STABILITY OF HYDROMAGNETIC FLOW BETWEEN POROUS PARALLEL PLATES UNDER TRANSVERSE MAGNETIC FIELD

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Effect of wall porosity is investigated on the stability of hydromagnetic flow between two parallel plates under uniform transverse magnetic field. It is shown that for weakly conducting fluids, critical Reynolds number for the onset of instability is lower for two-dimensional disturbances than for three-dimensional disturbances. A class of sufficient conditions is also obtained for stability of the system.

The Problem and Laminar Solution

Lock (1955) discussed the stability of a hydromagnetic flow between two parallel plates in the presence of a uniform transverse magnetic field. The main objective of the present investigation is to study the effect of wall porosity on the stability of flow. Laminar hydromagnetic flow between two parallel plates in the presence of uniform suction at one plate and uniform injection at another plate when the external field is uniform and transverse to the plate has been obtained by Mehta and Jain (1962). It is this laminar solution which is being subjected to investigation for stability. Squire (1933) established that it is sufficient to investigate two-dimensional disturbances for the stability of Poiseuille flow between two parallel plates and Synge (1938) obtained a class of sufficient conditions for the stability of these flows.

In the present paper, it is established that for weakly conducting fluids, Squire's result is still true when parallel plates are taken to be porous. A class of sufficient conditions is also obtained. The detailed investigation of two-dimensional disturbances and numerical work will be reported in a later communication.

We use Cartesian coordinates (x, y, z) taking the mid-point between two parallel plates as origin. The plates are at a distance $2a$ apart. The axis of x is in the main flow direction and the axis of y is perpendicular to the plates. v_0 is the constant velocity of injection and suction and H_0 is the uniform transverse magnetic field. It is assumed that there is uniform injection at the plate $y = -a$ and an equal suction at the plate $y = a$.

The basic equations (Cowling 1957) of hydromagnetics for incompressible, viscous and finitely conducting fluid flow yield the following steady state solution [Mehta and Jain (1962)] in non-dimensional form:

$$\omega = 1+\frac{\sinh m_1(e^{m_2\xi}-1)-\sinh m_2(e^{m_1\xi}-1)}{(\cosh m_1-1)\sinh m_2-(\cosh m_2-1)\sinh m_1} \quad .. \quad (1)$$

$$m_1, m_2 = \tfrac{1}{2}\left[R_c \pm \sqrt{R_c^2+4M^2}\right] \quad .. \quad .. \quad .. \quad (2)$$

where

$$\xi = \frac{y}{a}, \; -1 \leqslant \xi \leqslant 1, \; \omega = \frac{V_x}{U_0}$$

$$U_0 = \frac{Pa^2}{\rho\nu}\frac{1}{m_1-m_2}\left[\tanh\frac{m_1}{2}-\tanh\frac{m_2}{2}\right]$$

V_x, velocity in x-direction; U_0, velocity at $\xi = 0$; $R_c = \frac{v_0 a}{\nu}$, cross Reynolds number; $M = \mu H_0 a\sqrt{\frac{\sigma}{\rho\nu}}$, Hartmann number; ρ, density; ν, kinematic viscosity; σ, electrical conductivity; μ, permeability; $P = -\frac{\partial p}{\partial x}$, the constant pressure gradient. This solution is true under the condition that $\epsilon \ll 1$ where $\epsilon = \frac{\nu}{\nu_H}$, $\nu_H = \frac{1}{4\pi\mu\sigma}$ being the magnetic diffusivity. We use electromagnetic units.

