A GENERAL EXPANSION THEOREM FOR PRODUCTS OF GENERALIZED HYPERGEOMETRIC SERIES

by R. Y. DENIS, *Department of Mathematics, University of Gorakhpur, Gorakhpur*

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A general expansion theorem containing the product of three generalized hypergeometric functions has been established. Verma (1964) established an expansion theorem having the product of two generalized hypergeometric series, generalizing the previous results given by Chaundy and Ragab. The paper gives an extension of all these results. A number of series containing the products of three hypergeometric functions under summation sign has been evaluated as a single hypergeometric function as its particular cases. A further extension of this theorem can also be given in the same way under appropriate conditions of validity.

§ 1. INTRODUCTION

Chaundy (1942) and Ragab (1961) established certain relations expressing the summation of product of two ordinary hypergeometric series as a single hypergeometric series. Recently, Verma (1964) established a general theorem for basic hypergeometric series involving the product of two generalized basic hypergeometric series under summation sign. He, thus, generalized many of the results due to Chaundy and Ragab. In the present paper an attempt has been made to extend the general theorem given by Verma to one containing the product of three generalized basic hypergeometric series. This method, as is obvious from its method of proof, can be easily extended to the products of any finite number of basic hypergeometric series. A number of interesting particular cases of the theorem have been deduced.

The following notations will be followed throughout this paper:

$$[a]_n = a[a+1][a+2]\ldots[a+n-1]\,;\ [a]_0 = 1,$$

$$[a\,;n] = [1-q^a][1-q^{a+1}]\ldots[1-q^{a+n-1}],\ [a\,;0] = 1,\ |q| < 1.$$

We further denote by $(a_{M,N})$ the $N-M+1$ parameters a_M, a_{M+1}, $\ldots$, a_N and when $M = 1$ we write simply (a_N) instead of $(a_{1,N})$. Further, when there are A parameters $a_1, a_2, \ldots, a_A$, we simply write (a), the corresponding roman capital denoting the number of parameters.

Also $\sum_{t}$ shall mean $\sum_{t=0}^{\infty}$;

$G_s\begin{bmatrix}(a)\\(b)\end{bmatrix}$ stands for $\frac{[(a)]_s}{[(b)]_s}$;

$G_s^{(q)}\begin{bmatrix}(a);\\(b)\end{bmatrix}$ stands for $\frac{[(a);s]}{[(b);s]}$

and finally $\Gamma\begin{bmatrix}(a);\\(b)\end{bmatrix}$ denotes $\frac{\Gamma[(a)]}{\Gamma[(b)]}$

and $\Pi\begin{bmatrix}(a);\\(b)\end{bmatrix}$ denotes $\prod_{r=0}^{\infty}\frac{[1-(a)q^r]}{[1-(b)q^r]}$.

§ 2. In this section we give our main theorem.

Theorem: Whenever the series involved are absolutely convergent, we have

$$\sum_r \frac{(-)^{\alpha r}[\sigma;2r][(a);r][(c);2r][(h);2r][(u);2r]x^r y^r z^r q^{u(r)}}{[1;r][\sigma+r-1;r][(b);r][(d);2r][(m);2r][(v);2r]}$$

$$\times\Phi\begin{bmatrix}(e)+r,\ (c)+2r,\ (i);\ xq^{\gamma'}\\(f)+r,\ (d)+2r,\ (j)\end{bmatrix}\Phi\begin{bmatrix}(g)+r,\ (h)+2r,\ (k);\ yq^{\gamma'}\\(l)+r,\ (m)+2r,\ (n)\end{bmatrix}$$

$$\times\Phi\begin{bmatrix}(p)+r,\ (u)+2r,\ (k');\ zq^{\gamma''}\\(w)+r,\ (v)+2r,\ (n')\end{bmatrix}$$

$$=\sum_s\sum_t\sum_R G_s^{(q)}\begin{bmatrix}(e),\ (c),\ (i);\\(f),\ (d),\ (j),\ 1\end{bmatrix}G_t^{(q)}\begin{bmatrix}(g),\ (h),\ (k);\\(l),\ (m),\ (n),\ 1\end{bmatrix}G_R^{(q)}\begin{bmatrix}(p),\ (u),\ (k');\\(w),\ (v),\ (n'),\ 1\end{bmatrix}$$

