SCALAR DIFFERENTIAL INVARIANTS IN GENERAL RELATIVITY

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The fourteen scalar differential invariants of order two have been evaluated for axially symmetric cylindrically symmetric and plane symmetric metrics, for the product of two surfaces metric and for the Gödel metric. It is found that a certain tensor vanishes for all the well-known line elements of gravitational significance. But it is non-zero for the most general orthogonal metric. Various scalar invariant relations are satisfied by the metrics of different symmetries. These relations serve as necessary conditions for a general Riemannian fourfold to represent the corresponding symmetry.

1. Introduction

A scalar differential invariant $f(g_{ij[r]})$ of the general n-dimensional Riemannian metric is a function of the metric tensor g_{ij} and its derivatives $g_{ij,\,k_1k_2\ldots k_r} \equiv g_{ij[r]}$. The order of the invariant is the highest differentiation order r occurring in it. The absolute scalar invariants of second order associated with the four-dimensional Riemannian metric of general relativity have a special significance as the field equations are partial differential equations of the second order. It is well known that there are only fourteen independent scalar invariants of order two. The vanishing of the fourteen scalar invariants does not necessarily imply the vanishing of the curvature tensor which has twenty independent components. The fourteen scalar invariants were enumerated for the first time by Narlikar and Karmarkar (1949). It is known that for the general space-time with a perfect fluid distribution or an electromagnetic distribution, there are only nine functionally independent invariants (Ehlers and Kundt 1962). A study of these invariants has been made for the case of general spherically symmetric line-element by Narlikar and Karmarkar and some of the invariant features of spherically symmetric fourfolds have been brought out. In this paper we investigate the scalar invariants in the case of some other Riemannian fourfolds bearing certain symmetries which play an important role in general relativity. The line-elements of axial, cylindrical and plane symmetries, of the product of two surfaces and of the Gödel metric, have been examined and the invariant characteristics studied. It is found that in all these cases, including that of spherical symmetry, a certain fourth rank tensor D_{hijk} always vanishes. According to the scheme of enumeration followed here, one of the fourteen invariants turns out to be indeterminate. In the case of the various symmetries, certain invariant relations have been found out which serve as necessary criteria for the particular type of symmetry. Each of the axially symmetric and the cylindrically symmetric metrics has ten non-vanishing independent invariants. In the case of the plane symmetric metric, there are only nine independent invariants. For the case of the product of two surfaces, out of the ten non-vanishing invariants only two are independent. The Gödel metric admits only six non-zero invariants which are all expressible in terms of only one invariant. In the case of the most general orthogonal metric, it has been found that the tensor D_{hijk} is non-zero and all the fourteen invariants are non-vanishing.

2. Axially Symmetric Metric

The fourteen invariants are given explicitly as follows (Narlikar and Karmarkar 1949):

$$\left.\begin{aligned}
I_1 &= R^i_i, & I_2 &= {}^{(2)}R^i_i,\\
I_3 &= {}^{(3)}R^i_i, & I_4 &= {}^{(4)}R^i_i,\\
J_1 &= A_{hijk}g^{hj}g^{ik}, & J_2 &= B_{hijk}g^{hj}g^{ik},\\
J_3 &= E_{hijk}g^{hj}g^{ik}, & J_4 &= F_{hijk}g^{hj}g^{ik},\\
K_1 &= C_{hijk}R^{hj}R^{ik}, & K_2 &= A_{hijk}R^{hj}R^{ik},\\
K_3 &= \bar{D}_{hijk}R^{hj}R^{ik}, & K_4 &= C_{hijk}Q^{hj}Q^{ik},\\
K_5 &= A_{hijk}Q^{hj}Q^{ik}, & K_6 &= D_{hijk}Q^{hj}Q^{ik},
\end{aligned}\right\} \quad \ldots \quad (2.1)$$

where

$$A_{hijk} = C_{hipq}C_{jkrs}g^{pr}g^{qs},$$

$$B_{hijk} = C_{hipq}A_{jkrs}g^{pr}g^{qs},$$

$$D_{hijk} = B_{hijk}-\tfrac{1}{12}J_2(g_{hj}g_{ik}-g_{hk}g_{ij})-\tfrac{1}{4}J_1C_{hijk},$$

$$\bar{D}_{hijk} = (J_3)^{-\frac{1}{3}}D_{hijk},$$

$$E_{hijk} = C_{hipq}D_{rsjk}g^{pr}g^{qs},$$

$$F_{hijk} = C_{hipq}E_{rsjk}g^{pr}g^{qs},$$

$$Q^j_i = {}^{(2)}R^j_i,$$

R^j_i and C_{hijk} denote the Ricci and conformal curvature tensors respectively.

