LAMINAR FLOW IN AN ANNULUS WITH POROUS WALLS OF DIFFERENT PERMEABILITY

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The solution of Navier-Stokes equations in cylindrical coordinates is obtained for the flow of a viscous incompressible fluid in a porous annulus with different radial velocities v_a and v_b at the walls. The results of Berman (1958) and Bansal (1966) are obtained as particular cases of the investigation. In general, v_a and v_b are different and thus eight types of flow, in a porous annulus, arise on permuting the negative and positive signs of v_a and v_b, and their magnitudes. The case of $|v_b| \geqslant |v_a|$ is discussed fully in this paper. When $|v_a| \geqslant |v_b|$, the problem can be reduced by a simple transformation to the case of $|v_b| \geqslant |v_a|$ and is discussed in detail.

INTRODUCTION

In the last two decades a number of papers have been published on the flow problems of incompressible fluid through porous channels and pipes. They have given a deeper understanding of the problems of cooling rocket and jet motors by the diffusion of fluids through porous-metal combustion chamber liners, etc. Berman (1953) investigated the effect of wall porosity on the two-dimensional, steady state, incompressible, laminar flow of a fluid in a channel of rectangular cross-section. With the assumption of uniform suction at the walls, he arrived at a third order non-linear differential equation which has been solved by a perturbation method for small flows through the porous walls. Sellar (1955), Yuan (1956) and Berman (1956) have investigated by various approximate methods the laminar flow of a fluid in rectangular channels with porous walls for large suction and injection Reynolds numbers. Verma and Bansal (1966) have obtained the exact solution of Navier-Stokes equations for the flow of a viscous incompressible fluid between two parallel plates, one in uniform motion and the other at rest with uniform small suction at the stationary plate. Yuan and Finkelstein (1956) and Bansal (1966) have studied the laminar flow in a pipe with porous wall. Applying the method of averages, Morduchow (1957) discussed the laminar flow through a channel and a circular tube with injection. Berman (1958) gave the solution of Navier-Stokes equations for the case of steady state, laminar flow in an annulus with porous walls assuming the constant influx through one wall equal to the efflux through the other wall. Recently Terrill and Shrestha (1965) have studied the flow through parallel and uniform porous walls of different permeabilities for small Reynolds numbers of the cross-flow.

The present investigation represents part of a continuing effort to understand the effects on the flow field of fluid removal or injection through channel walls and ducts. The solution of Navier-Stokes equations in cylindrical coordinates is obtained for the flow of a viscous incompressible fluid in a porous annulus with different radial velocities v_a and v_b at the walls. The results of Berman (1958) and Bansal (1966) are obtained as particular cases of the investigation.

In general, v_a and v_b are different and thus eight types of flow in a porous annulus arise on permuting the negative and positive signs of v_a and v_b, and their magnitudes. When $|v_a| \geqslant |v_b|$, the suction parameter is defined by $\lambda_1 = \frac{av_a}{\nu}$, while for $|v_b| \geqslant |v_a|$, the suction parameter is $\lambda_2 = \frac{bv_b}{\nu}$, where a and b are the inner and outer radii respectively and ν is the coefficient of kinematic viscosity. The case of $|v_b| \geqslant |v_a|$ is discussed fully under the assumption that the flow is due to the pressure gradient of the Hagen-Poiseuille flow at the mouth and the suction at the wall is uniform and small. It is found that the axial pressure gradient and the axial velocity decrease along the axis of the annulus, and vanish simultaneously at a finite distance from the mouth of the annulus. In the present case it is interesting to note that an adverse pressure gradient is developed beyond the point where both axial velocity and axial pressure gradient vanish, which causes a back flow from infinity to this point. The axial velocity profiles, stream line patterns and the pressure-drop along the axis are shown graphically.

When $|v_a| \geqslant |v_b|$, the problem can be reduced by a simple transformation to the case of $|v_b| \geqslant |v_a|$ and is discussed in detail in Appendix I.

Equations of Motion

Let us consider the steady laminar flow of an incompressible fluid in the region bounded by the porous walls of two coaxial cylinders of radii a and b. It is assumed that the fluid is injected at the inner cylinder with a velocity v_a and is sucked at the outer cylinder with a velocity v_b and the rate of suction is not equal to the rate of injection. The Navier-Stokes equations in cylindrical polar coordinates (x, r, θ) for axially symmetrical flow are

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial r} = -\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\left(\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{\partial^2 u}{\partial x^2}\right), \quad \ldots \quad \ldots \quad (1)$$

$$u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial r} = -\frac{1}{\rho}\frac{\partial p}{\partial r}+\nu\left(\frac{\partial^2 v}{\partial r^2}+\frac{1}{r}\frac{\partial v}{\partial r}-\frac{v}{r^2}+\frac{\partial^2 v}{\partial x^2}\right), \quad \ldots \quad \ldots \quad (2)$$

and the equation of continuity is

$$\frac{\partial}{\partial x}(ru)+\frac{\partial}{\partial r}(rv) = 0, \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (3)$$

where u and v represent the axial and the radial components of the velocity in the directions of x- and r- respectively.

The boundary conditions are

$$\left.\begin{aligned} r = a,\ v = v_a,\ u = 0, \\ r = b,\ v = v_b,\ u = 0. \end{aligned}\right\} \qquad (4)$$

Since the velocities of suction and injection are constant along the axis of the annulus $\frac{\partial v}{\partial x} = 0$, therefore, v is a function of r only. Now, it is evident from eqn. (3) that $\frac{\partial^2 u}{\partial x^2} = 0$.

Thus equations (1) to (3) reduce to

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial r} = -\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\left(\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}\right), \qquad (5)$$

$$v\frac{\partial v}{\partial r} = -\frac{1}{\rho}\frac{\partial p}{\partial r}+\nu\left(\frac{\partial^2 v}{\partial r^2}+\frac{1}{r}\frac{\partial v}{\partial r}-\frac{v}{r^2}\right), \qquad (6)$$

and

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial r}+\frac{v}{r} = 0. \qquad (7)$$