$$\times x^s y^t z^R q^{\gamma s+\gamma' t+\gamma'' R}$$

$$\times\Phi\left[\begin{matrix}\sigma-1,\ \frac{\sigma+1}{2},\ -\left[\frac{\sigma+1}{2}\right]^*,\ (a),\ (f),\ (l),\ (w),\ (c)+s,\ (h)+t,\ (u)+R,\ -s,\\ -t,\ -R,\ 1-(j)-s,\ 1-(n)-t,\ 1-(n')-R;\ Q\\ \frac{\sigma-1}{2},\ -\left[\frac{\sigma-1}{2}\right]^*,\ (b),\ (e),\ (g),\ (p),\ (d)+s,\ (m)+t,\ (v)+R,\ 1-(i)-s,\\ 1-(k)-t,\ 1-(k')-R\end{matrix}\right] \quad \text{.. (2.1)}$$

where

$$u(r)=\lambda r^2+\mu r>0$$

and

$$Q=(-)^{\alpha+3+I+J+N+N'+K+K'}$$

$$\times q\left\{\begin{matrix}\frac{(r+1)}{2}(I+K+K'-J-N-N'-3+2\lambda)+\mu-\lambda-\gamma-\gamma'-\gamma''+3+s+t+R+s\\ \times(J-I)+R(N'-K')+t(N-K)+\Sigma j+\Sigma n+\Sigma n'-\Sigma i-\Sigma k-\Sigma k'\end{matrix}\right\}$$

and the asterisk (*) in the Φ-series on the right denotes the presence of a term $\left[-q^{\frac{\sigma+1}{2}};r\right]$.

The proof of this theorem is easy and follows by simple rearrangements of the series (cf. Verma 1964).

As $q \to 1$, in (2.1), we get the following general theorem for the ordinary hypergeometric series:

$$\sum_r \frac{(-)^{\alpha r}[\sigma]_{2r}[(a)]_r[(c)]_{2r}[(h)]_{2r}[(u)]_{2r}x^r y^r z^r}{[1]_r[\sigma+r-1]_r[(b)]_r[(d)]_{2r}[(m)]_{2r}[(v)]_{2r}}$$

$$\times F\begin{bmatrix}(e)+r,\ (c)+2r,\ (i);\ x \\ (f)+r,\ (d)+2r,\ (j)\end{bmatrix} F\begin{bmatrix}(g)+r,\ (h)+2r,\ (k);\ y \\ (l)+r,\ (m)+2r,\ (n)\end{bmatrix} F\begin{bmatrix}(p)+r,\ (u)+2r,\ (k');\ z \\ (w)+r,\ (v)+2r,\ (n')\end{bmatrix}$$

$$= \sum_s \sum_t \sum_R G_s\begin{bmatrix}(e),\ (c),\ (i); \\ (f),\ (d),\ (j),\ 1\end{bmatrix} G_t\begin{bmatrix}(g),\ (h),\ (k); \\ (l),\ (m),\ (n),\ 1\end{bmatrix} G_R\begin{bmatrix}(p),\ (u),\ (k'); \\ (w),\ (v),\ (n'),\ 1\end{bmatrix} x^s y^t z^R$$

$$\times F\begin{bmatrix}\sigma-1,\ \frac{\sigma+1}{2},\ (a),\ (f),\ (l),\ (w),\ (c)+s,\ (h)+t,\ (u)+R,\ -s,\ -t,\ -R,\ 1-(j)-s, \\ 1-(n)-t,\ 1-(n')-R;\ Q \\ \frac{\sigma-1}{2},\ (b),\ (e),\ (g),\ (p),\ (d)+s,\ (m)+t,\ (v)+R,\ 1-(i)-s,\ 1-(k)-t,\ 1-(k')-R\end{bmatrix} \quad \ldots \quad (2.2)$$

where

$$Q = (-)^{\alpha+3+I+J+N+N'+K+K'}.$$

§ 3. As illustrations we shall now derive certain special cases of (2.1) and (2.2).