Perturbation Equations

Normal mode analysis is followed to study the stability of laminar flow $V = (V_x, v_0, 0)$ and $H = (H_x, H_0, 0)$. If A, B and p_1 are the disturbances in velocity field, magnetic field and pressure, then linearized equations for perturbations are:

Momentum equation

$$\frac{\partial A}{\partial t}+(V.\nabla)A+(A.\nabla)V = -\frac{1}{\rho}\operatorname{grad} p_1+\nu\nabla^2 A + \frac{\mu}{4\pi\rho}\left[(\operatorname{curl} H)\times B+(\operatorname{curl} B)\times H\right]. \quad .. \quad .. \quad .. \quad (3)$$

Induction equation

$$\frac{\partial B}{\partial t}-\operatorname{curl}[V\times B+A\times H] = \nu_H\nabla^2 B. \quad .. \quad .. \quad (4)$$

Divergence relations

$$\operatorname{div} A = 0, \; \operatorname{div} B = 0. \quad .. \quad .. \quad .. \quad .. \quad (5)$$

We assume

$$\left.\begin{aligned} A &= [A_x(y), A_y(y), A_z(y)]\exp[i(\alpha_0 x+\beta_0 z-\alpha_0 c_1 t)] \\ B &= [B_x(y), B_y(y), B_z(y)]\exp[i(\alpha_0 x+\beta_0 z-\alpha_0 c_1 t)] \\ p_1 &= p_0(y)\exp[i(\alpha_0 x+\beta_0 z-\alpha_0 c_1 t)] \end{aligned}\right\} \quad .. \quad (6)$$

and, therefore, the eqns. (3) to (5) in the component form can be written as:

$$-i\alpha_0 c_1 A_x + i\alpha_0 V_x A_x + v_0 A_x' + A_y \frac{\partial V_x}{\partial y} = -\frac{1}{\rho} i\alpha_0 p_0 + \nu\left(-\alpha_0^2 A_x + A_x'' - \beta_0^2 A_x\right)$$

$$+\frac{\mu}{4\pi\rho}\left[B_y \frac{\partial H_x}{\partial y} - H_0\left(i\alpha_0 B_y - B_x'\right)\right] \quad \ldots \quad (7)$$

$$-i\alpha_0 c_1 A_y + i\alpha_0 V_x A_y + v_0 A_y' = -\frac{p_0'}{\rho} + \nu\left(-\alpha_0^2 A_y + A_y'' - \beta_0^2 A_y\right)$$

$$+\frac{\mu}{4\pi\rho}\left[-B_x \frac{\partial H_x}{\partial y} + H_x\left(i\alpha_0 B_y - B_x'\right)\right] \quad \ldots \quad (8)$$

$$-i\alpha_0 c_1 A_z + i\alpha_0 V_x A_z + v_0 A_z' = -\frac{1}{\rho} i\beta_0 p_0 + \nu\left[-\alpha_0^2 A_z + A_z'' - \beta_0^2 A_z\right]$$

$$+\frac{\mu}{4\pi\rho}\left[H_0\left(B_z' - i\beta_0 B_y\right) - H_x\left(i\beta_0 B_x - i\alpha_0 B_z\right)\right] \quad \ldots \quad (9)$$

$$-i\alpha_0 c_1 B_x - i\beta_0 V_x B_z + v_0 B_x' - V_x B_y' - B_y \frac{dV_x}{dy} + i\beta_0 H_x A_z$$

$$+H_x A_y' + A_y \frac{d}{dy} H_x - H_0 A_x' = \nu_H\left(-\alpha_0^2 B_x + B_x'' - \beta_0^2 B_x\right) \quad \ldots \quad (10)$$

$$-i\alpha_0 c_1 B_y - i\beta_0 v_0 B_z + i\alpha_0 (V_x B_y - v_0 B_x) + i\beta_0 H_0 A_z$$

$$-i\alpha_0 (H_x A_y - H_0 A_x) = \nu_H\left(-\alpha_0^2 B_y + B_y'' - \beta_0^2 B_y\right) \quad \ldots \quad (11)$$

$$-i\alpha_0 c_1 B_z + i\alpha_0 V_x B_z + v_0 B_z' - i\alpha_0 H_x A_z - H_0 A_z'$$

$$= \nu_H\left(-\alpha_0^2 B_z + B_z'' - \beta_0^2 B_z\right) \quad \ldots \quad (12)$$