Let us consider the axially symmetric metric (Synge 1960)

$$ds^2 = -e^{2\alpha}(dr^2+dz^2)-e^{2\beta}\,d\phi^2+e^{2\gamma}\,dt^2, \quad \ldots \quad \ldots \quad (2.2)$$

where α, β, γ are functions of r and z. Here r, z, ϕ, t correspond to x^1, x^2, x^3, x^4 respectively. The fourteen invariants for the metric (2.2) are

$$\left.\begin{aligned}
J_3 &= J_4 = K_6 = 0,\\
I_1 &= R,\\
I_2 &= (R^1_1)^2+(R^2_2)^2+(R^3_3)^2+(R^4_4)^2+2(R^2_1)^2,\\
I_3 &= (R^1_1)^3+(R^2_2)^3+(R^3_3)^3+(R^4_4)^3+3(R^2_1)^2(R^1_1+R^2_2),\\
I_4 &= (R^1_1)^4+(R^2_2)^4+(R^3_3)^4+(R^4_4)^4+2(R^2_1)^4+4(R^2_1)^2\left\{(R^1_1)^2+(R^2_2)^2+R^1_1R^2_2\right\},\\
J_1 &= 16\left\{(C^{12}_{12})^2+(C^{13}_{13})^2+C^{12}_{12}C^{13}_{13}+(C^{32}_{13})^2\right\},\\
J_2 &= -48C^{12}_{12}\left\{C^{13}_{13}(C^{12}_{12}+C^{13}_{13})+(C^{32}_{13})^2\right\},\\
K_1 &= 2C^{12}_{12}\left\{(R^1_1+R^3_3)(R^2_2-R^4_4)-(R^2_1)^2\right\}+2C^{13}_{13}\left\{(R^1_1-R^2_2)(R^3_3-R^4_4)\right\}+4C^{32}_{13}R^2_1(R^4_4-R^3_3),\\
K_2 &= 4(C^{12}_{12})^2\left\{(R^1_1+R^3_3)(R^2_2+R^4_4)-(R^2_1)^2\right\}+4(C^{13}_{13})^2\left\{(R^1_1+R^2_2)(R^3_3+R^4_4)\right\}\\
&\quad+4(C^{32}_{13})^2\left\{(R^1_1+R^2_2)(R^3_3+R^4_4)+8C^{12}_{12}C^{13}_{13}(R^1_1R^4_4+R^2_2R^3_3)+8C^{12}_{12}C^{32}_{13}R^2_1(R^3_3-R^4_4),\right.\\
K_4 &= 2C^{12}_{12}\left[\left\{(R^1_1)^2-(R^3_3)^2\right\}\left\{(R^2_2)^2-(R^4_4)^2\right\}+(R^2_1)^2\left\{(R^2_1)^2-(R^3_3)^2-(R^4_4)^2-2R^1_1R^2_2\right\}\right]\\
&\quad+2C^{13}_{13}\left\{(R^1_1)^2-(R^2_2)^2\right\}\left\{(R^3_3)^2-(R^4_4)^2\right\}+4C^{32}_{13}R^2_1(R^1_1+R^2_2)\left\{(R^4_4)^2-(R^3_3)^2\right\},\\
K_5 &= 4(C^{12}_{12})^2\left[\left\{(R^1_1)^2+(R^3_3)^2+(R^2_1)^2\right\}\left\{(R^2_2)^2+(R^4_4)^2+(R^2_1)^2\right\}-(R^2_1)^2(R^1_1+R^2_2)^2\right]\\
&\quad+4(C^{13}_{13})^2\left\{(R^1_1)^2+(R^2_2)^2+2(R^2_1)^2\right\}\left\{(R^3_3)^2+(R^4_4)^2\right\}\\
&\quad+4(C^{32}_{13})^2\left\{(R^1_1)^2+(R^2_2)^2+2(R^2_1)^2\right\}\left\{(R^3_3)^2+(R^4_4)^2\right\}\\
&\quad+8C^{12}_{12}C^{13}_{13}\left[(R^4_4)^2\left\{(R^1_1)^2+(R^2_1)^2\right\}+(R^3_3)^2\left\{(R^2_2)^2+(R^2_1)^2\right\}\right]\\
&\quad+8C^{12}_{12}C^{32}_{13}R^2_1(R^1_1+R^2_2)\left\{(R^3_3)^2-(R^4_4)^2\right\}.
\end{aligned}\right\} \quad (2.3)$$