It is convenient to introduce the following dimensionless variables:

$$\bar{u} = \frac{ub}{\nu},\ \bar{v} = \frac{vb}{\nu},\ \eta = \frac{r}{b},\ \bar{x} = \frac{x}{b},\ \bar{p} = \frac{pb^2}{\nu^2\rho},\ \lambda_1 = \frac{av_a}{\nu},$$

$$\lambda_2 = \frac{bv_b}{\nu},\ \alpha_1 = \frac{\lambda_2}{\lambda_1}-1,\ \alpha_2 = 1-\frac{\lambda_1}{\lambda_2} \text{ and } \sigma = \frac{a}{b},$$

where

$$\sigma \leqslant \eta \leqslant 1. \qquad (8)$$

Hence equations (5) to (7) in non-dimensional form are

$$\bar{u}\frac{\partial \bar{u}}{\partial \bar{x}}+\bar{v}\frac{\partial \bar{u}}{\partial \eta} = -\frac{\partial \bar{p}}{\partial \bar{x}}+\left(\frac{\partial^2 \bar{u}}{\partial \eta^2}+\frac{1}{\eta}\frac{\partial \bar{u}}{\partial \eta}\right), \qquad (9)$$

$$\bar{v}\frac{\partial \bar{v}}{\partial \eta} = -\frac{\partial \bar{p}}{\partial \eta}+\left(\frac{\partial^2 \bar{v}}{\partial \eta^2}+\frac{1}{\eta}\frac{\partial \bar{v}}{\partial \eta}-\frac{\bar{v}}{\eta^2}\right), \qquad (10)$$

and

$$\frac{\partial \bar{u}}{\partial \bar{x}}+\frac{\partial \bar{v}}{\partial \eta}+\frac{\bar{v}}{\eta} = 0, \qquad (11)$$

with the boundary conditions

$$\left.\begin{aligned} \eta = \sigma;\ \bar{v} = \frac{\lambda_1}{\sigma},\ \bar{u} = 0, \\ \eta = 1;\ \bar{v} = \lambda_2,\ \bar{u} = 0. \end{aligned}\right\} \qquad (12)$$

Let

$$\bar{p}(\bar{x}, \eta) = p_0+p'(\bar{x}, \eta),$$

$$\bar{u}(\bar{x}, \eta) = u_0+u'(\bar{x}, \eta),$$

$$\bar{v}(\eta) = v'(\eta), \qquad (13)$$

where the primed quantities are the perturbations caused due to porosity and p_0, u_0, the known quantities for flow with solid walls, are given by

$$\frac{\partial u_0}{\partial \bar{x}} = 0, \; \frac{\partial p_0}{\partial \eta} = 0, \qquad \ldots \quad (14)$$

and

$$\frac{\partial p_0}{\partial \bar{x}} = \frac{\partial^2 u_0}{\partial \eta^2} + \frac{1}{\eta}\frac{\partial u_0}{\partial \eta}, \qquad \ldots \quad (15)$$

with the boundary conditions

$$u_0 = 0 \text{ at } \eta = \sigma \text{ and } \eta = 1. \qquad \ldots \quad (16)$$

From (14) p_0 is independent of η and u_0 is independent of $\bar{x}$; thus from (15) we have

$$\frac{\partial p_0}{\partial \bar{x}} = \text{constant} = -a_0 \text{ (say)}$$

or

$$p_0 = -a_0 \bar{x} + \text{constant}, \qquad \ldots \quad (17)$$

and

$$u_0 = \frac{a_0}{4}\left[(1-\eta^2) - \frac{(1-\sigma^2)}{\log \sigma} \log \eta\right]. \qquad \ldots \quad (18)$$

The equations (9) to (11) using (13) to (15) reduce to

$$(u_0+u')\frac{\partial u'}{\partial \bar{x}} + v'\left(\frac{\partial u_0}{\partial \eta} + \frac{\partial u'}{\partial \eta}\right) = -\frac{\partial p'}{\partial \bar{x}} + \left(\frac{\partial^2 u'}{\partial \eta^2} + \frac{1}{\eta}\frac{\partial u'}{\partial \eta}\right), \qquad \ldots \quad (19)$$

$$v'\frac{\partial v'}{\partial \eta} = -\frac{\partial p'}{\partial \eta} + \left(\frac{\partial^2 v'}{\partial \eta^2} + \frac{1}{\eta}\frac{\partial v'}{\partial \eta} - \frac{v'}{\eta^2}\right), \qquad \ldots \quad (20)$$

and

$$\frac{\partial u'}{\partial \bar{x}} + \frac{\partial v'}{\partial \eta} + \frac{v'}{\eta} = 0, \qquad \ldots \quad (21)$$

with the boundary conditions

$$\left.\begin{aligned} \eta = \sigma\,;\; v' &= \frac{\lambda_1}{\sigma},\; u' = 0, \\ \eta = 1\,;\; v' &= \lambda_2,\; u' = 0. \end{aligned}\right\} \qquad \ldots \quad (22)$$

Method of Solution

Assuming $|v_b| \gg |v_a|$, let

$$v' = \frac{\lambda_2}{\eta} f(\eta). \qquad \ldots \quad (23)$$

From eqn. (21), we have

$$u' = -\frac{\lambda_2 \bar{x}}{\eta} f'(\eta) + F(\eta), \qquad \ldots \quad (24)$$

where $f(\eta)$ and $F(\eta)$ are unknown functions of η to be determined.

Substituting (23), (24) in (19) and (20), we get

$$\frac{\partial p'}{\partial \bar{x}} = F''+\frac{1}{\eta}F'-\frac{\lambda_2}{\eta}\left(f\frac{\partial u_0}{\partial \eta}-u_0 f'-Ff'+F'f\right)$$

$$-\frac{\lambda_2 \bar{x}}{\eta^3}\{\eta^2 f'''-\eta f''+f'+\lambda_2(\eta f'^2-\eta ff''+ff')\} \quad .. \quad (25)$$

and

$$\frac{\partial p'}{\partial \eta} = \frac{\lambda_2}{\eta^3}\{\eta^2 f''-\eta f'-\lambda_2(\eta ff'-f^2)\} \quad .. \quad .. \quad .. \quad (26)$$

with the boundary conditions

$$\left.\begin{array}{l}\eta = \sigma\,;\; f(\eta) = 1-\alpha_2, f'(\eta) = 0, F(\eta) = 0,\\ \eta = 1\,;\; f(\eta) = 1,\; f'(\eta) = 0,\; F(\eta) = 0.\end{array}\right\} \quad .. \quad .. \quad (27)$$

At the mouth of the annulus the pressure gradient along hte axis of the annulus is assumed to be the same as the pressure gradient of thə flow without porosity.

Therefore, at $\bar{x} = 0, \frac{\partial p'}{\partial \bar{x}} = 0$.

Hence from (25), we have

$$F''+\frac{1}{\eta}F'-\frac{\lambda_2}{\eta}\left(f\frac{\partial u_0}{\partial \eta}-u_0 f'-Ff'+F'f\right) = 0. \quad .. \quad .. \quad (28)$$

From (26), we have

$$\frac{\partial^2 p'}{\partial \bar{x}\partial \eta} = 0. \quad .. \quad .. \quad .. \quad .. \quad (29)$$

Using (28) and (29) from (25), we obtain

$$\frac{d}{d\eta}\left[\frac{1}{\eta^3}\{\eta^2 f'''-\eta f''+f'+\lambda_2(\eta f'^2-\eta ff''+ff')\}\right] = 0 \quad .. \quad (30)$$

or

$$\eta^3 f'''-\eta^2 f''+\eta f'+\lambda_2(\eta^2 f'^2-\eta^2 ff''+\eta ff') = K_2\eta^4, \quad .. \quad .. \quad (31)$$

where K_2 is the constant of integration to be determined.

