively. The surviving components of the Ricci tensor for the metric (2.6) are

$$\left.\begin{aligned} R_1^1 &= \tfrac{1}{2}e^{\mu-\lambda}\left(-\lambda_{11}+\lambda_{44}+\mu_{11}-\mu_{44}-\mu_1^2+\frac{\lambda_1}{\rho}+\frac{\mu_1}{\rho}\right)\\ R_2^2 &= -R_3^3 = \tfrac{1}{2}e^{\mu-\lambda}\left(\mu_{11}-\mu_{44}+\frac{\mu_1}{\rho}\right)\\ R_4^4 &= \tfrac{1}{2}e^{\mu-\lambda}\left(-\lambda_{11}+\lambda_{44}+\mu_{11}-\mu_{44}+\mu_4^2+\frac{\mu_1}{\rho}-\frac{\lambda_1}{\rho}\right)\\ R_1^4 &= -R_4^1 = \tfrac{1}{2}e^{\mu-\lambda}\left(\mu_1\mu_4-\frac{\lambda_4}{\rho}\right) \end{aligned}\right\}. \qquad \dots \quad (2.7)$$

The lower suffixes 1 and 4 after an unknown function denote partial differentiation with respect to ρ and t respectively. The algebraic relations (2.1) and (2.2) give rise to the following three possible cases of electromagnetic fields:

Case (a):

$$R_1^4 = 0, \quad R_1^1 = -R_4^4, \quad R_1^1 = R_2^2, \quad R_3^3 = R_4^4, \qquad \dots \quad (2.8)$$

Case (b):

$$R_1^4 = 0, \quad R_1^1 = -R_4^4, \quad R_1^1 = R_3^3, \quad R_2^2 = R_4^4, \qquad \dots \quad (2.9)$$

Case (c):

$$R_1^4 \neq 0, \quad R_2^2 = -R_3^3, \quad R_1^1 = -R_4^4, \quad (R_2^2)^2 = (R_1^1)^2-(R_1^4)^2. \qquad \dots \quad (2.10)$$

This is not an invariant classification as the above relations are not invariant under arbitrary coordinate transformation. We find that the complexion vector α_i vanishes in all the above three cases (*a*), (*b*) and (*c*) with the result that the field equation (2.4) is identically satisfied. So far as the other field equations are concerned we consider them with reference to cases (*a*) and (*b*) only in this paper.

Case (a): The relations (2.8) with the help of (2.7) provide the following three independent differential equations

$$\lambda_4-\rho\mu_1\mu_4 = 0, \qquad \dots \quad (2.11)$$

$$\lambda_{11}-\lambda_{44}-\frac{\lambda_1}{\rho}+\mu_1^2 = 0, \qquad \dots \quad (2.12)$$

$$\mu_{11}-\mu_{44}+\tfrac{1}{2}\mu_1^2+\tfrac{1}{2}\mu_4^2+\frac{\mu_1}{\rho}-\frac{\lambda_1}{\rho} = 0. \qquad \dots \quad (2.13)$$

Hereafter small Greek letters shall stand for arbitrary constants distinct for each solution. To obtain an exact solution of (2.11), (2.12) and (2.13), we put $\mu_1 = \lambda_4 = 0$. Equation (2.12) admits the solution

$$\lambda = w^2\rho^2+h. \qquad \text{(2.14)}$$

Substituting from (2.14) in (2.13), we get

$$\mu_{44}-\tfrac{1}{2}\mu_4^2+2w^2 = 0$$

which is transformed by the substitution $\psi = e^{-\mu/2}$ into

$$\psi_{44}-w^2\psi = 0.$$

Integrating this, we ultimately get

$$\mu = \log \{n^2 \operatorname{sech}^2 (wt+b)\}. \qquad \text{(2.15)}$$

From (2.14) and (2.15), we obtain after a change of scale the metric representing a non-null electromagnetic field *in vacuo*, viz.

$$ds^2 = \{e^{a^2\rho^2} \cosh^2 (at+b)\}(dt^2-d\rho^2)-\rho^2 \cosh^2 (at+b)\, d\phi^2-\operatorname{sech}^2 (at+b)\, dz^2. \qquad \text{(2.16)}$$

For the metric (2.16),

$$R_4^4 = -a^2e^{-a^2\rho^2} \operatorname{sech}^4 (at+b) < 0$$

which asserts that the energy density of the electromagnetic field is positive definite.