It is found that the tensor relation

$$D_{hijk} = 0 \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (2.4)$$

is necessarily satisfied by an axially symmetric metric. In consequence of (2.4), J_3 vanishes. Hence the invariant K_3, being equal to $(J_3)^{-\frac{1}{2}}D_{hijk}R^{hj}R^{ik}$, is indeterminate.

3. Cylindrically Symmetric Metric

The most general cylindrically symmetric metric given by Marder (1958) is

$$ds^2 = e^{2(\alpha-\beta)}(dt^2-d\rho^2)-\rho^2e^{-2\beta}\,d\phi^2-e^{2(\beta+\gamma)}\,dz^2, \quad \ldots \quad \ldots \quad (3.1)$$

where α, β, γ are arbitrary functions of ρ and t only. Here ρ, ϕ, z and t correspond to x^1, x^2, x^3 and x^4 respectively. The fourteen invariants have been given by Singh *et al.* (1965) in the case when (3.1) represents a non-null electromagnetic field. For the metric (3.1), in general, the invariants are

$$\left.\begin{aligned}
J_3 &= J_4 = K_6 = 0,\\
I_1 &= R,\\
I_2 &= (R^1_1)^2+(R^2_2)^2+(R^3_3)^2+(R^4_4)^2-2(R^4_1)^2,\\
I_3 &= (R^1_1)^3+(R^2_2)^3+(R^3_3)^3+(R^4_4)^3-3(R^4_1)^2(R^1_1+R^4_4),\\
I_4 &= (R^1_1)^4+(R^2_2)^4+(R^3_3)^4+(R^4_4)^4+2(R^4_1)^4-4(R^4_1)^2\{(R^1_1)^2+(R^4_4)^2+R^1_1R^4_4\},\\
J_1 &= 16\{(C^{12}_{12})^2+(C^{13}_{13})^2+C^{12}_{12}C^{13}_{13}-(C^{24}_{12})^2\},\\
J_2 &= -48(C^{12}_{12}+C^{13}_{13})\{C^{12}_{12}C^{13}_{13}+(C^{24}_{12})^2\},\\
K_1 &= 2C^{12}_{12}\{(R^1_1-R^3_3)(R^2_2-R^4_4)-(R^4_1)^2\}+2C^{13}_{13}\{(R^1_1-R^2_2)(R^3_3-R^4_4)-(R^4_1)^2\}\\
&\quad +4C^{24}_{12}R^4_1(R^2_2-R^3_3),\\
K_2 &= 4(C^{12}_{12})^2\{(R^1_1+R^3_3)(R^2_2+R^4_4)+(R^4_1)^2\}+4(C^{13}_{13})^2\{(R^1_1+R^2_2)(R^3_3+R^4_4)+(R^4_1)^2\}\\
&\quad -4(C^{24}_{12})^2(R^1_1+R^4_4)(R^2_2+R^3_3)+8C^{12}_{12}C^{13}_{13}\{R^1_1R^4_4+R^2_2R^3_3+(R^4_1)^2\}\\
&\quad +8C^{24}_{12}(C^{12}_{12}+C^{13}_{13})R^4_1(R^2_2-R^3_3),\\
K_4 &= 2C^{12}_{12}\Big[\{(R^1_1)^2-(R^3_3)^2\}\{(R^2_2)^2-(R^4_4)^2\}-(R^4_1)^2\{(R^2_2)^2+(R^3_3)^2+(R^4_1)^2+2R^1_1R^4_4\}\Big[\\
&\quad +2C^{13}_{13}\Big[\{(R^1_1)^2-(R^2_2)^2\}\{(R^3_3)^2-(R^4_4)^2\}-(R^4_1)^2\{(R^2_2)^2+(R^3_3)^2+(R^4_1)^2+2R^1_1R^4_4\}\Big]\\
&\quad +4C^{24}_{12}R^4_1(R^1_1+R^4_4)\{(R^2_2)^2-(R^3_3)^2\},\\
K_5 &= 4(C^{12}_{12})^2\Big[\{(R^1_1)^2+(R^3_3)^2\}\{(R^2_2)^2+(R^4_4)^2\}-(R^4_1)^2\{(R^2_2)^2+(R^3_3)^2-(R^4_1)^2-2R^1_1R^4_4\}\Big]\\
&\quad +4(C^{13}_{13})^2\Big[\{(R^1_1)^2+(R^2_2)^2\}\{(R^3_3)^2+(R^4_4)^2\}\\
&\quad -(R^4_1)^2\{(R^2_2)^2+(R^3_3)^2-(R^4_1)^2-2R^1_1R^4_4\}\Big]\\
&\quad +8C^{12}_{12}C^{13}_{13}\Big[(R^1_1)^2(R^4_4)^2+(R^2_2)^2(R^3_3)^2+(R^4_1)^2\{(R^4_1)^2+2R^1_1R^4_4\}\Big]\\
&\quad -4(C^{24}_{12})^2\{(R^1_1)^2+(R^4_4)^2-2(R^4_1)\ \}\{(R^2_2)^2+(R^3_3)^2\}\\
&\quad +8C^{24}_{12}(C^{12}_{12}+C^{13}_{13})R^4_1(R^1_1+R^4_4)\{(R^2_2)^2-(R^3_3)^2\},
\end{aligned}\right\} \quad (3.2)$$