$$i\alpha_0 A_x + i\beta_0 A_z + A_y' = 0 \quad \ldots \quad (13)$$

$$i\alpha_0 B_x + i\beta_0 B_z + B_y' = 0. \quad \ldots \quad (14)$$

Equations (10), (12) and (14) give

$$\alpha_0 c_1 B_y' + i v_0 B_y'' - \alpha_0 V_x B_y' - \alpha_0 \frac{dV_x}{dy} B_y + \alpha_0 H_x A_y'$$

$$+\alpha_0 \frac{dH_x}{dy} A_y - i H_0 A_y'' = -i\nu_H\left(\alpha_0^2 B_y' + \beta_0^2 B_y' - B_y'''\right). \quad (15)$$

Equations (7), (9) and (13) give

$$\alpha_0 c_1 A_y' - \alpha_0 V_x A_y' + i v_0 A_y'' + \alpha_0 \frac{dV_x}{dy} A_y$$

$$= -i \frac{\alpha_0^2 + \beta_0^2}{\rho} p_0 - i\nu\left(\alpha_0^2 A_y' + \beta_0^2 A_y' - A_y'''\right)$$

$$+\frac{\mu}{4\pi\rho}\left[\alpha_0 \frac{dH_x}{dy} B_y - i(\alpha_0^2 + \beta_0^2) H_0 B_y + i H_0 B_y'' - i H_x\left(\beta_0^2 B_x - \alpha_0 \beta_0 B_z\right)\right].$$

$$\ldots \quad (16)$$

Eliminating p_0 from eqns. (8) and (16) one gets

$$\alpha_0 c_1\left[(\alpha_0^2+\beta_0^2)A_y-A_y''\right]-\alpha_0 V_x\left[(\alpha_0^2+\beta_0^2)A_y-A_y''\right]+iv_0\left[(\alpha_0^2+\beta_0^2)A_y'-A_y'''\right]$$

$$-\alpha_0\frac{d^2V_x}{dy^2}A_y = -i\nu\left[(\alpha_0^2+\beta_0^2)^2A_y-2(\alpha_0^2+\beta_0^2)A_y''+A_y^{iv}\right]$$

$$+\frac{\mu}{4\pi\rho}\left[-\alpha_0\frac{d^2H_x}{dy^2}B_y+i(\alpha_0^2+\beta_0^2)H_0B_y'-iH_0B_y'''-\alpha_0(\alpha_0^2+\beta_0^2)H_xB_y+\alpha_0H_xB_y''\right]. \quad \ldots \quad (17)$$

Equations (15) and (17) are two equations in A_y and B_y. We shall express them in non-dimensional form with the help of the following substitution:

$$A_y = U_0\psi, \quad B_y = H_0\phi, \quad h = \frac{H_x}{H_0}$$

$$\theta = \frac{v_0}{U_0}, \quad R = \frac{U_0 a}{\nu}, \quad R_M = \frac{U_0 a}{\nu_H}$$

$$\alpha_0 = \frac{\alpha}{a}, \quad \beta_0 = \frac{\beta}{a}, \quad c_1 = U_0 c$$

$$S = \frac{\mu}{4\pi\rho}\frac{H_0^2}{U_0^2}, \quad \xi = \frac{y}{a}, \quad \frac{d}{dy} = \frac{1}{a}\frac{d}{d\xi}.$$

The basic equations in non-dimensional form are reduced to

$$\alpha c\phi'+i\theta\phi''-\alpha\omega\phi'-\alpha\omega'\phi+\alpha h\psi'+\alpha h'\psi-i\psi''$$
$$= -\frac{i}{R_M}(\alpha^2\phi'+\beta^2\phi'-\phi''') \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (18)$$

$$\alpha(c-\omega)[(\alpha^2+\beta^2)\psi-\psi'']+i\theta[(\alpha^2+\beta^2)\psi'-\psi''']-\alpha\omega''\psi$$
$$= -\frac{i}{R}[(\alpha^2+\beta^2)^2\psi-2(\alpha^2+\beta^2)\psi''+\psi^{iv}]$$
$$+S[-\alpha h''\phi+i(\alpha^2+\beta^2)\phi'-i\phi'''-\alpha(\alpha^2+\beta^2)h\phi+\alpha h\phi'']. \quad (19)$$

