CERTAIN INTEGRALS INVOLVING MEIJER'S G-FUNCTION OF TWO VARIABLES

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In the course of an attempt to give extension of certain results of MacRobert (1958) and Sharma (1964), the infinite integral

$$\int_0^\infty x^{\beta-1}(x+y)^{-\alpha-\beta} G^{o,\,q,\,s,\,r,\,t}_{A,\,[C,\,E],\,B,\,[D,\,F]}\left[\begin{array}{l|c} u(x+y)^n x^{m-n} & (a) \\ & (c)\,;\ (e) \\ v(x+y)^n x^{m-n} & (b) \\ & (d)\,;\ (f) \end{array}\right] dx$$

is evaluated, in terms of Agarwal's $G\begin{bmatrix} X \\ Y \end{bmatrix}$, for positive integral values of m and n, whether $m > n$ or $m < n$, and its numerous interesting special cases are discussed.

1. INTRODUCTION

Recently, Agarwal (1965) gave a generalization of Meijer's G-function (Meijer 1946) to two variables by means of a double Mellin-Barnes contour integral in the form†

$$G^{p,\,q,\,s,\,r,\,t}_{A,\,[C,\,E],\,B,\,[D,\,F]}\left[\begin{array}{c|c} x & (a) \\ & (c)\,;\ (e) \\ y & (b) \\ & (d)\,;\ (f) \end{array}\right] = \frac{1}{(2\pi i)^2}\int_{-i\infty}^{i\infty}\int_{-i\infty}^{i\infty} \Phi(\xi+\eta)\Psi(\xi,\,\eta)x^{\xi}y^{\eta}\,d\xi\,d\eta, \tag{1.1}$$

$$\Phi(\xi+\eta) = \frac{\prod_{j=1}^{p} \Gamma[1-a_j+\xi+\eta]}{\prod_{j=p+1}^{A} \Gamma[a_j-\xi-\eta] \prod_{j=1}^{B} \Gamma[b_j+\xi+\eta]},$$

where

$$\Psi(\xi,\,\eta) = \frac{\prod_{j=1}^{q} \Gamma[c_j+\xi] \prod_{j=1}^{r} \Gamma[d_j-\xi] \prod_{j=1}^{s} \Gamma[e_j+\eta] \prod_{j=1} \Gamma[f_j-\eta]}{\prod_{j=q+1}^{C} \Gamma[1-c_j-\xi] \prod_{j=r+1}^{D} \Gamma[1-d_j+\xi] \prod_{j=s+1}^{E} \Gamma[1-e_j-\eta] \prod_{j=t+1}^{F} \Gamma[1-f_j+\eta]},$$

and

$$0 \le p \le A,\ 0 \le q \le C,\ 0 \le s \le E,\ 0 \le r \le D,\ 0 \le t \le F.$$

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† The G-function in two arguments that we discuss here is a slight variant of the one given by Professor Agarwal, though in essence the function is the same.

Here as well as in what follows it is supposed that there are A of the a parameters, B of the b parameters, and so on. Thus (a) denotes the sequence of A parameters

$$a_1, a_2, \ldots, a_A$$

with similar interpretations for (b), etc.

The sequences of parameters $a_1, a_2, \ldots, a_p$; $c_1, c_2, \ldots, c_q$; $d_1, d_2, \ldots, d_r$; $e_1, e_2, \ldots, e_s$ and $f_1, f_2, \ldots, f_t$ are such that none of the poles of the integrand coincide. The paths of the integration are indented, if necessary, in such a manner that all the poles of $\Gamma[d_j-\xi]$, $j = 1, 2, \ldots, r$ and $\Gamma[f_k-\eta]$, $k = 1, 2, \ldots, t$ lie to the right, and those of $\Gamma[c_j+\xi]$, $j = 1, 2, \ldots, q$, $\Gamma[e_k+\eta]$, $k = 1, 2, \ldots, s$ and $\Gamma[1-a_j+\xi+\eta]$, $j = 1, 2, \ldots, p$ lie to the left of the imaginary axis.