Now we proceed to obtain another solution of (2.11), (2.12) and (2.13). When $\lambda_4 = \mu_4 = 0$, eqns. (2.12) and (2.13) give

$$2\mu_{11}+\frac{2\mu_1}{\rho}-\lambda_{11}-\frac{\lambda_1}{\rho} = 0$$

of which the integral is

$$\lambda = 2\mu-q \log (p\rho). \qquad \text{(2.17)}$$

Substituting for λ_1 and λ_{11} from (2.17) in (2.12), we get

$$\mu_{11}-\frac{\mu_1}{\rho}+\tfrac{1}{2}\mu_1^2+\frac{q}{\rho^2} = 0$$

which is transformed by the substitution $y = e^{\mu/2}$ into

$$y_{11}-\frac{y_1}{\rho}+\frac{qy}{2\rho^2} = 0. \qquad \text{(2.18)}$$

From (2.17) and (2.18), we ultimately get the solution of (2.11), (2.12) and (2.13) as

$$\left.\begin{aligned}\lambda &= \log \{h^2(w\rho^{2n}+k)^4\rho^{2(n-1)^2}\}\\ \mu &= \log \{(w\rho^{2n}+k)^2\rho^{2-2n}\}\end{aligned}\right\} \qquad \text{(2.19)}$$

which after a change of scale yields the metric representing an electromagnetic field, viz.

$$ds^2 = (m^2\rho^{2n}+v^2)^2\rho^{2n^2-2n}(dt^2-d\rho^2)-\frac{\rho^{2n}}{(m^2\rho^{2n}+v^2)^2}d\phi^2-(m^2\rho^{2n}+v^2)^2\rho^{2-2n}\, dz^2. \qquad \text{(2.20)}$$

For the metric (2.20),

$$R_4^4 = -\frac{4m^2n^2v^2}{(m^2\rho^{2n}+v^2)^4\rho^{2(n-1)^2}} < 0$$

which is in accordance with (2.3).

Case (b): The substitution from (2.7) in (2.9) leads to the following three independent differential equations:

$$\lambda_4 - \rho\mu_1\mu_4 = 0 \qquad \ldots \ldots \ldots \ldots \ldots (2.21)$$

$$\lambda_{11} - \lambda_{44} + \frac{\lambda_1}{\rho} - \mu_4^2 = 0 \qquad \ldots \ldots \ldots \ldots (2.22)$$

$$\mu_{11} - \mu_{44} - \tfrac{1}{2}\mu_1^2 - \tfrac{1}{2}\mu_4^2 + \frac{\mu_1}{\rho} + \frac{\lambda_1}{\rho} = 0 \qquad \ldots \ldots (2.23)$$

To obtain an exact solution of (2.21), (2.22) and (2.23), we take λ as well as μ as a sum of a function of ρ and a function of t. From (2.21), we have

$$\frac{\lambda_4}{\mu_4} = \rho\mu_1 = m \qquad \ldots \ldots \ldots \ldots (2.24)$$

by virtue of which (2.22) provides

$$\lambda_{11} + \frac{\lambda_1}{\rho} = m\mu_{44} + \mu_4^2 = v. \qquad \ldots \ldots \ldots (2.25)$$

Equation (2.25) gives

$$\lambda_1 = \frac{v\rho}{2} + \frac{q}{\rho}. \qquad \ldots \ldots \ldots \ldots (2.26)$$

Substituting from (2.24), (2.25) and (2.26) in (2.23), we find that (2.23) is satisfied if $m = q = 2$. Finally the solution of (2.21), (2.22) and (2.23) is found to be

$$\lambda = \frac{c^2\rho^2}{4} + \log\left\{n^2\rho^2 \cosh^4\left(\frac{c}{2}t + w\right)\right\}$$

$$\mu = \log\left\{k^2\rho^2 \cosh^2\left(\frac{c}{2}t + w\right)\right\}$$

which after a change of scale gives the line-element representing an electromagnetic field, viz.

$$ds^2 = \left\{\cosh^2\left(\frac{c}{2}t+w\right) e^{c^2\rho^2/4}\right\}(dt^2 - d\rho^2) - \operatorname{sech}^2\left(\frac{c}{2}t+w\right) d\phi^2 - \rho^2 \cosh^2\left(\frac{c}{2}t+w\right) dz^2. \qquad \ldots (2.27)$$

For the metric (2.27),

$$R_4^4 = -\frac{c^2 e^{-c^2\rho^2/4}}{4} \operatorname{sech}^4\left(\frac{c}{2}t+w\right) < 0$$

which is in keeping with (2.3).

Another solution of (2.21), (2.22) and (2.23) is obtained by putting $\lambda_4 = \mu_4 = 0$. Equations (2.22) and (2.23) give

$$\mu_{11} - \tfrac{1}{2}\mu_1^2 + \frac{\mu_1}{\rho} + \frac{v}{\rho^2} = 0. \qquad \ldots \quad \ldots \quad \ldots \quad (2.28)$$

From (2.22) and (2.28), we obtain the solution of (2.21), (2.22) and (2.23) as

$$\left.\begin{aligned} \lambda &= \frac{c^2}{2}\log\rho + h \\ \mu &= \log\frac{\rho^c}{(k\rho^c + w)^2} \end{aligned}\right\}. \qquad \ldots \quad \ldots \quad \ldots \quad (2.29)$$

Hence the metric representing an electromagnetic field *in vacuo* may be written finally in the form

$$ds^2 = (a^2\rho^c + b^2)^2 \rho^{\frac{c^2}{2} - c}(dt^2 - d\rho^2) - (a^2\rho^c + b^2)^2 \rho^{2-c} d\phi^2 - \frac{\rho^c}{(a^2\rho^c + b^2)^2} dz^2. \quad (2.30)$$

For the metric (2.30),

$$R_4^4 = -\frac{a^2b^2c^2}{(a^2\rho^c + b^2)^4 \rho^{\frac{1}{2}(c-2)^2}} < 0$$

which is in agreement with (2.3).