K_3 is indeterminate. The tensor D_{hijk} vanishes for a cylindrically symmetric metric.

4. Plane Symmetric Metric

The fourteen invariants for the plane symmetric metric, viz.

$$ds^2 = e^{2u}(dt^2-dx^2)-e^{2v}(dy^2+dz^2), \quad \ldots \quad (4.1)$$

u, v being functions of x and t, are obtained as

$$\left.\begin{aligned}
J_3 &= J_4 = K_6 = 0,\\
I_1 &= R,\\
I_2 &= (R^1_1)^2+(R^4_4)^2+2(R^2_2)^2-2(R^4_1)^2,\\
I_3 &= (R^1_1)^3+(R^4_4)^3+2(R^2_2)^3-3(R^4_1)^2(R^1_1+R^4_4),\\
I_4 &= (R^1_1)^4+(R^4_4)^4+2(R^2_2)^4+2(R^4_1)^4-4(R^4_1)^2\left\{(R^1_1)^2+(R^4_4)^2+R^1_1R^4_4\right\},\\
J_1 &= 48(C^{12}_{12})^2,\\
J_2 &= -96(C^{12}_{12})^3,\\
K_1 &= 4C^{12}_{12}\left\{R^2_2(R^1_1+R^4_4)-R^1_1R^4_4-(R^4_1)^2-(R^2_2)^2\right\},\\
K_2 &= 8(C^{12}_{12})^2\left\{2R^1_1R^4_4+2(R^2_2)^2+2(R^4_1)^2+R^2_2(R^1_1+R^4_4)\right\},\\
K_4 &= 4C^{12}_{12}\left[\left\{(R^1_1)^2-(R^2_2)^2\right\}\left\{(R^2_2)^2-(R^4_4)^2\right\}-(R^4_1)^2\left\{2(R^2_2)^2+2R^1_1R^4_4+(R^4_1)^2\right\}\right],\\
K_5 &= 8(C^{12}_{12})^2\left[2(R^1_1)^2(R^4_4)^2+2(R^2_2)^4+(R^2_2)^2\left\{(R^1_1)^2+(R^4_4)^2\right\}\right.\\
&\quad \left.+2(R^4_1)^2\left\{2R^1_1R^4_4+(R^4_1)^2-(R^2_2)^2\right\}\right],
\end{aligned}\right\} \qquad (4.2)$$