The integral (1.1) converges if

$$\left.\begin{aligned} &2(p+q+r) > A+B+C+D \\ &2(p+s+t) > A+B+E+F \\ &|\arg x| < [p+q+r-\tfrac{1}{2}A-\tfrac{1}{2}B-\tfrac{1}{2}C-\tfrac{1}{2}D]\pi \\ &|\arg y| < [p+s+t-\tfrac{1}{2}A-\tfrac{1}{2}B-\tfrac{1}{2}E-\tfrac{1}{2}F]\pi. \end{aligned}\right\} \quad \ldots \quad \ldots \quad (1.2)$$

The behaviour of $G\begin{bmatrix} X \\ Y \end{bmatrix}$ for small values of X and Y is given by

$$G\begin{bmatrix} X \\ Y \end{bmatrix} = O(|X|^{d_j}|Y|^{k}) \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (1.3)$$

where

$$j = 1, 2, \ldots, r \text{ and } k = 1, 2, \ldots, t.$$

Similarly, when X and $Y \to \infty$, the associated function $G^{(1)}\begin{bmatrix} X \\ Y \end{bmatrix}$ which corresponds to the case $p = 0$ of $G\begin{bmatrix} X \\ Y \end{bmatrix}$ has the behaviour

$$G^{(1)}\begin{bmatrix} X \\ Y \end{bmatrix} = O(|X|^{-c_j}|Y|^{-e_k}) \quad \ldots \quad \ldots \quad \ldots \quad (1.4)$$

where

$$j = 1, 2, \ldots, q \text{ and } k = 1, 2, \ldots, s.$$

In the present paper we evaluate certain infinite integrals, involving the generalized function, which include the results of MacRobert (1958) and Sharma (1964) as special cases.

2. The General Integrals

Consider the integral

$$\int_0^\infty x^{\beta-1}(x+y)^{-\alpha-\beta} G^{o,\,q,\,s,\,r,}_{A,\,[C,\,E],\,B,\,[D,\,F]}\left[\begin{matrix} u(x+y)^n x^{m-n} \\ v(x+y)^n x^{m-n} \end{matrix}\,\middle|\, \begin{matrix} (a) \\ (c);\ (e) \\ (b) \\ (d);\ (f) \end{matrix}\right] dx$$

where m and n are positive integers.

On substituting the double contour integral for $G\begin{bmatrix} X \\ Y \end{bmatrix}$, if we change the order of integration, which is permissible under the conditions stated with (2.1) and (2.2), and then interpret the inner integral with the aid of (Erdélyi *et al.* 1954),

$$\int_0^\infty x^{\nu-1}(x+y)^{-\sigma}\,dx = \frac{\Gamma(\nu)\Gamma(\sigma-\nu)}{\Gamma(\sigma)}\,y^{\nu-\sigma}$$

where

$$\mathrm{Re}\,(\nu) > 0,\ \mathrm{Re}\,(\nu-\sigma) < 0,$$

we shall get:

(i) if m, n are positive integers, and $m > n$,

$$2(q+r) > A+B+C+D$$
$$2(s+t) > A+B+E+F$$
$$|\arg\,(uy^m)| < [q+r-\tfrac{1}{2}A-\tfrac{1}{2}B-\tfrac{1}{2}C-\tfrac{1}{2}D]\pi$$
$$|\arg\,(vy^m)| < [s+t-\tfrac{1}{2}A-\tfrac{1}{2}B-\tfrac{1}{2}E-\tfrac{1}{2}F]\pi$$
$$\mathrm{Re}\,[\beta+(m-n)(d_j+f_k)] > 0,\ j = 1, 2, \ldots, r;\ k = 1, 2, \ldots, t,$$

and

$$\mathrm{Re}\,[\alpha+m(c_\lambda+e_\mu)] > 0,\ \lambda = 1, 2, \ldots, q;\ \mu = 1, 2, \ldots, s,$$

then

$$\int_0^\infty x^{\beta-1}(x+y)^{-\alpha-\beta} G^{o,\,q,\,s,\,r,\,t}_{A,\,[C,\,E],\,B,\,[D,\,F]}\left[\begin{matrix} u(x+y)^n x^{m-n} \\ v(x+y)^n x^{m-n} \end{matrix}\left|\begin{matrix} (a) \\ (c);\ (e) \\ (b) \\ (d);\ (f) \end{matrix}\right.\right] dx$$