The plates are taken to be non-conducting so that perturbations in the magnetic field due to its continuity vanish on the boundaries. This condition with the help of divergence relation yields

$$\phi = \phi' = 0 \text{ at } \xi = \pm 1. \quad \ldots \quad \ldots \quad \ldots \quad (20)$$

Similarly no slip conditions along with continuity equation give

$$\psi = \psi' = 0 \text{ at } \xi = \pm 1. \quad \ldots \quad \ldots \quad \ldots \quad (21)$$

Integration of equation (18) gives

$$h\psi-\frac{i\psi'}{\alpha} = (\omega-c)\phi-\frac{i\theta\phi'}{\alpha}+\frac{i}{\alpha R_M}(\phi''-\alpha^2\phi-\beta^2\phi). \quad \ldots \quad \ldots \quad (22)$$

The constant of integration vanishes because of boundary conditions.

For most of the practical fluids $\frac{R_M}{R} = \frac{\nu}{\nu_H} \ll 1$ and h contains R_M as a direct factor (Mehta and Jain 1962), therefore, it is permissible to neglect all the terms in (19) and (22) involving h and also the first and second terms on the right-hand side of eqn. (22) (Lock 1955). Eliminating ϕ from simplified form of eqns. (19) and (22), one gets

$$(\omega-c)[\psi''-(\alpha^2+\beta^2)\psi]-\omega''\psi-\frac{iR_c}{\alpha R}[\psi'''-(\alpha^2+\beta^2)\psi']$$

$$+\frac{i}{\alpha R}[\psi^{iv}-2(\alpha^2+\beta^2)\psi''+\alpha^4\psi] = \frac{iM^2}{\alpha R}\psi''. \qquad (23)$$

Equations (21) and (23) constitute the basic equation and boundary conditions for the present problem. We note that the equations have been obtained for weakly conducting fluids.

A Theorem

If one substitutes

$$\bar{\alpha}^2 = \alpha^2+\beta^2 \text{ and } \bar{\alpha}\bar{R} = \alpha R \qquad (24)$$

in eqn. (23), then it has the same structure as with $\beta = 0$. But $\beta = 0$ corresponds to a two-dimensional disturbance. Thus each three-dimensional problem is equivalent to a two-dimensional problem. Hence it is sufficient to solve this two-dimensional problem and supplement it with the conversion (24). In fact (24) shows that the equivalent two-dimensional problem is associated with a lower Reynolds number, since $\bar{\alpha} > \alpha$. Thus the minimum critical Reynolds number is given directly by the two-dimensional problem. So we shall take $\beta = 0$ and the basic eqn. (23) reduces to

$$(\omega-c)(\psi''-\alpha^2\psi)-\omega''\psi-\frac{iR_c}{\alpha R}(\psi'''-\alpha^2\psi')$$

$$+\frac{i}{\alpha R}\left[\psi^{iv}-2\alpha^2\psi''+\alpha^4\psi\right] = \frac{iM^2}{\alpha R}\psi''. \qquad (25)$$

Hence we have established that 'for weakly conducting fluids, three-dimensional problem is equivalent to a two-dimensional problem with a lower critical Reynolds number when porosity of plates is taken into consideration'.