If $\lambda_1 = \lambda_2 = R$ (say), i.e. the rate of injection is equal to the rate of suction, then $\alpha_2 = 0$ and $f(\sigma) = 1$, giving the obvious solution $f(\eta) = 1$. Hence from (28), we have

$$\eta F''-(R-1)F' = R\frac{\partial u_0}{\partial \eta}. \quad .. \quad .. \quad .. \quad (32)$$

Using (13) to (15) and (24), eqn. (32) reduces to

$$\bar{u}''-(R-1)\frac{\bar{u}'}{\eta} = -a_0, \quad .. \quad .. \quad .. \quad (33)$$

with the boundary conditions

$$\bar{u}(1) = \bar{u}(\sigma) = 0. \quad .. \quad .. \quad .. \quad .. \quad (34)$$

This agrees with the result of Berman (1958).

Following Bansal (1966) it can be easily seen that if we put

$$F(\eta) = -u_0 - \frac{a_0}{k_2}\frac{f'(\eta)}{\eta}, \quad \ldots \quad \ldots \quad \ldots \quad (35)$$

then eqn. (28) reduces to eqn. (31) satisfying the proper boundary conditions.

Hence we have to determine the solution of $f(\eta)$ from eqn. (31) with the help of the following boundary conditions

$$\begin{aligned} &f(1) = 1\,;\ f'(1) = 0, \\ &f(\sigma) = 1-\alpha_2\,;\ f'(\sigma) = 0. \end{aligned} \quad \ldots \quad \ldots \quad \ldots \quad (36)$$

and then $F(\eta)$ is given by (35).

Perturbation Solution for small λ_2

The solution of eqn. (31) can be expressed for small values of λ_2, by a power series developed near $\lambda_2 = 0$, as follows

$$f(\eta) = \sum f_r(\eta)\lambda_2^r \text{ and } K_2 = \sum C_r\lambda_2^r, \quad \ldots \quad \ldots \quad \ldots \quad (37)$$

where the f_r's and C_r's are independent of λ_2. Substituting (37) into (31) and equating the coefficients of like powers of λ_2 leads to the general equation

$$\eta^3 f_r''' - \eta^2 f_r'' + \eta f_r' = \eta^4 C_r - \eta^2 \sum_{s=0}^{r-1}\left(f_s' f_{r-1-s}' - f_s f_{r-1-s}'' + \frac{1}{\eta} f_s f_{r-1-s}'\right). \quad (38)$$

The boundary conditions to be satisfied by the f_r's are

$$f_0(1) = 1, f_0(\sigma) = 1-\alpha_2,$$

$$f_r(1) = f_r(\sigma) = 0, \text{ for } r \geqslant 1,$$

and

$$f_r'(1) = f_r'(\sigma) = 0, \text{ for } r \geqslant 0. \quad \ldots \quad \ldots \quad \ldots \quad (39)$$

The equation for $f_0(\eta)$ is

$$\eta^3 f_0''' - \eta^2 f_0'' + \eta f_0' = C_0 \eta^4. \quad \ldots \quad \ldots \quad \ldots \quad (40)$$

The solution of eqn. (40) subject to the boundary conditions (39) is

$$f_0(\eta) = L_1 + (L_2 \log \eta + L_3)\eta^2 + \frac{C_0\eta^4}{16}, \quad \ldots \quad \ldots \quad \ldots \quad (41)$$

where

$$L_1 = \frac{(1-\alpha_2)-\sigma^2}{1-\sigma^2} - \frac{C_0\sigma^2}{16}, \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (42)$$

$$L_2 = -\frac{2\alpha_2}{1-\sigma^2} - \frac{C_0}{8}(1+\sigma^2), \quad \ldots \quad \ldots \quad \ldots \quad (43)$$

$$L_3 = \frac{\alpha_2}{1-\sigma^2} - \frac{C_0}{16}(1-\sigma^2), \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (44)$$

and

$$C_0 = -\frac{16\alpha_2 \log \sigma}{(1-\sigma^2)[(1-\sigma^2)+(1+\sigma^2)\log \sigma]}. \quad \ldots \quad \ldots \quad (45)$$

The equation for $f_1(\eta)$ is

$$\eta^3 f_1''' - \eta^2 f_1'' + \eta f_1' = C_1\eta^4 - \eta^2 f_0'^2 + \eta^2 f_0 f_0'' - \eta f_0 f_0'. \quad \ldots \quad (46)$$

Using (41), the solution of (46) is

$$f_1(\eta) = M_1 + (M_2 \log \eta + M_3)\eta^2 + \frac{C_1\eta^4}{16} - \frac{C_0^2}{256\times 36}\eta^8 - \frac{C_0}{96\times 24}(12L_3 + 12L_2 \log \eta - 5L_2)\eta^6 + \tfrac{1}{32}\Big\{C_0L_1 + 16L_2L_3 - 14L_2^2 - 8L_3^2 - 8L_2^2(\log \eta)^2 + 16L_2^2 \log \eta - 16L_2L_3 \log \eta\Big\}\eta^4 + \tfrac{1}{4}L_1L_2\{2(\log \eta)^2 - 2 \log \eta + 1\}\eta^2 \quad \ldots \quad (47)$$

and

$$f_1'(\eta) = (2M_2 \log \eta + 2M_3 + M_2)\eta + \frac{C_1}{4}\eta^3 - \frac{C_0^2}{32\times 36}\eta^7 - \frac{C_0}{128}(4L_3 + 4L_2 \log \eta - L_2)\eta^5 + \tfrac{1}{8}\Big\{C_0L_1 + 12L_2L_3 - 10L_2^2 - 8L_3^2 - 8L_2^2(\log \eta)^2 + 12L_2^2 \log \eta - 16L_2L_3 \log \eta\Big\}\eta^3 + L_1L_2(\log \eta)^2\eta, \quad \ldots \quad (48)$$

where M_1, M_2, M_3 and C_1 are constants of integration to be determined. In order to determine these four constants we use the following boundary conditions

and

$$\left.\begin{aligned} f_1(1) &= f_1'(1) = 0,\\ f_1(\sigma) &= f_1'(\sigma) = 0.\end{aligned}\right\} \quad \ldots \quad (49)$$

The boundary condition $f_1(1) = 0$, gives

$$M_1 + M_3 + \frac{C_1}{16} = \frac{C_0^2}{256\times 36} + \frac{C_0}{96\times 24}(12L_3 - 5L_2) - \tfrac{1}{32}(C_0L_1 + 16L_2L_3 - 14L_2^2 - 8L_3^2) - \frac{L_1L_2}{4} = P \text{ (say)}. \quad \ldots \quad (50)$$

The boundary conditions $f_1'(1) = 0$ gives

$$2M_3 + M_2 + \frac{C_1}{4} = \frac{C_0^2}{32\times 36} + \frac{C_0}{128}(4L_3 - L_2) - \tfrac{1}{8}(C_0L_1 + 12L_2L_3 - 10L_2^2 - 8L_3^2) = Q \text{ (say)}. \quad \ldots \quad (51)$$