(i) Let us take
$C = H = U = V = K = N = N' = F = L = 0$; $\alpha = J = I = K' = D = M = E = G = 1$; $B = W$, $A = P+2$, $y = x$, $\lambda = 3/2$ $(b_B) = (w_B)$, $(a_P) = (p)$, $a_{P+1} = e$, $a_{P+2} = g$, $e_1 = e$, $g_1 = g$, $\gamma - \gamma' = g$, $\mu = \sigma + 2\gamma' + \gamma'' + g - 3/2$, $j_1 = 2 - \sigma$, $d_1 = 1$, $m_1 = \sigma$, $i_1 = 1$, $k_1' = 1$; and summing the inner well-poised ${}_6\Phi_4$ on the right-hand side of (2.1) by a known result due to Sears (1951, 9.2), the right-hand side of (2.1) reduces to

$$\sum_s \sum_t \sum_R \frac{[e;\, s][g;\, t][(p);\, R]x^{s+t}z^R q^{\gamma s+\gamma' t+\gamma'' R}}{[2-\sigma;\, s][1;\, t][(w);\, R][1;\, s+t]}.$$

Next, setting $s+t = r$, changing the order of summation and summing the inner ${}_2\Phi_1$ by basic analogue of Gauss's theorem (Bailey 1935, 8.4 (3)), we get the following transformation:

$$\sum_r \frac{(-)^r[(p);\, r][e;\, r][g;\, r]x^{2r}z^r q^{r(\sigma+g+2\gamma'+\gamma'')+\frac{3}{2}r^2}}{[1;\, r][\sigma+r-1;\, r][(w);\, r][1;\, 2r]}$$

$$\times {}_2\Phi_2\begin{bmatrix}e+r,\ 1;\ xq^{\gamma'+g} \\ 1+2r,\ 2-\sigma\end{bmatrix} {}_1\Phi_1\begin{bmatrix}g+r;\ xq^{\gamma'} \\ \sigma+2r\end{bmatrix} {}_{P+1}\Phi_W\begin{bmatrix}(p)+r,\ 1;\ zq^{\gamma''} \\ (w)+r\end{bmatrix}$$

$$= \Pi\begin{bmatrix}1-e-g,\ 2-e-\sigma; \\ 1-e,\ 2-e-g-\sigma\end{bmatrix} {}_1\Phi_1\begin{bmatrix}e+g;\ xq^{\gamma'} \\ 2-\sigma\end{bmatrix} {}_{P+1}\Phi_W\begin{bmatrix}(p),\ 1;\ zq^{\gamma''} \\ (w)\end{bmatrix} \quad \ldots \quad \ldots \quad (3.1)$$

(ii) Next, in (2.1), let us put

$y = x$, $C = H = U = J = N = N' = I = K = V = F = 0$, $K' = D = M = L = 1$, $G = 3$, $E = 2 = \alpha$, $A = P+6$, $B = W+2$, $\lambda = 1$, $1+a+\mu = \sigma+\gamma+\gamma'+\gamma''$, $\gamma'-\gamma = e$, $k_1' = 1$, $d_1 = m_1 = \sigma$, $(a_P) = (p)$, $a_{P+1} = 1$, $a_{P+2} = a_{P+3} = \sigma-a$, $a_{P+4} = g$, $a_{P+5} = e$, $a_{P+6} = a$, $(b_W) = (w_W)$, $b_{W+1} = l$, $b_{W+2} = \sigma-a$, $l_1 = l$, $g_1 = 1$, $g_2 = \sigma-a$, $g_3 = g$, $e_1 = \sigma-a$, $e_2 = e$; and summing the inner well-poised ${}_6\Phi_5$, on the right-hand side of (2.1), by a known result due to Bailey (1947, 8.5 (3)), we get

$$\sum_s \sum_t \sum_R \frac{[\sigma-a\,;\, s+t][e\,;\, s][g\,;\, t][(p)\,;\, R]x^{s+t}z^R q^{\gamma s+\gamma' t+\gamma'' R}}{[1\,;\, s][l\,;\, t][(w)\,;\, R][\sigma\,;\, s+t]}.$$

Now, setting $s+t = r$, changing the order of summation and summing the inner ${}_2\Phi_1$ by Bailey (1935, 8.4 (3)), we get

$$\sum_r \frac{[(p)\,;\, r][\sigma-a\,;\, r][g\,;\, r][e\,;\, r][a\,;\, r]x^{2r}z^R q^{r^2+\mu r}}{[\sigma+r-1 : r][(w)\,;\, r][l\,;\, r][\sigma\,;\, 2r]}\, {}_2\Phi_1\left[\begin{matrix}\sigma-a+r,\ e+r\,;\ xq^{\gamma}\\ \sigma+2r\end{matrix}\right]$$