Sufficient Conditions for Stability

We multiply eqn. (25) by $\psi^* d\xi$ where ψ^* is complex conjugate of ψ and integrate from -1 to 1, one gets

$$I_2^2+2\alpha^2 I_1^2+\alpha^4 I_0^2 = -i\alpha RQ+i\alpha Rc(I_1^2+\alpha^2 I_0^2)$$

$$+M^2 I_1^2+R_c\int_{-1}^{1}(\psi'''\psi^*-\alpha^2\psi'\psi^*)\,d\xi \qquad (26)$$

where

$$I_0^2 = \int_{-1}^{1} |\psi|^2\, d\xi$$

$$I_1^2 = \int_{-1}^{1} |\psi'|^2\, d\xi$$

$$I_2^2 = \int_{-1}^{1} |\psi''|^2\, d\xi$$

$$Q = \int_{-1}^{1} \{\omega|\psi'|^2 + (\alpha^2\omega + \omega'')|\psi|^2\}\, d\xi + \int_{-1}^{1} \omega'\psi'\psi^*\, d\xi.$$

If we add to (26) its complex conjugate, then

$$2(I_2^2 + 2\alpha^2 I_1^2 + \alpha^4 I_0^2) = -i\alpha R(Q - Q^*) - 2\alpha R c_i(I_1^2 + \alpha^2 I_0^2) + 2M^2 I_1^2, \quad \ldots \quad (27)$$

where c_i is the imaginary part of c. Moreover the contribution from the coefficient of R_c is nil as can easily be seen.

$$\int_{-1}^{1} \psi'''\psi^*\, d\xi = -\int_{-1}^{1} \psi''\psi^{*\prime}\, d\xi = \int_{-1}^{1} \psi'\psi^{*\prime\prime}\, d\xi.$$

Hence

$$\int_{-1}^{1} (\psi''\psi^{*\prime} + \psi^{*\prime\prime}\psi')\, d\xi = 0.$$

Also

$$\int_{-1}^{1} \psi'\psi^*\, d\xi + \int_{-1}^{1} \psi^{*\prime}\psi\, d\xi = \int_{-1}^{1} \frac{d}{d\xi}(\psi\psi^*)\, d\xi = [\psi\psi^*]_{-1}^{1} = 0.$$

Thus there is no contribution from the coefficient of R_c. Here

$$Q - Q^* = \int_{-1}^{1} \omega'(\psi'\psi^* - \psi^{*\prime}\psi)\, d\xi.$$

Let the maximum of $|\omega'|$ in $(-1, 1)$ be q then Schwarz's inequality gives

$$|Q - Q^*| \leqslant 2\int_{-1}^{1} |\omega'||\psi'||\psi|\, d\xi < 2qI_0I_1.$$

Hence eqn. (27) reduces to

$$\alpha R c_i(I_1^2 + \alpha^2 I_0^2) \leqslant \alpha R q I_0 I_1 - (I_2^2 + 2\alpha^2 I_1^2 + \alpha^4 I_0^2) + M^2 I_1^2.$$

Flow will be stable provided $c_i < 0$ which in turn is satisfied if

$$R < \frac{1}{\alpha q I_0 I_1}\left[(I_2^2 + 2\alpha^2 I_1^2 + \alpha^4 I_0^2) - M^2 I_1^2\right]$$

and

$$M^2 < \frac{1}{I_1^2}\left(I_2^2+2\alpha^2 I_1^2+\alpha^4 I_0^2\right).$$

It is to be noted here that in the absence of magnetic field ($M = 0$), the form of sufficient condition is not at all affected by wall porosity except that q will be numerically different in this situation.

References

Cowling, T. G. (1957). Magnetohydrodynamics. Interscience Publishers, New York.

Lock, R. C. (1955). The stability of the flow of an electrically conducting fluid between parallel planes under a transverse magnetic field. *Proc. R. Soc.*, A **233**, 105.

Mehta, K. N., and Jain, R. K. (1962). Hydromagnetic flow in rectangular channels with porous walls. *Proc. natn. Inst. Sci. India*, A **28**, 846.

Squire, H. B. (1933). On the stability of the three-dimensional disturbances of viscous flow between parallel walls. *Proc. R. Soc.*, A **142**, 621.

Synge, J. L. (1938). Hydrodynamic Stability. Semi-centennial Pub. *Am. math. Soc.*, **2**, 227.