K_3 is indeterminate. From the above, it is clear that the invariant relation

$$(J_1)^3 = 12(J_2)^2 \qquad \text{.. .. .. ..} \quad (4.3)$$

holds for a plane symmetric metric. Besides this invariant relation, the tensor relation $D_{hijk} = 0$ is also satisfied by a plane symmetric metric.

5. Product of two Surfaces

Since every Riemannian V_2 is conformal to a flat space S_2 the metric representing the product of two surfaces may be written in the form

$$ds^2 = -A(dx^2+dy^2)-B(dz^2-dt^2), \qquad \text{.. .. ..} \quad (5.1)$$

where $A = A(x, y)$, $B = B(z, t)$. As the class of each of the two V_2's is one, the class of the metric (5.1) is two. The fourteen invariants for (5.1) are given by the following:

$$J_3 = J_4 = K_6 = 0, \qquad \text{.. .. ..} \quad (5.2)$$

$$I_1 = R, \qquad \text{.. .. ..} \quad (5.3)$$

$$I_2 = 2\left\{(R^1_1)^2+(R^3_3)^2\right\}. \qquad \text{.. .. .. ..} \quad (5.4)$$

$$I_3 = \tfrac{3}{8}I_1(2I_2-J_1), \qquad \text{.. .. ..} \quad (5.5)$$

$$I_4 = \tfrac{1}{16}\{4(I_2)^2+12J_1I_2-9(J_1)^2\}, \qquad \text{.. .. ..} \quad (5.6)$$

$$J_1 = \tfrac{1}{8}(I_1)^2, \qquad \text{.. .. ..} \quad (5.7)$$

$$J_2 = -\tfrac{1}{18}(I_1)^3, \quad \ldots \quad (5.8)$$

$$K_1 = \tfrac{1}{12} I_1(3J_1 - 4I_2), \quad \ldots \quad (5.9)$$

$$K_2 = \tfrac{1}{24} J_1(3J_1 + 2I_2), \quad \ldots \quad (5.10)$$

$$K_4 = \tfrac{3}{4} J_1 K_1, \quad \ldots \quad (5.11)$$

$$K_5 = \tfrac{1}{64} J_1\{4(I_2)^2 + 4J_1 I_2 - 3(J_1)^2\}, \quad \ldots \quad (5.12)$$

K_3 is indeterminate. Thus out of the ten non-zero invariants, only two are independent. For the metric (5.1)

$$D_{hijk} = 0. \quad \ldots \quad (5.13)$$

Each of the relations (5.7), (5.8), (5.11) and (5.13) serves as a necessary condition for a general Riemannian fourfold to be decomposable into a product of two surfaces.

When the metric (5.1) represents a non-null electromagnetic field, we find that K_3 is indeterminate and only two invariants are non-vanishing which are given by

$$I_2 = 4(R^1_1)^2, \quad \ldots \quad (5.14)$$

$$I_4 = \tfrac{1}{4}(I_2)^2. \quad \ldots \quad (5.15)$$

Thus out of the two non-zero invariants, only one is independent. It may be pointed out that the invariant relation (5.15) does not always hold for a product of two surfaces in general.

6. Gödel Metric

Gödel (1949) showed that the metric

$$ds^2 = -(dx^1)^2 + \tfrac{1}{2}e^{2\alpha x^1}(dx^2)^2 - (dx^3)^2 + (dx^4)^2 + 2e^{\alpha x^1}\, dx^2\, dx^4, \quad \ldots \quad (6.1)$$

α being a constant, is compatible with incoherent matter distribution. In this case the fourteen scalar differential invariants are found to be

$$\left.\begin{aligned} J_3 = J_4 = K_1 = K_2 = K_4 = K_5 = K_6 = 0,\\ I_1 = -\alpha^2,\\ I_2 = (I_1)^2,\\ I_3 = (I_1)^3,\\ I_4 = (I_1)^4,\\ J_1 = \tfrac{4}{3}(I_1)^2,\\ J_2 = \tfrac{4}{9}(I_1)^3, \end{aligned}\right\} \quad \ldots \quad (6.2)$$