3. Electromagnetic Field Tensor

In order to obtain explicitly the components of the electromagnetic field tensor F_{ij} associated with the metrics representing electromagnetic fields in the preceding section, we consider the Einstein-Maxwell field equations *in vacuo*, viz.

$$R_i^j = -8\pi E_i^j \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (3.1)$$

$$F_{ij,k} + F_{jk,i} + F_{ki,j} = 0 \qquad \ldots \quad \ldots \quad \ldots \quad (3.2)$$

$$(\sqrt{-g}F^{ij})_{,j} = 0 \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (3.3)$$

where

$$E_i^j = -F_{ik}F^{jk} + \tfrac{1}{4}g_i^j F_{ab}F^{ab}. \qquad \ldots \quad \ldots \quad \ldots \quad (3.4)$$

Here E_i^j is the electromagnetic energy tensor, and a comma followed by a lower suffix denotes partial differentiation with respect to the corresponding variable.

The classification of the electromagnetic fields given in (2.8), (2.9) and (2.10) based on the Rainich algebraic relations (2.1) and (2.2) facilitates a lot the handling of the Einstein-Maxwell field equations as is clear from what follows.

Case (a): In view of (3.1) and (3.4), $R_1^1 = -R_4^4$ implies

$$F_{14} = F_{23} = 0.$$

Similar results hold for other pairs of the components of the Ricci tensor. Consequently by virtue of (2.8), F_{12} and F_{34} are found to be the only non-zero components of F_{ij}, so that the electromagnetic field corresponds to an electromagnetic wrench, in the terminology of flat space-time, along z-direction. From (2.8), (3.1) and (3.4), we get

$$R^1_1 = R^2_2 = -R^3_3 = -R^4_4 = 4\pi(F_{12}F^{12} - F_{34}F^{34}). \quad \quad (3.5)$$

Further, a close examination of (3.2) and (3.3) reveals that F_{12} and F_{34} are functions of ρ and t respectively and can be determined by considering only the equations

$$(\sqrt{-g}F^{12}),_1 = 0 \quad \quad (3.6)$$

and

$$(\sqrt{-g}F^{34}),_4 = 0 \quad \quad (3.7)$$

respectively.

For the metric (2.16), we get from (3.5), (3.6) and (3.7)

$$F_{12} = m\rho \quad \quad (3.8)$$

$$F_{34} = c \operatorname{sech}^2 (at+b) \quad \quad (3.9)$$

where

$$m^2 + c^2 = \frac{a^2}{4\pi}. \quad \quad (3.10)$$

Giving c the value zero, we get

$$F_{12} = \frac{a}{2\sqrt{\pi}}\rho \quad \quad (3.11)$$

as the only non-vanishing component of F_{ij} which corresponds to purely magnetic field.

Putting $m = 0$, the only surviving component of F_{ij} corresponding to purely electric field is found to be

$$F_{34} = \frac{a}{2\sqrt{\pi}} \operatorname{sech}^2 (at+b). \quad \quad (3.12)$$

Now we consider the line-element (2.20). Equations (3.5), (3.6) and (3.7) provide

$$F_{12} = a(m^2\rho^{2n} + v^2)^{-2}\rho^{2n-1} \quad \quad (3.13)$$

$$F_{34} = b \quad \quad (3.14)$$

where

$$a^2 + b^2 = \frac{m^2 n^2 v^2}{\pi}. \quad \quad (3.15)$$

When $b = 0$, it corresponds to a purely magnetic field and when $a = 0$, it corresponds to a purely electric field.

Case (b): From (2.9), (3.1) and (3.4), it follows that F_{13} and F_{24} are the only surviving components of F_{ij} so that the electromagnetic field in this case corresponds to an electromagnetic wrench, in the terminology of flat space-time, along ϕ-direction. From (2.9), (3.1) and (3.4), we get

$$R^1_1 = -R^2_2 = R^3_3 = -R^4_4 = 4\pi(F_{13}F^{13} - F_{24}F^{24}). \quad \quad (3.16)$$

A close examination of (3.2) and (3.3) leads to the conclusion that F_{13} and F_{24} are functions of ρ and t respectively and can be determined by considering only the equations

$$(\sqrt{-g}F^{13})_{,1} = 0 \quad \ldots \quad (3.17)$$

and

$$(\sqrt{-g}F^{24})_{,4} = 0 \quad \ldots \quad (3.18)$$

respectively.