The boundary condition $f_1(\sigma) = 0$ gives

$$M_1 + (M_2 \log \sigma + M_3)\sigma^2 + \frac{C_1\sigma^4}{16} = \frac{C_0^2}{256\times 36}\sigma^8 + \frac{C_0}{96\times 24}(12L_3 + 12L_2 \log \sigma - 5L_2)\sigma^6 - \tfrac{1}{32}\Big\{C_0L_1 + 16L_2L_3 - 14L_2^2 - 8L_3^2 - 8L_2^2(\log \sigma)^2 + 16L_2^2 \log \sigma - 16L_2L_3 \log \sigma\Big\}\sigma^4 - \tfrac{1}{4}L_1L_2\{2(\log \sigma)^2 - 2 \log \sigma + 1\}\sigma^2 = R \text{ (say)}, \quad \ldots \quad (52)$$

and the boundary condition $f_1'(\sigma) = 0$ gives

$$(2M_2 \log \sigma + 2M_3 + M_2)\sigma + \frac{C_1\sigma^3}{4} = \frac{C_0^2}{32\times 36}\sigma^7 + \frac{C_0}{128}(4L_3 + 4L_2 \log \sigma - L_2)\sigma^5$$

$$-\tfrac{1}{8}\Big\{C_0L_1 + 12L_2L_3 - 10L_2^2 - 8L_3^2 - 8L_2^2(\log \sigma)^2$$

$$+12L_2^2 \log \sigma - 16L_2L_3 \log \sigma\Big\}\sigma^3 - L_1L_2(\log \sigma)^2\sigma$$

$$= S \text{ (say)}. \qquad \qquad (53)$$

Solving (50) to (53) we find

$$M_1 = \frac{(2R - S\sigma) - (2P - Q)\sigma^2}{2(1-\sigma^2)} - \frac{C_1}{16}\sigma^2, \qquad (54)$$

$$M_2 = \frac{(2R - S\sigma) - (2P - Q)}{(1-\sigma^2)} - \frac{C_1}{8}(1+\sigma^2), \qquad (55)$$

$$M_3 = \frac{(2P - Q\sigma^2) - (2R - S\sigma)}{2(1-\sigma^2)} - \frac{C_1}{16}(1-\sigma^2), \qquad (56)$$

and

$$C_1 = \frac{4(Q\sigma - S)}{\sigma[(1-\sigma^2) + (1+\sigma^2)\log \sigma]} + \frac{8\{(2R - S\sigma) - (2P - Q)\}\log \sigma}{(1-\sigma^2)[(1-\sigma^2) + (1+\sigma^2)\log \sigma]}. \qquad (57)$$

The equation for $f_2(\eta)$ is

$$\eta^3 f_2''' - \eta^2 f_2'' + \eta f_2' = C_2\eta^4 - 2\eta^2 f_0' f_1' + \eta^2 f_1 f_0'' + \eta^2 f_0 f_1'' - \eta f_1 f_0' - \eta f_0 f_1'. \qquad (58)$$

Using (41) and (47), the solution of eqn. (58) is

$$f_2(\eta) = N_1 + (N_2 \log \eta + N_3)\eta^2 + \frac{C_2\eta^4}{16} + \frac{C_0^3}{256\times 86400}\eta^{12} + \frac{C_0^2}{256\times 72\times 12800}[A_1$$

$$+199L_2(20 \log \eta - 7)]\eta^{10} + \frac{C_0}{1152\times 288}[A_2 + A_3(\log \eta - \tfrac{11}{24})$$

$$+423L_2^2\{(\log \eta)^2 - \tfrac{11}{12}\log \eta + \tfrac{9}{32}\}]\eta^8 + \frac{1}{128\times 96}[A_4 + A_5(\log \eta - \tfrac{2}{3})$$

$$+A_6\{(\log \eta)^2 - \tfrac{4}{3}\log \eta + \tfrac{43}{72}\} + 256L_2^3\{(\log \eta)^3 - 2(\log \eta)^2 + \tfrac{43}{24}\log \eta$$

$$-\tfrac{97}{144}\}]\eta^6 + \tfrac{1}{64}[A_7 + A_8(\log \eta - \tfrac{5}{4}) - A_9\{(\log \eta)^2 - \tfrac{5}{2}\log \eta + \tfrac{17}{8}\}$$

$$-16L_1L_2^2\{(\log \eta)^3 - \tfrac{15}{4}(\log \eta)^2 + \tfrac{51}{8}\log \eta - \tfrac{147}{32}\}]\eta^4 + \tfrac{1}{24}[6A_{10}\{2(\log \eta)^2$$

$$-2\log \eta + 1\} + L_1^2L_2\{4(\log \eta)^3 - 6(\log \eta)^2 + 6\log \eta - 3\}]\eta^2, \qquad (59)$$

where

$$A_1 = 20(64L_3 - 24L_2), \qquad (60)$$

$$A_2 = (288L_3^2 - 480L_2L_3 + 383L_2^2 - 22C_0L_1 - 72C_1), \qquad (61)$$

$$A_3 = (711L_2L_3 - 480L_2^2), \qquad (62)$$

$$A_4 = -8(6C_1L_2 + C_0L_1L_2 - 26L_2^3 + 56L_2L_3^2 + 8C_0M_3 + 6C_0L_1L_3$$

$$+8C_1L_3 - 32L_3^3 + 6C_0M_2 - 32L_3L_2^2), \qquad (63)$$

$$A_5 = \left(-64C_1L_3-49C_0L_1L_2-1024L_2^2L_3+256L_2^3+1280L_2L_3^2-64C_0M_2\right), \quad \ldots \quad (64)$$

$$A_6 = 64\left(16L_2^2L_3-7L_2^3-\tfrac{1}{2}C_0L_1L_2\right), \quad \ldots \quad (65)$$

$$A_7 = \left(C_0L_1^2+2C_1L_1+4L_1L_2L_3-2L_1L_2^2-8L_1L_3^2-32L_3M_3-8M_3L_2-8L_2M_2\right), \quad \ldots \quad (66)$$