$$= \frac{(2\pi)^{\frac{1}{2}}\Gamma(\alpha)\Gamma(1-\alpha)(m-n)^{\beta-\frac{1}{2}}y^{-\alpha}}{\Gamma(\alpha+\beta)\Gamma(1-\alpha-\beta)m^{\frac{1}{2}-\alpha}n^{-\frac{1}{2}+\alpha+\beta}}$$

$$\times G^{m,\,q,\,s,\,r,\,t}_{A+m,\,[C,\,E],\,B+m,\,[D,\,F]}\left[\begin{matrix} \delta u \\ \delta v \end{matrix}\left|\begin{matrix} \nabla(m-n,\beta),\ \nabla(n,1-\alpha-\beta),\ (a) \\ (c);\ (e) \\ \Delta(m,1-\alpha),\ (b) \\ (d);\ (f) \end{matrix}\right.\right] \quad \ldots \quad (2.1)$$

where

$$\delta = \frac{(-1)^{m-n}(m-n)^{m-n}n^n y^m}{m^m},$$

$\nabla(k, \alpha)$ denotes the set of k parameters

$$1-\frac{\alpha}{k}, \ldots, 1-\frac{\alpha+k-1}{k},$$

and as usual, $\Delta(k, \alpha)$ stands for the set of k parameters

$$\frac{\alpha}{k}, \ldots, \frac{\alpha+k-1}{k};$$

and

(ii) if m, n are positive integers, but $m < n$,

$$2(q+r) > A+B+C+D$$
$$2(s+t) > A+B+E+F$$
$$|\arg (uy^m)| < [q+r-\tfrac{1}{2}A-\tfrac{1}{2}B-\tfrac{1}{2}C-\tfrac{1}{2}D]\pi$$
$$|\arg (vy^m)| < [s+t-\tfrac{1}{2}A-\tfrac{1}{2}B-\tfrac{1}{2}E-\tfrac{1}{2}F]\pi$$
$$\text{Re}\,[\beta+(n-m)(d_j+f_k)] > 0, j = 1, 2, \ldots, r;\; k = 1, 2, \ldots, t,$$
$$\text{Re}\,[\alpha+m(c_\lambda+e_\mu)] > 0, \lambda = 1, 2, \ldots, q;\; \mu = 1, 2, \ldots, s,$$

then

$$\int_0^\infty x^{\beta-1}(x+y)^{-\alpha-\beta} G^{o,\, q,\, s,\, r,\, t}_{A,\, [C,\, E],\, B,\, [D,\, F]}\left[\begin{matrix} u(x+y)^n x^{m-n} \\ v(x+y)^n x^{m-n} \end{matrix}\middle| \begin{matrix} (a) \\ (c);\ (e) \\ (b) \\ (d);\ (f) \end{matrix}\right] dx$$

$$= \frac{(2\pi)^{-\frac{1}{2}}\Gamma(\alpha)\Gamma(1-\alpha)\Gamma(\beta)\Gamma(1-\beta)n^{\frac{1}{2}-\alpha-\beta}y^{-\alpha}}{\Gamma(\alpha+\beta)\Gamma(1-\alpha-\beta)m^{\frac{1}{2}-\alpha}(n-m)^{\frac{1}{2}-\beta}}$$

$$\times G^{n,\, q,\, s,\, r,\, t}_{A+n,\, [C,\, E],\, B+n,\, [D,\, F]}\left[\begin{matrix} \delta' u \\ \delta' v \end{matrix}\middle| \begin{matrix} \nabla(n,\, 1-\alpha-\beta),\, (a) \\ (c);\ (e) \\ \Delta(m,\, 1-\alpha),\ \Delta(n-m,\, 1-\beta),\, (b) \\ (d);\ (f) \end{matrix}\right], \quad \ldots \quad (2.2)$$

where

$$\delta' = \frac{n^n y^m}{m^m (n-m)^{n-m}}.$$

3. Particular Cases

The function $G\begin{bmatrix} X \\ Y \end{bmatrix}$ not only includes Meijer's G-function (Meijer 1946) or the product of two G-functions as particular cases but also most of the known functions in two arguments, e.g. the Appell functions F_1, F_2, F_3 and F_4, the Whittaker function of two variables, etc. The formulae of reduction stated by Agarwal (1965) give us the following interesting special cases of (2.1) and (2.2).