Using (3.16), (3.17) and (3.18), we find that the components of F_{ij} corresponding to the line-element (2.27) are given by

$$F_{13} = a\rho \quad \ldots \quad (3.19)$$

$$F_{24} = h \operatorname{sech}^2 \left(\frac{c}{2}t+w\right) \quad \ldots \quad (3.20)$$

where

$$a^2+h^2 = \frac{c^2}{16\pi}. \quad \ldots \quad (3.21)$$

Equating h or a to zero, we get a field which corresponds to a purely magnetic field or a purely electric field respectively.

Considering the metric (2.30), eqns. (3.16), (3.17) and (3.18) provide

$$F_{13} = \frac{p\rho^{c-1}}{(a^2\rho^c+b^2)^2} \quad \ldots \quad (3.22)$$

$$F_{24} = n \quad \ldots \quad (3.23)$$

where

$$p^2+n^2 = \frac{a^2b^2c^2}{4\pi}. \quad \ldots \quad (3.24)$$

This field can also be made to correspond to either a purely magnetic or a purely electric field by proper choice of p and n.

4. Product Space

Since the metric for the three-space can in general be diagonalized, the two cases admitted by the product space ($V_3 \times V_1$) may be taken without loss of generality in the following forms:

Case (1):

$$ds^2 = -Adx^2-Bdy^2-Cdz^2+dt^2 \quad \ldots \quad (4.1)$$

where A, B, C are functions of x, y, z only.

Case (2):

$$ds^2 = -Adx^2-Bdy^2-dz^2+Ddt^2 \quad \ldots \quad (4.2)$$

where A, B, D are functions of x, y, t only.

It is easy to see that each of the metrics (4.1) and (4.2) is of class 3. In what follows the lower suffixes 1, 2, 3, 4 after an unknown function denote partial differentiation with respect to x, y, z, t respectively.

First we consider the metric (4.1). The non-zero components of R_{hijk} for (4.1) are

$$\left.\begin{aligned}
R_{1212} &= \tfrac{1}{2}(A_{22}B_{11}) - \frac{1}{4A}(A_2^2+A_1B_1) - \frac{1}{4B}(B_1^2+A_2B_2) + \frac{1}{4C}A_3B_3\\
R_{1313} &= \tfrac{1}{2}(A_{33}+C_{11}) - \frac{1}{4A}(A_3^2+A_1C_1) - \frac{1}{4C}(C_1^2+A_3C_3) + \frac{1}{4B}A_2C_2\\
R_{2323} &= \tfrac{1}{2}(B_{33}+C_{22}) - \frac{1}{4B}(B_3^2+B_2C_2) - \frac{1}{4C}(C_2^2+B_3C_3) + \frac{1}{4A}B_1C_1\\
R_{1213} &= \tfrac{1}{2}A_{23} - \frac{1}{4A}A_2A_3 - \frac{1}{4B}A_2B_3 - \frac{1}{4C}A_3C_2\\
R_{1223} &= -\tfrac{1}{2}B_{13} + \frac{1}{4B}B_1B_3 + \frac{1}{4A}A_3B_1 + \frac{1}{4C}B_3C_1\\
R_{1323} &= \tfrac{1}{2}C_{12} - \frac{1}{4C}C_1C_2 - \frac{1}{4A}A_2C_1 - \frac{1}{4B}B_1C_2
\end{aligned}\right\} \qquad (4.3)$$

while the surviving components of R_{ij} are

$$\left.\begin{aligned}
R_{11} &= \frac{1}{B}R_{1212} + \frac{1}{C}R_{1313} = \tfrac{1}{2}\left(\frac{A_{22}+B_{11}}{B} + \frac{A_{33}+C_{11}}{C}\right) - \tfrac{1}{4}\left(\frac{A_2^2+A_1B_1}{AB} + \frac{A_3^2+A_1C_1}{AC}\right.\\
&\quad \left. + \frac{B_1^2+A_2B_2}{B^2} + \frac{C_1^2+A_3C_3}{C^2} - \frac{A_3B_3+A_2C_2}{BC}\right)\\
R_{22} &= \frac{1}{C}R_{2323} + \frac{1}{A}R_{1212} = \tfrac{1}{2}\left(\frac{B_{33}+C_{22}}{C} + \frac{A_{22}+B_{11}}{A}\right) - \tfrac{1}{4}\left(\frac{B_3^2+B_2C_2}{BC} + \frac{B_1^2+A_2B_2}{AB}\right.\\
&\quad \left. + \frac{A_2^2+A_1B_1}{A^2} + \frac{C_2^2+B_3C_3}{C^2} - \frac{A_3B_3+B_1C_1}{AC}\right)\\
R_{33} &= \frac{1}{A}R_{1313} + \frac{1}{B}R_{2323} = \tfrac{1}{2}\left(\frac{A_{33}+C_{11}}{A} + \frac{B_{33}+C_{22}}{B}\right) - \tfrac{1}{4}\left(\frac{C_1^2+A_3C_3}{AC} + \frac{C_2^2+B_3C_3}{BC}\right.\\
&\quad \left. + \frac{A_3^2+A_1C_1}{A^2} + \frac{B_3^2+B_2C_2}{B^2} - \frac{A_2C_2+B_1C_1}{AB}\right)\\
R_{12} &= \frac{1}{C}R_{1323} = \frac{C_{12}}{2C} - \tfrac{1}{4}\left(\frac{C_1C_2}{C^2} + \frac{A_2C_1}{AC} + \frac{B_1C_2}{BC}\right)\\
R_{13} &= -\frac{1}{B}R_{1223} = \frac{B_{13}}{2B} - \tfrac{1}{4}\left(\frac{B_1B_3}{B^2} + \frac{A_3B_1}{AB} + \frac{B_3C_1}{BC}\right)\\
R_{23} &= \frac{1}{A}R_{1213} = \frac{A_{23}}{2A} - \tfrac{1}{4}\left(\frac{A_2A_3}{A^2} + \frac{A_2B_3}{AB} + \frac{A_3C_2}{AC}\right)
\end{aligned}\right\}. \qquad (4.4)$$