$$A_8 = -8(L_1L_2L_3+4L_2M_3-2L_2M_2-4M_2L_3), \quad \ldots \quad (67)$$

$$A_9 = 4L_2(L_1L_2+8M_2+4L_1L_3), \quad \ldots \quad (68)$$

and

$$A_{10} = (L_2M_1+L_1M_2). \quad \ldots \quad (69)$$

Differentiating each term of (59) we get

$$f_2'(\eta) = (2N_2 \log \eta+2N_3+N_2)\eta+\frac{C_2}{4}\eta^3+\frac{C_0^3}{256\times 7200}\eta''+\frac{C_0^2}{256\times 72\times 1280}$$
$$\times[A_1+199L_2(20 \log \eta-5)]\eta^9+\frac{C_0}{144\times 288}[A_2+A_3(\log \eta-\tfrac{1}{3})$$
$$+423L_2^2\{(\log \eta)^2-\tfrac{1}{3} \log \eta+\tfrac{1}{6}\}]\eta^7+\frac{1}{128\times 16}[A_4+A_5 (\log \eta-\tfrac{1}{2})$$
$$+A_6\{(\log \eta)^2-\log \eta+\tfrac{3}{8}\}+256L_2^3\{(\log \eta)^3-\tfrac{3}{2}(\log \eta)^2+\tfrac{9}{8} \log \eta$$
$$-\tfrac{3}{8}\}]\eta^5+\tfrac{1}{16}[A_7+A_8(\log \eta-1)-A_9\{(\log \eta)^2-2 \log \eta+\tfrac{3}{2}\}$$
$$-16L_1L_2^2\{(\log \eta)^3-3(\log \eta)^2+\tfrac{9}{2} \log \eta-3\}]\eta^3$$
$$+\left[A_{10}(\log \eta)^2+\frac{L_1^2L_2}{3}(\log \eta)^3\right]\eta, \quad \ldots \quad (70)$$

where N_1, N_2, N_3 and C_2 are constants of integration and A_1 to A_{10} are known constants. In order to determine the constants of integration we use the following boundary conditions

$$f_2(1) = f_2'(1) = 0,$$

and

$$f_2(\sigma) = f_2'(\sigma) = 0. \quad \ldots \quad (71)$$

The boundary condition $f_2(1) = 0$ gives

$$N_1+N_3+\frac{C_2}{16} = -\frac{C_0^3}{256\times 86400}-\frac{C_0^2}{256\times 72\times 12800}(A_1-1393L_2)$$
$$-\frac{C_0}{1152\times 288}\left(A_2-\tfrac{11}{24}A_3+\tfrac{3807}{32}L_2^2\right)-\frac{1}{128\times 96}$$
$$\times\left(A_4-\tfrac{2}{3}A_5+\tfrac{43}{72}A_6-\tfrac{1552}{9}L_2^3\right)-\tfrac{1}{64}\left(A_7-\tfrac{5}{4}A_8-\tfrac{17}{8}A_9\right.$$
$$\left.+\tfrac{147}{2}L_1L_2^2\right)-\tfrac{1}{8}\left(2A_{10}-L_1^2L_2\right)$$
$$= T \text{ (say)}. \quad \ldots \quad (72)$$

The boundary condition $f_2'(1) = 0$ gives

$$
\begin{aligned}
2N_3+N_2+\frac{C_2}{4} = &-\frac{C_0^3}{256\times 7200}-\frac{C_0^2}{256\times 72\times 1280}(A_1-195L_2)-\frac{C_0}{144\times 288}(A_2\\
&-\tfrac{1}{3}A_3+\tfrac{14}{2}L_2^2)-\frac{1}{128\times 16}(A_4-\tfrac{1}{2}A_5+\tfrac{3}{8}A_6-96L_2^3)\\
&-\tfrac{1}{16}(A_7-A_8-\tfrac{3}{2}A_9+48L_1L_2^2)\\
=\ &U \text{ (say).} \qquad (73)
\end{aligned}
$$

The boundary condition $f_2(\sigma) = 0$ gives

$$
\begin{aligned}
N_1+(N_2\log\sigma+N_3)\sigma^2+\frac{C_2\sigma^4}{16} = &-\frac{C_0^3}{256\times 86400}\sigma^{12}-\frac{C_0^2}{256\times 72\times 12800}[A_1\\
&+199L_2(20\log\sigma-7)]\sigma^{10}-\frac{C_0}{1152\times 288}[A_2\\
&+A_3(\log\sigma-\tfrac{11}{24})+423L_2^2\{(\log\sigma)^2\\
&-\tfrac{11}{12}\log\sigma+\tfrac{9}{32}\}]\sigma^8-\frac{1}{128\times 96}[A_4+A_5(\log\sigma\\
&-\tfrac{2}{3})+A_6\{(\log\sigma)^2-\tfrac{4}{3}\log\sigma+\tfrac{43}{72}\}\\
&+256L_2^3\{(\log\sigma)^3-2(\log\sigma)^2+\tfrac{43}{24}\log\sigma-\tfrac{97}{144}\}]\sigma^6\\
&-\tfrac{1}{64}[A_7+A_8(\log\sigma-\tfrac{5}{4})-A_9\{(\log\sigma)^2\\
&-\tfrac{5}{2}\log\sigma+\tfrac{17}{8}\}-16L_1L_2^2\{(\log\sigma)^3-\tfrac{15}{4}(\log\sigma)^2\\
&+\tfrac{51}{8}\log\sigma-\tfrac{147}{32}\}]\sigma^4-\tfrac{1}{24}[6A_{10}\{2(\log\sigma)^2\\
&-2\log\sigma+1\}+L_1^2L_2\{4(\log\sigma)^3-6(\log\sigma)^2\\
&+6\log\sigma-3\}]\sigma^2\\
=\ &V \text{ (say),} \qquad (74)
\end{aligned}
$$

and the boundary condition $f_2'(\sigma) = 0$ gives

$$
\begin{aligned}
(2N_2\log\sigma+2N_3+N_2)\sigma+\frac{C_2\sigma^3}{4} = &-\frac{C_0^3}{256\times 7200}\sigma^{11}-\frac{C_0^2}{256\times 72\times 1280}[A_1\\
&+199L_2(20\log\sigma-5)]\sigma^9-\frac{C_0}{144\times 288}[A_2\\
&+A_3(\log\sigma-\tfrac{1}{3})+423L_2^2\{(\log\sigma)^2\\
&-\tfrac{1}{3}\log\sigma+\tfrac{1}{6}\}]\sigma^7-\frac{1}{128\times 16}[A_4+A_5(\log\sigma\\
&-\tfrac{1}{2})+A_6\{(\log\sigma)^2-\log\sigma+\tfrac{3}{8}\}\\
&+256L_2^3\{(\log\sigma)^3-\tfrac{3}{2}(\log\sigma)^2+\tfrac{9}{8}\log\sigma-\tfrac{3}{8}\}]\sigma^5\\
&-\tfrac{1}{16}[A_7+A_8(\log\sigma-1)-A_9\{(\log\sigma)^2\\
&-2\log\sigma+\tfrac{3}{2}\}-16L_1L_2^2\{(\log\sigma)^3-3(\log\sigma)^2\\
&+\tfrac{9}{2}\log\sigma-3\}]\sigma^3-\Big[A_{10}(\log\sigma)^2\\
&+\frac{L_1^2L_2}{3}(\log\sigma)^3\Big]\sigma\\
=\ &W \text{ (say).} \qquad (75)
\end{aligned}
$$