(i) In (2.1) if we set $A = 0 = B$, we get

$$\int_0^\infty x^{\beta-1}(x+y)^{-\alpha-\beta} G^{r,\, q}_{C,\, D}\left(u(x+y)^n x^{m-n}\middle| \begin{matrix} (c) \\ (d) \end{matrix}\right) G^{t,\, s}_{E,\, F}\left(v(x+y)^n x^{m-n}\middle| \begin{matrix} (e) \\ (f) \end{matrix}\right) dx$$

$$= \frac{(2\pi)^{\frac{1}{2}}\Gamma(\alpha)\Gamma(1-\alpha)(m-n)^{\beta-\frac{1}{2}}y^{-\alpha}}{\Gamma(\alpha+\beta)\Gamma(1-\alpha-\beta)m^{\frac{1}{2}-\alpha}n^{-\frac{1}{2}+\alpha+\beta}}$$

$$\times G^{m,\, q,\, s,\, r,\, t}_{m,\, [C,\, E],\, m,\, [D,\, F]}\left[\begin{matrix} \delta u \\ \delta v \end{matrix}\middle| \begin{matrix} \nabla(m-n,\, \beta),\ \nabla(n,\, 1-\alpha-\beta) \\ \{1-(c)\};\ \{1-(e)\} \\ \Delta(m,\, 1-\alpha) \\ (d):\ (f) \end{matrix}\right] \quad \ldots \quad \ldots \quad \ldots \quad (3.1)$$

valid under the same conditions as for (2.1) with $A = 0 = B$.

Similarly (2.2) yields

$$\int_0^\infty x^{\beta-1}(x+y)^{-\alpha-\beta}G^{r,q}_{C,D}\left(u(x+y)^n x^{m-n}\,\middle|\,\begin{matrix}(c)\\(d)\end{matrix}\right)G^{t,s}_{E,F}\left(v(x+y)^n x^{m-n}\,\middle|\,\begin{matrix}(e)\\(f)\end{matrix}\right)dx$$

$$=\frac{(2\pi)^{-\frac{1}{2}}\Gamma(\alpha)\Gamma(1-\alpha)\Gamma(\beta)\Gamma(1-\beta)n^{\frac{1}{2}-\alpha-\beta}y^{-\alpha}}{\Gamma(\alpha+\beta)\Gamma(1-\alpha-\beta)m^{\frac{1}{2}-\alpha}(n-m)^{\frac{1}{2}-\beta}}$$

$$\times G^{n,q,s,r,}_{n,[C,E],n,[D,F]}\left[\begin{matrix}\delta' u\\ \delta' v\end{matrix}\,\middle|\,\begin{matrix}\nabla(n,1-\alpha-\beta)\\ \{1-(c)\};\{1-(e)\}\\ \Delta(m,1-\alpha),\ \Delta(n-m,1-\beta)\\ (d);(f)\end{matrix}\right] \quad \ldots \quad (3.2)$$

valid under the conditions stated earlier.

(ii) Next we put $A = 0$, $E = s$, $t = 1$, $f_1 = 0$ and make use of the formula (Agarwal 1965, p. 539 (3.2)). Since

$$\Gamma(z)\Gamma(1-z)=\frac{\pi}{\sin \pi z},$$

for non-integral values of z, a slight variation in our notations will yield

$$\int_0^\infty x^{\beta-1}(x+y)^{-\alpha-\beta}G^{\lambda,\mu}_{r,s}\left(zx^{l-m}(x+y)^m\,\middle|\,\begin{matrix}a_1,\ldots,a_r\\b_1,\ldots,b_s\end{matrix}\right)dx$$

$$=(2\pi)^{\frac{1}{2}+m-l}(l-m)^{\beta-\frac{1}{2}}l^{\alpha-\frac{1}{2}}m^{\frac{1}{2}-\alpha-\beta}y^{-\alpha}$$