The non-vanishing components of the energy-momentum tensor T_i^j for (4.1) are given by

$$\left.\begin{aligned}
-8\pi T_1^1 &= \frac{B_{33}+C_{22}}{2BC}-\tfrac{1}{4}\left(\frac{B_3C_3+C_2^2}{BC^2}+\frac{B_2C_2+B_3^2}{B^2C}-\frac{B_1C_1}{ABC}\right)\\
-8\pi T_2^2 &= \frac{A_{33}+C_{11}}{2AC}-\tfrac{1}{4}\left(\frac{A_3C_3+C_1^2}{AC^2}+\frac{A_1C_1+A_3^2}{A^2C}-\frac{A_2C_2}{ABC}\right)\\
-8\pi T_3^3 &= \frac{A_{22}+B_{11}}{2AB}-\tfrac{1}{4}\left(\frac{A_2B_2+B_1^2}{AB^2}+\frac{A_1B_1+A_2^2}{A^2B}-\frac{A_3B_3}{ABC}\right)\\
-8\pi T_4^4 &= \tfrac{1}{2}\left(\frac{A_{22}+B_{11}}{AB}+\frac{A_{33}+C_{11}}{AC}+\frac{B_{33}+C_{22}}{BC}\right)-\tfrac{1}{4}\left(\frac{A_2B_2+B_1^2}{AB^2}\right.\\
&\quad+\frac{A_1B_1+A_2^2}{A^2B}+\frac{A_3C_3+C_1^2}{AC^2}+\frac{A_1C_1+A_3^2}{A^2C}+\frac{B_3C_3+C_2^2}{BC^2}\\
&\quad\left.+\frac{B_2C_2+B_3^2}{B^2C}-\frac{B_1C_1+A_2C_2+A_3B_3}{ABC}\right)\\
-8\pi T_2^1 &= -8\pi T_1^2\frac{B}{A}=-\frac{C_{12}}{2AC}+\tfrac{1}{4}\left(\frac{C_1C_2}{AC^2}+\frac{A_2C_1}{A^2C}+\frac{B_1C_2}{ABC}\right)\\
-8\pi T_3^1 &= -8\pi\frac{C}{A}T_1^3=-\frac{B_{13}}{2AB}+\tfrac{1}{4}\left(\frac{B_1B_3}{AB^2}+\frac{A_3B_1}{A^2B}+\frac{B_3C_1}{ABC}\right)\\
-8\pi T_3^2 &= -8\pi\frac{C}{B}T_2^3=-\frac{A_{23}}{2AB}+\tfrac{1}{4}\left(\frac{A_2A_3}{A^2B}+\frac{A_2B_3}{AB^2}+\frac{A_3C_2}{ABC}\right)
\end{aligned}\right\}. \qquad (4.5)$$

From (4.5), we observe that

$$T_4^4 = T_1^1+T_2^2+T_3^3. \qquad (4.6)$$

In view of (4.3) and (4.4), the empty space-time conditions

$$R_{ij} = 0 \qquad (4.7)$$

may be regarded as 6 homogeneous equations in 6 unknowns R_{1212}, R_{1313}, R_{2323}, R_{1213}, R_{1223}, R_{1323}. Since the determinant of this set of equations does not vanish, the set admits only the zero solution. Consequently the field equations (4.7) lead to flat space-time.

The eigenvalues of T_i^j for (4.1) are given by

$$\left|T_i^j-\lambda g_i^j\right| = 0$$

i.e.

$$(T^4_4-\lambda)\Big[-\lambda^3+\lambda^2T^4_4+\lambda\{-T^1_1T^2_2-T^1_1T^3_3-T^2_2T^3_3+\frac{C}{B}(T^3_2)^2+\frac{B}{A}(T^2_1)^2+\frac{C}{A}(T^3_1)^2\}$$
$$+T^1_1T^2_2T^3_3-\frac{C}{B}T^1_1(T^3_2)^2-\frac{B}{A}(T^2_1)^2T^3_3+\frac{2C}{A}T^2_1T^3_1T^3_2-\frac{C}{A}T^2_2(T^3_1)^2\Big]=0.$$

Hence T^4_4 is one of the eigenvalues of T^j_i. For electromagnetic distribution, we have

$$T=0. \qquad \ldots \qquad (4.8)$$

From (4.6) and (4.8), it follows that

$$T^4_4=0. \qquad \ldots \qquad (4.9)$$

Consequently the metric (4.1) is incompatible with non-null electromagnetic field.