Solving (72) to (75), we find

$$N_1 = \frac{(2V-W\sigma)-(2T-U)\sigma^2}{2(1-\sigma^2)} - \frac{C_2}{16}\sigma^2, \quad \ldots \quad (76)$$

$$N_2 = \frac{(2V-W\sigma)-(2T-U)}{(1-\sigma^2)} - \frac{C_2}{8}(1+\sigma^2), \quad \ldots \quad (77)$$

$$N_3 = \frac{(2T-U\sigma^2)-(2V-W\sigma)}{2(1-\sigma^2)} - \frac{C_2}{16}(1-\sigma^2), \quad \ldots \quad (78)$$

and

$$C_2 = \frac{4(U\sigma-W)}{\sigma[(1-\sigma^2)+(1+\sigma^2)\log\sigma]} + \frac{8\{(2V-W\sigma)-(2T-U)\}\log\sigma}{(1-\sigma^2)[(1-\sigma^2)+(1+\sigma^2)\log\sigma]}. \quad \ldots \quad (79)$$

Hence the second order perturbation solution of eqn. (31) satisfying the boundary conditions given by eqn. (36) is

$$f(\eta) = f_0(\eta)+\lambda_2 f_1(\eta)+\lambda_2^2 f_2(\eta), \quad \ldots \quad (80)$$

where $f_0(\eta)$, $f_1(\eta)$ and $f_2(\eta)$ are given by eqns. (41), (47) and (59) respectively, and

$$K_2 = C_0+\lambda_2 C_1+\lambda_2^2 C_2, \quad \ldots \quad (81)$$

where C_0, C_1, C_2 are given by eqns. (45), (57) and (79) respectively.

The particular case of laminar flow through a uniform circular pipe with small suction can be obtained by taking $v_a = 0$ and $a \to 0$, this implies $\alpha_2 = 1$ and $\sigma \to 0$.

In this case

$$L_1 = 0, \quad P = \tfrac{31}{36}, \quad M_1 = 0, \quad A_1 = 2560,$$
$$L_2 = 0, \quad Q = \tfrac{29}{9}, \quad M_2 = 0, \quad A_2 = 288,$$
$$L_3 = 2, \quad R = 0, \quad M_3 = \tfrac{1}{9}, \quad A_4 = \frac{64\times 88}{9},$$
$$C_0 = -16, \quad S = 0, \quad C_1 = 12, \quad A_7 = -\tfrac{64}{9},$$
$$A_3 = A_5 = A_6 = A_8 = A_9 = A_{10} = 0,$$
$$T = \tfrac{386}{5400}, \quad N_1 = 0,$$
$$U = \tfrac{202}{900}, \quad N_2 = 0,$$
$$V = 0, \quad N_3 = \tfrac{166}{5400},$$
$$W = 0, \text{ and } C_2 = \tfrac{88}{135}. \quad \ldots \quad (82)$$

Therefore from (41), (47) and (59), we have

$$f_0(\eta) = (2\eta^2-\eta^4),$$
$$f_1(\eta) = \tfrac{1}{36}(4\eta^2-9\eta^4+6\eta^6-\eta^8),$$
$$f_2(\eta) = \tfrac{1}{5400}(166\eta^2-380\eta^4+275\eta^6-75\eta^8+15\eta^{10}-\eta^{12}), \quad \ldots \quad (83)$$

which agrees with the result of Bansal (1966).

Stream Function

From (13), (14), (25), (28) and (31), the axial pressure gradient is

$$-\frac{\partial \bar{p}}{\partial \bar{x}} = a_0 + \lambda_2 K_2 \bar{x}, \qquad (84)$$

where K_2 is given by (81).
Using (13), (18), (23), (24), (35) and (84), the velocity components are

$$\bar{u} = \frac{1}{K_2}\frac{\partial \bar{p}}{\partial \bar{x}}\frac{1}{\eta} f'(\eta) \qquad (85)$$

and

$$\bar{v} = \frac{\lambda_2}{\eta} f(\eta). \qquad (86)$$

If ψ be the stream function, then

$$\bar{u} = -\frac{1}{\eta}\frac{\partial \psi}{\partial \eta} \text{ and } \bar{v} = \frac{1}{\eta}\frac{\partial \psi}{\partial \bar{x}}. \qquad (87)$$

We have

$$d\psi = \frac{\partial \psi}{\partial \bar{x}} d\bar{x} + \frac{\partial \psi}{\partial \eta} d\eta. \qquad (88)$$

From (85) to (88), we obtain

$$\psi = -\frac{1}{K_2}\frac{\partial \bar{p}}{\partial \bar{x}} f(\eta) + \text{constant} \qquad (89)$$

or

$$\frac{\psi}{a_0} = \frac{1}{K_2}\left(1 + \lambda_2 K_2 \frac{\bar{x}}{a_0}\right) f(\eta). \qquad (90)$$

Pressure Distribution

Substituting eqns. (28) and (31) in (25) and (26), we get

$$\frac{\partial p'}{\partial \bar{x}} = -K_2 \lambda_2 \bar{x} \qquad (91)$$

and

$$\frac{\partial p'}{\partial \eta} = \lambda_2 \left[\frac{\eta f'' - f'}{\eta^2} - \lambda_2 \left(\frac{\eta f f' - f^2}{\eta^3}\right)\right]. \qquad (92)$$

From eqns. (14) and (17), we have

$$\frac{\partial p_0}{\partial \eta} = 0 \text{ and } \frac{\partial p_0}{\partial \bar{x}} = -a_0. \qquad (93)$$

Therefore, from eqns. (91), (92) and (93), on integrating, we get

$$\bar{p}(\bar{x}, \eta) = a_0 \bar{x} + \frac{K_2 \lambda_2}{2} \bar{x}^2 - \frac{\lambda_2}{\eta} f' + \frac{\lambda_2^2}{2\eta^2} f^2 + \text{constant}. \qquad (94)$$

From this general expression, the pressure distribution in the $\bar{x}$ and η directions can be immediately deduced. The pressure drop in the $\bar{x}$-direction is given by

$$\bar{p}(0, \eta) - \bar{p}(\bar{x}, \eta) = a_0 \bar{x} + \frac{K_2 \lambda_2}{2} \bar{x}^2, \qquad (95)$$

and that in the η-direction by

$$\bar{p}(\bar{x}, \sigma) - \bar{p}(\bar{x}, \eta) = \frac{\lambda_2^2}{2}\left\{\frac{f^2}{\eta^2} - \frac{(1-\alpha_2)^2}{\sigma^2}\right\} - \frac{\lambda_2 f'}{\eta}. \qquad (96)$$