$$\times\, {}_3\Phi_2\left[\begin{matrix}1+r,\ \sigma-a+r,\ g+r\,;\ xq^{\gamma'}\\ l+r,\ \sigma+2r\end{matrix}\right] {}_{P+1}\Phi_W\left[\begin{matrix}(p)+r,\ 1;\ zq^{\gamma''}\\ (w)+r\end{matrix}\right]$$

$$= \Pi\left[\begin{matrix}1-g-e,\ l-g\,;\\ 1-g,\ l-g-e\end{matrix}\right] {}_3\Phi_2\left[\begin{matrix}\sigma-a,\ g+e,\ 1\,;\ xq^{\gamma'-e}\\ \sigma,\ l\end{matrix}\right] {}_{P+1}\Phi_W\left[\begin{matrix}(p),\ 1\,;\ zq^{\gamma''}\\ (w)\end{matrix}\right]$$

with $\gamma'-\gamma = e$ and $1+a+\mu = \sigma+\gamma+\gamma'+\gamma''$. .. (3.2)

(iii) Now, in (2.2), taking

$L = G = N = K = C = H = U = 0$, $\alpha = K' = I = J = N' = M = V = D = P = W = E = F = 1$, $A = B = 2$, $x = y = z$, $k_1' = i_1 = f_1 = b_1 = 1$, $d_1 = d$, $m_1 = \sigma$, $v_1 = v$, $n_1' = p_1 = a_2 = 1+v-\sigma$, $w_1 = b_2 = \mu$, $j_1 = a_1 = e_1 = 1+d-\sigma$ and summing the inner well-poised ${}_5F_4$ by Bailey (1935, 4·3 (3)), we get

$$\sum_s \sum_t \sum_R \frac{[d+v-\sigma]_{s+t+R}x^{s+t+R}}{[1]_s[1]_t[\mu]_R[d]_{s+t}[v]_{t+R}[d+v-\sigma]_{s+R}}.$$

Now, setting $t+R = r$, changing the order of summation and summing the inner ${}_2F_1$ by Gauss's theorem (Bailey 1935, 1.3 (1)), we get

$$\left[\begin{matrix}d,\ 2d-2+\mu-\sigma+v\,;\\ d+\mu-1,\ 2d+v-\sigma-1\end{matrix}\right] \sum_s \sum_r \frac{[2d+\mu-2-\sigma+v]_{2s+2r}x^{s+r}}{[1]_s[\mu]_r[v]_r[d+\mu-1]_{s+r}[2d+v-\sigma-1]_{2s+r}},$$

provided $Rl(2d+\mu+v-2-\sigma) > 0$.

Again, setting $r+s = n$, changing the order of summation and summing the inner ${}_2F_1$ by Gauss's theorem (Bailey 1935, 1.3 (1)), for $Rl(2d+2v-\sigma+\mu-3) > 0$, we get

$$\sum_r \frac{(-)^r[1+d-\sigma]_r[1+v-\sigma]_r x^{3r}}{\{[1]_r\}^2[\sigma+r-1]_r[\mu]_r[d]_{2r}[v]_{2r}}\, {}_2F_3\left[\begin{matrix}1+d-\sigma+r,\ 1;\ x\\ 1+r,\ d+2r,\ 1+d-\sigma\end{matrix}\right]$$

$$\times {}_0F_1\left[\begin{matrix};\ x\\ \sigma+2r\end{matrix}\right] {}_2F_3\left[\begin{matrix}1+v-\sigma+r,\ 1;\ x\\ \mu+r,\ v+2r,\ 1+v-\sigma\end{matrix}\right]$$

$$= \Gamma\left[\begin{matrix}d,\ 2d+2v-\sigma+\mu-3;\\ d+\mu-1,\ 2d+2v-\sigma-2\end{matrix}\right]$$

$$\times {}_4F_5\left[\begin{matrix}1,\ \frac{2d+2v-\sigma+\mu-3}{3},\ \frac{2d+2v-\sigma+\mu-2}{3},\ \frac{2d+2v-\sigma+\mu-1}{3};\ 27x/4\\ \mu,\ v,\ d+\mu-1,\ \frac{2d+2v-\sigma-2}{2},\ \frac{2d+2v-\sigma-1}{2}\end{matrix}\right], \quad \text{.. (3.3)}$$

provided $Rl[2d+\mu+v-\sigma-2] > 0$ and $Rl[2d+2v-\sigma+\mu-3] > 0$.