The necessary conditions for null-electromagnetic field are

$$R^j_iR^k_j=0. \qquad \ldots \qquad (4.10)$$

For $i=k=1$, (4.10) gives

$$\frac{1}{A^2}(R_{11})^2+\frac{1}{AB}(R_{12})^2+\frac{1}{AC}(R_{13})^2=0$$

which implies that

$$R_{11}=R_{12}=R_{13}=0.$$

Similarly on taking $i=k=2$ and $i=k=3$, we get

$$R_{22}=R_{33}=R_{23}=0.$$

Hence the necessary conditions for null-electromagnetic field imply $R_{ij}=0$ which lead to flat space-time. Thus the product space (4.1) is incompatible with null as well as non-null electromagnetic field.

Now we consider the product space (4.2). The surviving components of R_{hijk} for (4.2) are

$$\left.\begin{aligned}
R_{1212}&=\tfrac{1}{2}(A_{22}+B_{11})-\frac{1}{4A}(A_2^2+A_1B_1)-\frac{1}{4B}(B_1^2+A_2B_2)-\frac{1}{4D}A_4B_4\\
R_{1414}&=\tfrac{1}{2}(A_{44}-D_{11})-\frac{1}{4A}(A_4^2-A_1D_1)+\frac{1}{4D}(D_1^2-A_4D_4)-\frac{1}{4B}A_2D_2\\
R_{2424}&=\tfrac{1}{2}(B_{44}-D_{22})-\frac{1}{4B}(B_4^2-B_2D_2)+\frac{1}{4D}(D_2^2-B_4D_4)-\frac{1}{4A}B_1D_1\\
R_{1214}&=\tfrac{1}{2}A_{24}-\frac{1}{4A}A_2A_4-\frac{1}{4B}A_2B_4-\frac{1}{4D}D_2A_4\\
R_{1224}&=-\tfrac{1}{2}B_{14}+\frac{1}{4B}B_1B_4+\frac{1}{4A}B_1A_4+\frac{1}{4D}D_1B_4\\
R_{1424}&=-\tfrac{1}{2}D_{12}+\frac{1}{4A}D_1A_2+\frac{1}{4B}D_2B_1+\frac{1}{4D}D_1D_2
\end{aligned}\right\} \qquad (4.11)$$

and the non-zero components of R_{ij} are

$$\left.\begin{aligned}
R_{11} &= \frac{1}{B}R_{1212}-\frac{1}{D}R_{1414} = \tfrac{1}{2}\left(\frac{A_{22}+B_{11}}{B}+\frac{D_{11}-A_{44}}{D}\right)+\tfrac{1}{4}\left(-\frac{A_2^2+A_1B_1}{AB}+\frac{A_4^2-A_1D_1}{AD}\right.\\
&\qquad\left.-\frac{B_1^2+A_2B_2}{B^2}-\frac{D_1^2-A_4D_4}{D^2}+\frac{-A_4B_4+A_2D_2}{BD}\right)\\
R_{22} &= \frac{1}{A}R_{1212}-\frac{1}{D}R_{2424} = \tfrac{1}{2}\left(\frac{A_{22}+B_{11}}{A}+\frac{D_{22}-B_{44}}{D}\right)+\tfrac{1}{4}\left(-\frac{B_1^2+A_2B_2}{AB}+\frac{B_4^2-B_2D_2}{BD}\right.\\
&\qquad\left.-\frac{A_2^2+A_1B_1}{A^2}-\frac{D_2^2-B_4D_4}{D^2}+\frac{B_1D_1-A_4B_4}{AD}\right)\\
R_{44} &= \frac{1}{A}R_{1414}+\frac{1}{B}R_{2424} = \tfrac{1}{2}\left(\frac{A_{44}-D_{11}}{A}+\frac{B_{44}-D_{22}}{B}\right)+\tfrac{1}{4}\left(\frac{D_1^2-A_4D_4}{AD}+\frac{D_2^2-B_4D_4}{BD}\right.\\
&\qquad\left.-\frac{A_4^2-A_1D_1}{A^2}-\frac{B_4^2-B_2D_2}{B^2}-\frac{A_2D_2+B_1D_1}{AB}\right)\\
R_{12} &= -\frac{1}{D}R_{1424} = \frac{D_{12}}{2D}-\tfrac{1}{4}\left(\frac{A_2D_1}{AD}+\frac{B_1D_2}{BD}+\frac{D_1D_2}{D^2}\right)\\
R_{14} &= -\frac{1}{B}R_{1224} = \frac{B_{14}}{2B}-\tfrac{1}{4}\left(\frac{A_4B_1}{AB}+\frac{B_4D_1}{BD}+\frac{B_1B_4}{B^2}\right)\\
R_{24} &= \frac{1}{A}R_{1214} = \frac{A_{24}}{2A}-\tfrac{1}{4}\left(\frac{A_2B_4}{AB}+\frac{A_4D_2}{AD}+\frac{A_2A_4}{A^2}\right)
\end{aligned}\right\}. \tag{4.12}$$