FRICTION—COEFFICIENT

The shearing stress τ_b at the outer cylinder is

$$\tau_b = -\frac{\mu\nu}{b^2}\left(\frac{\partial \bar{u}}{\partial \eta}\right)_{\eta=1},$$

$$= -\frac{\mu\nu}{b^2}\frac{1}{K_2}\frac{\partial \bar{p}}{\partial \bar{x}}[f''(\eta)]_{\eta=1}, \qquad (97)$$

and the shearing stress τ_a at the inner cylinder is

$$\tau_a = -\frac{\mu\nu}{a^2}\left(\frac{\partial \bar{u}}{\partial \eta}\right)_{\eta=\sigma},$$

$$= -\frac{\mu\nu}{a^2}\frac{1}{\sigma K_2}\frac{\partial \bar{p}}{\partial \bar{x}}[f''(\eta)]_{\eta=\sigma}. \qquad (98)$$

The expressions for coefficient of skin-friction at $\eta = 1$ and $\eta = \sigma$, with the help of (80), (97) and (98), are given by

$$C_f^{(b)} = \frac{\tau_b b^2}{\mu\nu} = -\frac{1}{K_2}\frac{\partial \bar{p}}{\partial \bar{x}}[f_0''(1) + \lambda_2 f_1''(1) + \lambda_2^2 f_2''(1)] \qquad (99)$$

and

$$C_f^{(a)} = \frac{\tau_a a^2}{\mu\nu} = -\frac{1}{\sigma k_2}\frac{\partial \bar{p}}{\partial \bar{x}}[f_0''(\sigma) + \lambda_2 f_1''(\sigma) + \lambda_2^2 f''(\sigma)]. \qquad (100)$$

Hence it can be inferred that the coefficient of skin-friction at the walls changes sign with the axial pressure gradient.

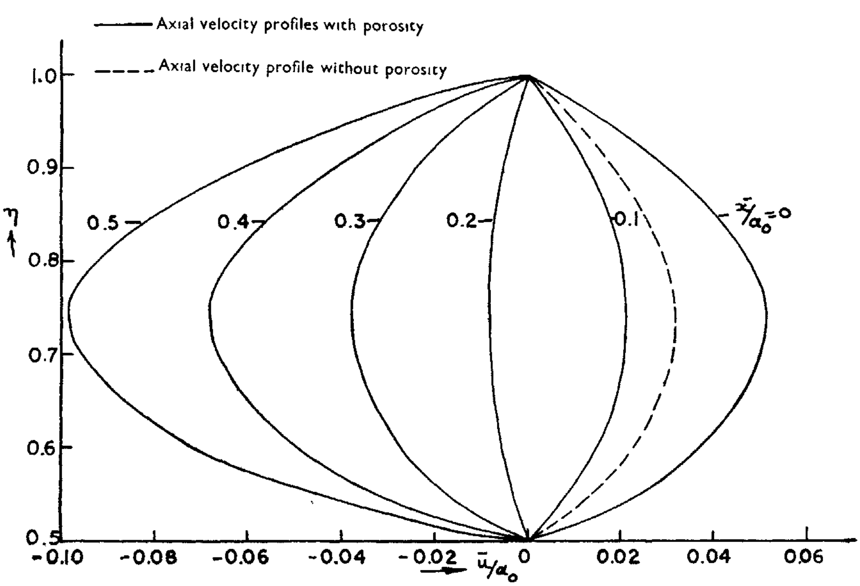

FIG. 1. Axial velocity profiles plotted against η in an annulus with porous walls of different permeability. $[\alpha_2 = 0{\cdot}75,\ \sigma = 0{\cdot}5,\ \lambda_2 = 0{\cdot}1]$

Numerical Discussion

The axial velocity profiles for various values of $\frac{\bar{x}}{a_0}$ along the axis of the annulus are calculated from eqns. (85) and (86) for $\alpha_2 = 0{\cdot}75$, $\sigma = 0{\cdot}5$ and

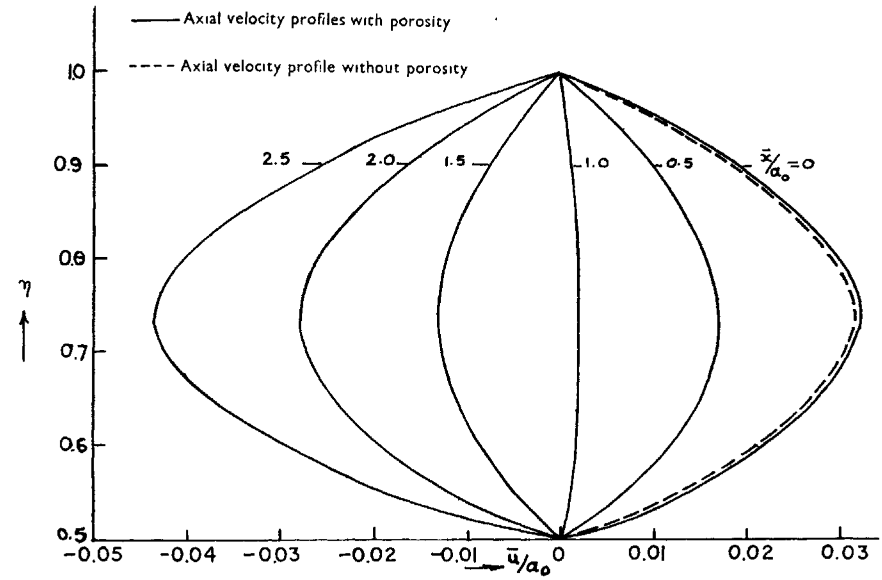

FIG. 2. Axial velocity profiles plotted against η in an annulus with porous walls of different permeability. $[\alpha_2 = 0{\cdot}75,\ \sigma = 0{\cdot}5,\ \lambda_2 = 0{\cdot}01]$

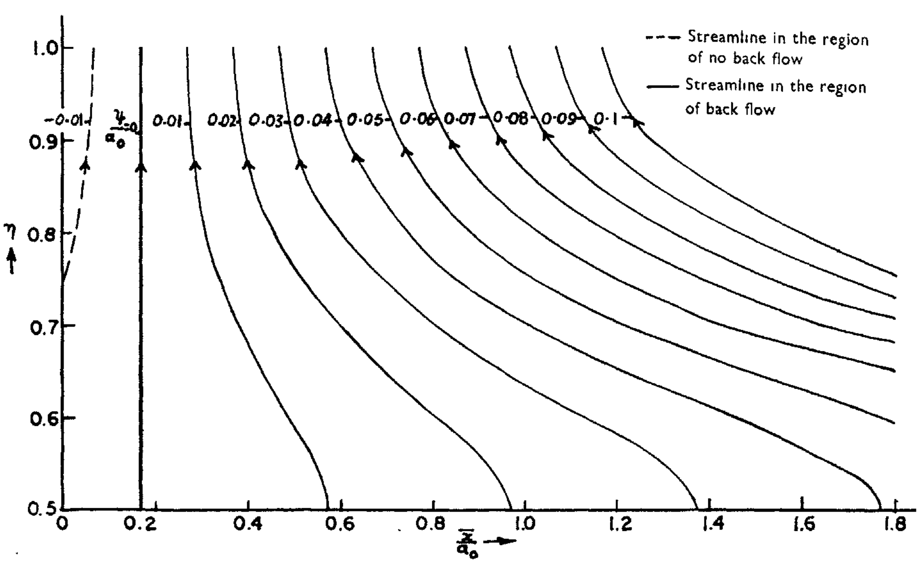

FIG. 3. Streamlines pattern in an annulus with porous walls of different permeability. $[\alpha_2 = 0{\cdot}75,\ \sigma = 0.5,\ \lambda_2 = 0{\cdot}1]$