(iv) Further, if in (2.2) we take $A = B = K' = I = \alpha = J = N' = D = M = V = P = W = 1$, $E = F = N = K = C = H = U = L = G = 0$, $k_1' = i_1 = d_1 = w_1 = b_1 = 1$, $m_1 = \sigma$, $v_1 = v$, $j_1 = 2-\sigma$, $n_1' = p_1 = a_1 = 1+v-\sigma$, $z = y = x$ and sum the inner well-poised ${}_5F_4$ by Bailey (1935, 4.3 (3)), we get, after some simplification,

$$\sum_s \sum_t \sum_R \frac{[1+v-\sigma]_{s+t+R}\, x^{s+t+R}}{[2-\sigma]_s[1]_t[1]_R[v]_{R+t}[1]_{s+t}[1+v-\sigma]_{s+R}}.$$

Now, setting $t+R = r$, changing the order of summation and summing the inner ${}_2F_1$ by Gauss's theorem, we get, on some simplification, for $Rl[1+v-\sigma] > 0$,

$$\sum_s \sum_r \frac{[1+v-\sigma]_{2s+2r}\, x^{s+r}}{[2-\sigma]_s[1]_r[v]_r[1]_{s+r}[1+v-\sigma]_{2s+r}}.$$

Again, putting $s+r = n$, changing the order of summation and summing the inner ${}_2F_1$ by Gauss's theorem, we get, for $Rl[2v-2\sigma+1] > 0$,

$$\sum_r \frac{(-)^r[1+v-\sigma]_r x^{3r}}{\{[1]_r\}^2[\sigma+r-1]_r[1]_{2r}[v]_{2r}}\, {}_1F_2\left[\begin{matrix}1;\ x\\ 1+2r,\ 2-\sigma\end{matrix}\right] {}_0F_1\left[\begin{matrix};\ x\\ \sigma+2r\end{matrix}\right]$$

$$\times {}_2F_3\left[\begin{matrix}1+v-\sigma+r,\ 1;\ x\\ 1+r,\ v+2r,\ 1+v-\sigma\end{matrix}\right]$$

$$= \Gamma\left[\begin{matrix}v,\ 2v-2\sigma+1;\\ v-\sigma+1,\ 2v-\sigma\end{matrix}\right] {}_3F_4\left[\begin{matrix}\frac{2v-2\sigma+1}{3},\ \frac{2v-2\sigma+2}{3},\ \frac{2v-2\sigma+3}{3};\ 27x/4\\ 2-\sigma,\ v-\sigma+1,\ \frac{2v-\sigma}{2},\ \frac{2v-\sigma+1}{2}\end{matrix}\right], \quad (3.4)$$

provided $Rl[1+v-\sigma] > 0$ and $Rl[2v-2\sigma+1] > 0$.

(v) Lastly, if we make the following substitution in (2.2): $C = H = U = J = N = N' = V = I = K = P = 0$, $K' = D = M = \alpha = L = W = F = 1$, $G = E = B = 2$, $A = 3$; $b_2 = l_1 = \beta$, $w_1 = g_1 = k_1' = 1$, $m_1 = d_1 = \sigma$, $g_2 =$

$a_3 = \alpha,\ e_1 = a_1 = \sigma-\alpha,\ e_2 = a_2 = \delta,\ f_1 = b_1 = \mu,\ y = -z = x$ and sum the inner well-poised ${}_4F_3\,(-1)$ by a known result (Bailey 1935, 4.4 (3)), we get, after some simplification,

$$\sum_s \sum_t \sum_R \frac{[\delta]_s[\sigma-\alpha]_s[\alpha]_t(-)^R x^{s+t+R}}{[1]_s[\mu]_s[\beta]_t[1]_R[\sigma]_{s+t}}.$$

Further, setting $t+R = r$, changing the order of summation and summing the inner ${}_2F_1$ by Gauss's theorem, we get, after some simplification,

$$\begin{bmatrix} 1-\alpha,\ \sigma+\beta-\alpha-1; \\ \beta-\alpha,\ \sigma-\alpha \end{bmatrix} \sum_s \sum_r \frac{[\delta]_s[\sigma+\beta-\alpha-1]_{s+r}(-)^r x^{s+r}}{[1]_s[\mu]_s[\beta]_r[\sigma]_{r+s}},$$

for $Rl[\beta+\sigma-\alpha-1] > 0$.