The non-vanishing components of the energy-momentum tensor T_i^j for (4.2) are given by

$$\left.\begin{aligned}
-8\pi T_1^1 &= \frac{D_{22}-B_{44}}{2BD}-\tfrac{1}{4}\left(\frac{D_2^2-B_4D_4}{BD^2}+\frac{B_2D_2-B_4^2}{B^2D}-\frac{B_1D_1}{ABD}\right)\\
-8\pi T_2^2 &= \frac{D_{11}-A_{44}}{2AD}-\tfrac{1}{4}\left(\frac{D_1^2-A_4D_4}{AD^2}+\frac{A_1D_1-A_4^2}{A^2D}-\frac{A_2D_2}{ABD}\right)\\
-8\pi T_3^3 &= \tfrac{1}{2}\left(\frac{A_{22}+B_{11}}{AB}+\frac{D_{11}-A_{44}}{AD}+\frac{D_{22}-B_{44}}{BD}\right)-\tfrac{1}{4}\left(\frac{A_2B_2+B_1^2}{AB^2}\right.\\
&\qquad+\frac{A_1B_1+A_2^2}{A^2B}+\frac{D_1^2-A_4D_4}{AD^2}+\frac{A_1D_1-A_4^2}{A^2D}+\frac{D_2^2-B_4D_4}{BD^2}\\
&\qquad\left.+\frac{B_2D_2-B_4^2}{B^2D}+\frac{A_4B_4-A_2D_2-B_1D_1}{ABD}\right)\\
-8\pi T_4^4 &= \frac{A_{22}+B_{11}}{2AB}-\tfrac{1}{4}\left(\frac{A_2B_2+B_1^2}{AB^2}+\frac{A_1B_1+A_2^2}{A^2B}+\frac{A_4B_4}{ABD}\right)\\
-8\pi T_2^1 &= -8\pi\frac{B}{A}T_1^2 = -\frac{D_{12}}{2AD}+\tfrac{1}{4}\left(\frac{D_1D_2}{AD^2}+\frac{A_2D_1}{A^2D}+\frac{B_1D_2}{ABD}\right)\\
-8\pi T_4^1 &= 8\pi\frac{D}{A}T_1^4 = -\frac{B_{14}}{2AB}+\tfrac{1}{4}\left(\frac{B_1B_4}{AB^2}+\frac{A_4B_1}{A^2B}+\frac{B_4D_1}{ABD}\right)\\
-8\pi T_4^2 &= 8\pi\frac{D}{B}T_2^4 = -\frac{A_{24}}{2AB}+\tfrac{1}{4}\left(\frac{A_2A_4}{A^2B}+\frac{A_2B_4}{AB^2}+\frac{A_4D_2}{ABD}\right)
\end{aligned}\right\}. \tag{4.13}$$

From (4.13), it is clear that

$$T^3_3 = T^1_1 + T^2_2 + T^4_4. \qquad \ldots \quad (4.14)$$

Just as in the preceding case, the field equations $R_{ij} = 0$ may be regarded as 6 homogeneous equations in 6 unknowns R_{1212}, R_{1414}, R_{2424}, R_{1214}, R_{1224}, R_{1424}. The determinant of these equations being non-zero, they admit only the zero solution. Thus $R_{ij} = 0$ lead to $R_{hijk} = 0$ implying flat space-time.

The eigenvalues of T^j_i for (4.2) are given by

$$\left| T^j_i - \lambda g^j_i \right| = 0$$

i.e.

$$(T^3_3 - \lambda)\Big[-\lambda^3 + \lambda^2 T^3_3 + \lambda\Big\{-T^1_1T^2_2 - T^1_1T^4_4 - T^2_2T^4_4 - \frac{D}{B}(T^4_2)^2 + \frac{B}{A}(T^2_1)^2 - \frac{D}{A}(T^4_1)^2\Big\}$$

$$+T^1_1T^2_2T^4_4 + \frac{D}{B}T^1_1(T^4_2)^2 - \frac{B}{A}(T^2_1)^2T^4_4 - \frac{2D}{A}T^2_1T^4_1T^4_2 + \frac{D}{A}T^2_2(T^4_1)^2\Big] = 0. \quad \ldots \quad (4.15)$$

Hence T^3_3 is one of the eigenvalues of T^j_i. Now for electromagnetic distribution, we have

$$T = 0. \qquad \ldots \quad (4.16)$$

In view of (4.14) and (4.16), we get

$$T^3_3 = 0. \qquad \ldots \quad (4.17)$$

Consequently the product space (4.2) cannot represent non-null electromagnetic field.