$\lambda_2 = 0{\cdot}1$, $0{\cdot}01$ and are shown in Figs. 1 and 2. It is noted that the maximum axial velocity exists near the middle of the annulus and that at the mouth it is greater than the velocity of the flow without porosity. The axial velocity distribution is parabolic and it decreases along the axis of the annulus with the axial pressure gradient and vanishes for $\lambda_2 = 0{\cdot}1$ and $0{\cdot}01$, $\alpha_2 = 0{\cdot}75$, $\sigma = 0{\cdot}5$

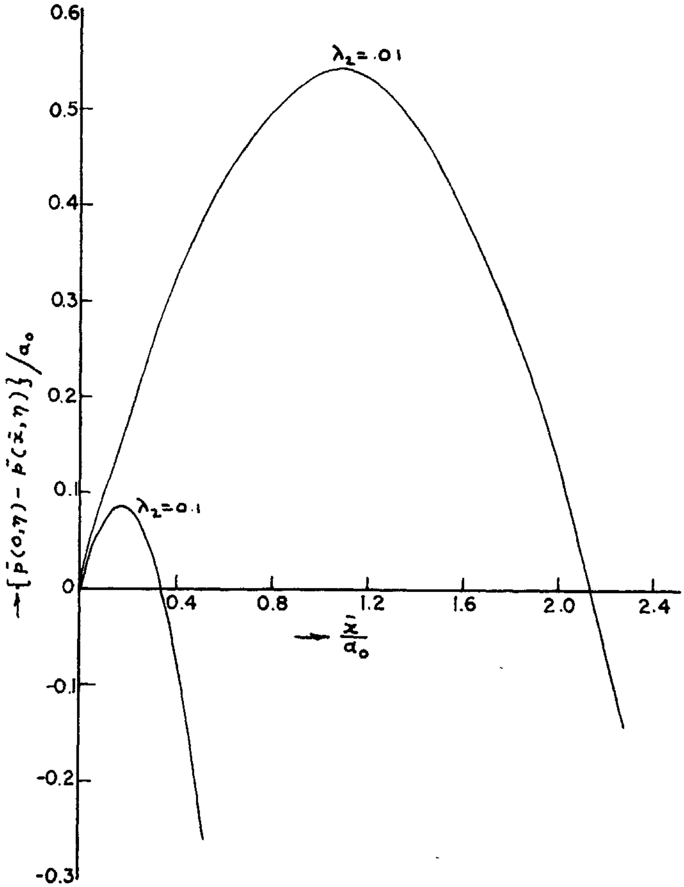

FIG. 4. Axial pressure drop against the axis of the annulus with porous walls of different permeability. [$\alpha_2 = 0{\cdot}75$, $\sigma = 0{\cdot}5$, $\lambda_2 = 0{\cdot}1$, $0{\cdot}01$]

at $\dfrac{\bar{x}}{a_0} = 0{\cdot}17$ and $1{\cdot}07$ respectively. It is interesting to note that due to porosity ($|v_b| \geqslant |v_a|$) an adverse pressure gradient is developed beyond $\dfrac{\bar{x}}{a_0} = 0{\cdot}17$ and $1{\cdot}07$ for $\lambda_2 = 0{\cdot}1$ and $0{\cdot}01$ respectively, which causes a back flow from infinity to these points. This phenomenon follows from the law of conservation of matter. To have a clear insight into the pattern of the flow the streamlines are plotted in Fig. (3) for $\alpha_2 = 0{\cdot}75$, $\sigma = 0{\cdot}5$ and $\lambda_2 = 0{\cdot}1$. Since there is injection at the inner wall and suction at the outer wall, the effect of back flow is nearer the outer wall, as it is suction that causes the back flow. For $\frac{\bar{x}}{a_0} = 0{\cdot}17$, the streamline $\frac{\psi}{a_0} = 0$ emerges out radially and thus separates the two regions. The pressure drop along the axis of the annulus is parabolic and is shown in Fig. 4 for $\alpha_2 = 0{\cdot}75$, $\sigma = 0{\cdot}5$, $\lambda_2 = 0{\cdot}1$ and $0{\cdot}01$.

Appendix I

It can be seen that a certain transformation will reduce the problem for $|v_a| \geqslant |v_b|$ to $|v_b| \geqslant |v_a|$.

For

$$|v_a| \geqslant |v_b|,$$

let

$$v' = \frac{\lambda_1}{\eta}\phi(\eta), \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \ (101)$$

and therefore from (21) we have

$$u' = -\frac{\lambda_1}{\eta}\bar{x}\phi'(\eta)+\xi(\eta). \qquad \ldots \quad \ldots \quad \ldots \ (102)$$

In this case the equation

$$\eta^3\phi'''-\eta^2\phi''+\eta\phi'+\lambda_1(\eta^2\phi'^2-\eta^2\phi\phi''+\eta\phi\phi') = K_1\eta^4, \qquad \ldots \quad \ldots \ (103)$$

where K_1 is an arbitrary constant, has to be solved subject to the boundary conditions

$$\begin{aligned} \eta = 1\,;\ \phi(\eta) &= 1+\alpha_1,\ \phi'(\eta) = 0, \\ \eta = \sigma\,;\ \phi(\eta) &= 1,\ \phi'(\eta) = 0, \end{aligned} \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \ (104)$$

and $\xi(\eta)$ is given by

$$\xi(\eta) = -u_0 - \frac{a_0}{K_1}\frac{\phi'(\eta)}{\eta}. \qquad \ldots \quad \ldots \quad \ldots \ (105)$$

Now, let us apply the transformation $\phi(\eta) = \frac{\lambda_2}{\lambda_1}f(\eta)$ and $K_1 = \frac{\lambda_2}{\lambda_1}K_2$.

Then eqn. (103) reduces to

$$\eta^3f'''-\eta^2f''+\eta f'+\lambda_2(\eta^2f'^2-\eta^2ff''+\eta ff') = K_2\eta^4, \qquad \ldots \quad \ldots \ (106)$$

with the boundary conditions

$$\left.\begin{aligned} \eta = 1,\ f(\eta) &= 1,\ f'(\eta) = 0, \\ \eta = \sigma,\ f(\eta) &= 1-\alpha_2,\ f'(\eta) = 0, \end{aligned}\right\} \qquad \ldots \quad \ldots \quad \ldots \ (107)$$

which is the same equation as (31) with the boundary conditions (36) governing the flow with $|v_b| \geqslant |v_a|$.

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