Now, putting $s+r = n$, changing the order of summation and summing the inner ${}_2F_1$ by Gauss's theorem (provided $Rl[\mu+\beta-\delta-1] > 0$), we get, after some simplification, the following transformation:

$$\sum_r \frac{[\sigma-\alpha]_r[\delta]_r[\alpha]_r x^{3r}}{[\sigma]_{2r}[1]_r[\sigma+r-1]_r[\mu]_r[\beta]_r}\, {}_2F_2\begin{bmatrix} \sigma-\alpha+r,\ \delta+r;\ x \\ \mu+r,\ \sigma+2r \end{bmatrix} {}_2F_2\begin{bmatrix} 1+r,\ \alpha+r;\ x \\ \beta+r,\ \sigma+2r \end{bmatrix}$$

$$\times\, {}_1F_1\begin{bmatrix} 1;\ -x \\ 1+r \end{bmatrix}$$

$$= \Gamma\begin{bmatrix} 1-\alpha,\ \sigma+\beta-\alpha-1,\ \mu,\ \mu+\beta-\delta-1; \\ \beta-\alpha,\ \sigma-\alpha,\ \mu-\delta,\ \mu+\beta-1 \end{bmatrix} {}_3F_3\begin{bmatrix} 1,\ \sigma+\beta-\alpha-1,\ \mu+\beta-\delta-1;\ -x \\ \sigma,\ \beta,\ \mu+\beta-1 \end{bmatrix}, \quad \text{.. (3.5)}$$

provided $Rl[\beta+\sigma-\alpha-1] > 0$ and $Rl[\beta+\mu-\delta-1] > 0$.

A number of other interesting particular cases of (2.1) and (2.2) can also be cited in a similar manner.

§ 4. Finally, we briefly discuss the convergence conditions of our main theorem (2.1). For the sake of convenience we take all the parameters to be real, positive and the base q to be real and $0 < q < 1$. The result for complex values follows by analytic extension.

LEMMA:

$$\Phi\begin{bmatrix} (e)+r,\ (c)+2r,\ (i);\ x \\ (f)+r,\ (d)+2r,\ (j) \end{bmatrix} \leqslant \frac{1}{[(e);\ r][(c);\ 2r]}\,\Phi\begin{bmatrix} (e),\ (c),\ (i);\ x \\ (f),\ (d),\ (j) \end{bmatrix}. \quad \text{.. (4.1)}$$

PROOF: We first prove that

$$\Phi\begin{bmatrix} (e)+1,\ (c)+2,\ (i);\ x \\ (f)+1,\ (d)+2,\ (j) \end{bmatrix} \leqslant A\Phi\begin{bmatrix} (e),\ (c),\ (i);\ x \\ (f),\ (d),\ (j) \end{bmatrix}, \quad \text{.. .. (4.2)}$$

where

$$1/A = [1-q^{(e)}][1-q^{(c)}][1-q^{(c)+1}].$$

If U_n and V_n are the coefficients of x^n on the left-hand side and right-hand side of (4.1) respectively, we get

$$\frac{U_n}{V_n} = \frac{1}{A} \times \frac{[(e)+1;\ n][(c)+2;\ n][(f);\ n][(d);\ n]}{[(f)+1;\ n][(d)+2;\ n][(e);\ n][(c);\ n]}$$

$$= \frac{[1-q^{(f)}][1-q^{(d)}][1-q^{(d)+1}][1-q^{(c)+n}][1-q^{(c)+n+1}]}{[1-q^{(f)+n}][1-q^{(d)+n+1}]}$$

$\leqslant 1$, for all values of n, which proves (4.2). A repeated application of (4.2) gives the inequality (4.1).

Applying lemma (4.1) to (2.1) we get the following conditions for its validity:

$|xq^{\gamma}| < 1$, $|yp^{\gamma'}| < 1$, $|zq^{\gamma''}| < 1$ and with an additional condition $|xyzq^{\mu}| < 1$, if $\lambda = 0$.

The corresponding convergence conditions for (2.2) are as follows:

$E+2C < F+2D$,

$G+2H < L+2M$,

$P+2U < W+2V$,

$E+C+I < F+D+J$, $G+H+K < L+M+N$, $P+U+K' < W+V+N'$, and $A+L+F+W < B+G+E+P$.

In case $E+C+I = F+D+J+1$, $P+U+K' = W+V+N'+1$ and $G+H+K = L+M+N+1$ we have additional conditions $|x| < 1$, $|y| < 1$, $|z| < 1$.

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