5. Liouville Form of the Line-element

The generalized Liouville form of the linear element is (Eisenhart 1960)

$$ds^2 = (X_1 + X_2 + \ldots + X_n)\sum_i e_i(dx^i)^2,\ i = 1, 2, \ldots, n, \qquad \ldots \quad (5.1)$$

where X_i is a function of x^i alone. In a Riemannian fourfold corresponding to the space-time of general relativity, the metric (5.1) reduces to the following form:

$$ds^2 = (X + Y + Z + \tau)(-dx^2 - dy^2 - dz^2 + dt^2) \qquad \ldots \quad (5.2)$$

where X, Y, Z, τ are functions of x, y, z, t respectively. An investigation of the relativistic significance of this line-element seems to be interesting as in such a space-time a family of hypersurfaces is immediately available which is normal to a geodesic congruence.

The non-zero components of the Ricci tensor R_i^j for the metric (5.2) are

$$\left.\begin{aligned}
R_1^1 &= \frac{3X_1^2}{2F^3}+\frac{1}{2F^2}(-3X_{11}-Y_{22}-Z_{33}+\tau_{44})\\
R_2^2 &= \frac{3Y_2^2}{2F^3}+\frac{1}{2F^2}(-X_{11}-3Y_{22}-Z_{33}+\tau_{44})\\
R_3^3 &= \frac{3Z_3^2}{2F^3}+\frac{1}{2F^2}(-X_{11}-Y_{22}-3Z_{33}+\tau_{44})\\
R_4^4 &= -\frac{3\tau_4^2}{2F^3}+\frac{1}{2F^2}(-X_{11}-Y_{22}-Z_{33}+3\tau_{44})\\
R_1^2 &= R_2^1 = \frac{3X_1Y_2}{2F^3}, \quad R_1^3 = R_3^1 = \frac{3X_1Z_3}{2F^3}\\
R_2^3 &= R_3^2 = \frac{3Y_2Z_3}{2F^3}, \quad R_4^1 = -R_1^4 = \frac{3X_1\tau_4}{2F^3}\\
R_4^2 &= -R_2^4 = \frac{3Y_2\tau_4}{2F^3}, \quad R_4^3 = -R_3^4 = \frac{3Z_3\tau_4}{2F^3}
\end{aligned}\right\} \quad \ldots \quad \ldots \quad (5.3)$$

where

$$F = X+Y+Z+\tau.$$

Giving (i, k) the values (1, 2), (1, 3), (1, 4), (2, 3) in the equation (2.2), we get the following relations for the metric (5.2).

$$\frac{3}{2F}(X_1^2+Y_2^2+Z_3^2-\tau_4^2)+(-2X_{11}-2Y_{22}-Z_{33}+\tau_{44}) = 0 \quad \ldots \quad (5.4)$$

$$\frac{3}{2F}(X_1^2+Y_2^2+Z_3^2-\tau_4^2)+(-2X_{11}-Y_{22}-2Z_{33}+\tau_{44}) = 0 \quad \ldots \quad (5.5)$$

$$\frac{3}{2F}(X_1^2+Y_2^2+Z_3^2-\tau_4^2)+(-2X_{11}-Y_{22}-Z_{33}+2\tau_{44}) = 0 \quad \ldots \quad (5.6)$$

$$\frac{3}{2F}(X_1^2+Y_2^2+Z_3^2-\tau_4^2)+(-X_{11}-2Y_{22}-2Z_{33}+\tau_{44}) = 0. \quad \ldots \quad (5.7)$$

Subtracting eqns. (5.5), (5.6), (5.7) from (5.4), we get

$$-Y_{22}+Z_{33} = 0, \quad -Y_{22}-\tau_{44} = 0, \quad -X_{11}+Z_{33} = 0$$

respectively. These equations imply that

$$X_{11} = Y_{22} = Z_{33} = -\tau_{44}. \quad \ldots \quad \ldots \quad \ldots \quad (5.8)$$

For the metric (5.2), the equation (2.1) gives

$$\frac{3}{2F}(X_1^2+Y_2^2+Z_3^2-\tau_4^2)+3(-X_{11}-Y_{22}-Z_{33}+\tau_{44}) = 0. \quad \ldots \quad (5.9)$$

Subtracting (5.9) from (5.4), we obtain

$$X_{11}+Y_{22}+2Z_{33}-2\tau_{44} = 0$$

which, by virtue of (5.8), provides

$$X_{11} = Y_{22} = Z_{33} = \tau_{44} = 0. \quad \ldots \quad \ldots \quad \ldots \quad (5.10)$$

The solution admitted by (5.10) is

$$X = ax+l, \quad Y = by+m, \quad Z = cz+n, \quad \tau = ht+p \quad \ldots \quad (5.11)$$

which on substituting in (5.9) gives the relation between a, b, c, h, namely

$$a^2+b^2+c^2 = h^2. \quad \ldots \quad (5.12)$$

The solution (5.11) represents a null electromagnetic field. Thus the metric (5.2) is incompatible with non-null electromagnetic field. The case of material distribution will be dealt with separately.

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