$$\times G^{\lambda+l,\mu+l-m}_{r+l,s+l}\left[\frac{m^m(l-m)^{l-m}}{l^l}zy^l\,\middle|\,\begin{matrix}\frac{1-\beta}{l-m},\ldots,\frac{l-m-\beta}{l-m},a_1,\ldots,a_r,\frac{\alpha+\beta}{m},\ldots,\frac{\alpha+\beta+m-1}{m}\\ \frac{\alpha}{l},\ldots,\frac{\alpha+l-1}{l},b_1,\ldots b_s\end{matrix}\right] \quad \ldots \quad (3.3)$$

which holds when l, m are positive integers, $l > m$, and

$2(\lambda+\mu) > r+s$, $|\arg(zy^l)| < (\lambda+\mu-\frac{1}{2}r-\frac{1}{2}s)\pi$,

$\operatorname{Re}\left(b_h+\frac{\beta}{l-m}\right) > 0$, and $\operatorname{Re}\left(1-a_j+\frac{\alpha}{l}\right) > 0$, ($h = 1, 2, \ldots, \lambda$;

$j = 1, 2, \ldots, \mu$),

and

$$\int_0^\infty x^{\beta-1}(x+y)^{-\alpha-\beta}G^{\lambda,\mu}_{r,s}\left(zx^{l-m}(x+y)^m\,\middle|\,\begin{matrix}a_1,\ldots,a_r\\b_1,\ldots,b_s\end{matrix}\right)dx$$

$$=(2\pi)^{\frac{1}{2}}l^{\alpha-\frac{1}{2}}m^{\frac{1}{2}-\alpha-\beta}(m-l)^{\beta-\frac{1}{2}}y^{-\alpha}$$

$$\times G^{\lambda+m,\mu}_{r+m,s+m}\left[\frac{m^m(m-l)^{l-m}}{l^l}zy^l\,\middle|\,\begin{matrix}a_1,\ldots,a_r,\frac{\alpha+\beta}{m},\ldots,\frac{\alpha+\beta+m-1}{m}\\ \frac{\beta}{m-l},\ldots,\frac{\beta+m-l-1}{m-l},\frac{\alpha}{l},\ldots,\frac{\alpha+l-1}{l},b_1,\ldots,b_s\end{matrix}\right] \quad \ldots \quad (3.4)$$

valid if l, m are positive integers, $l < m$, and $2(\lambda+\mu) > r+s$,

$$|\arg (zy^l)| < (\lambda+\mu-\tfrac{1}{2}r-\tfrac{1}{2}s)\pi, \text{ Re}\left(1-a_j+\frac{\alpha}{l}\right) > 0,$$

$$\text{Re}\left(1-a_j+\frac{\beta}{m-l}\right) > 0, \ (j = 1, 2, \ldots, \mu).$$

The integrals in (3.3) and (3.4), given earlier by Sharma (1964), lead us to the evaluation of MacRobert's integral (1958)

$$\int_0^1 \lambda^{\alpha-1}(1-\lambda)^{\beta-1}E[p\,;\ \alpha_r:\ q\,;\ \beta_s:\ z\lambda^m(1-\lambda)^n]\,d\lambda$$

for every permutation of integral values of m and n, whether positive or negative, since (Erdélyi *et al.* 1953)

$$\begin{aligned} E[p\,;\alpha_r:q\,;\beta_s:z] &= G^{p,1}_{q+1,p}\left(z\,\middle|\,\begin{matrix}1, \beta_1, \ldots, \beta_q\\ \alpha_1, \ldots, \alpha_p\end{matrix}\right)\\ &= G^{1,p}_{p,q+1}\left(z^{-1}\,\middle|\,\begin{matrix}1-\alpha_1, \ldots, 1-\alpha_p\\ 0, 1-\beta_1, \ldots, 1-\beta_q\end{matrix}\right).\end{aligned}$$

We conclude with the remark that some of the recent formulae of the first author (Srivastava 1966; *see also* Srivastava and Singhal 1968) admit themselves of rather heuristic generalizations in terms of Agarwal's $G\begin{bmatrix}X\\Y\end{bmatrix}$. This as well as numerous other aspects of the analysis will, therefore, form the subject-matter of a subsequent communication.

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