K_3 is indeterminate. Thus we find that only one invariant is independent for the Gödel metric (6.1). The tensor D_{hijk} vanishes in this case as well. It is already known that this tensor vanishes in the case of the spherically symmetric line element (Narlikar and Karmarkar 1949). Thus we find that for all the well-known line elements of gravitational significance the tensor D_{htjk} is zero.

We have verified that D_{htjk} is non-zero for the most general orthogonal line element

$$ds^2 = -A(dx^1)^2 - B(dx^2)^2 - C(dx^3)^2 + D(dx^4)^2, \quad \ldots \quad \ldots \quad (6.3)$$

A, B, C, D being functions of x^1, x^2, x^3, x^4. In this case it is found that all the fourteen scalar invariants are non-vanishing. The values of the components of the curvature tensor and the Ricci tensor for this line element are given in the Appendix.

APPENDIX

The non-zero components of R_{htjk} for (6.3):

$$R_{1212} = \frac{A,_{22}+B,_{11}}{2} - \frac{(A,_2)^2+A,_1B,_1}{4A} - \frac{(B,_1)^2+A,_2B,_2}{4B} + \frac{A,_3B,_3}{4C} - \frac{A,_4B,_4}{4D},$$

$$R_{1213} = \frac{A,_{23}}{2} - \frac{A,_2A,_3}{4A} - \frac{A,_2B,_3}{4B} - \frac{C,_2A,_3}{4C},$$

$$R_{1214} = \frac{A,_{24}}{2} - \frac{A,_2A,_4}{4A} - \frac{A,_2B,_4}{4B} - \frac{D,_2A,_4}{4D},$$

$$R_{1223} = -\frac{B,_{13}}{2} + \frac{B,_1B,_3}{4B} + \frac{B,_1A,_3}{4A} + \frac{C,_1B,_3}{4C},$$

$$R_{1224} = -\frac{B,_{14}}{2} + \frac{B,_1B,_4}{4B} + \frac{B,_1A,_4}{4A} + \frac{D,_1B,_4}{4D},$$

$$R_{1313} = \frac{A,_{33}+C,_{11}}{2} - \frac{(A,_3)^2+A,_1C,_1}{4A} - \frac{(C,_1)^2+A,_3C,_3}{4C} + \frac{A,_2C,_2}{4B} - \frac{A,_4C,_4}{4D},$$

$$R_{1314} = \frac{A,_{34}}{2} - \frac{A,_3A,_4}{4A} - \frac{A,_3C,_4}{4C} - \frac{D,_3A,_4}{4D},$$

$$R_{1323} = \frac{C,_{12}}{2} - \frac{C,_1C,_2}{4C} - \frac{C,_1A,_2}{4A} - \frac{B,_1C,_2}{4B},$$

$$R_{1334} = -\frac{C,_{14}}{2} + \frac{C,_1C,_4}{4C} + \frac{C,_1A,_4}{4A} + \frac{D,_1C,_4}{4D},$$

$$R_{1414} = \frac{A,_{44}-D,_{11}}{2} - \frac{(A,_4)^2-A,_1D,_1}{4A} + \frac{(D,_1)^2-A,_4D,_4}{4D} - \frac{A,_3D,_3}{4C} - \frac{A,_2D,_2}{4B},$$

$$R_{1424} = -\frac{D,_{12}}{2} + \frac{D,_1A,_2}{4A} + \frac{D,_2B,_1}{4B} + \frac{D,_1D,_2}{4D},$$

$$R_{1434} = -\frac{D,_{13}}{2} + \frac{D,_1A,_3}{4A} + \frac{D,_3C,_1}{4C} + \frac{D,_1D,_3}{4D},$$

$$R_{2323} = \frac{B,_{33}+C,_{22}}{2} - \frac{(B,_3)^2+B,_2C,_2}{4B} - \frac{(C,_2)^2+B,_3C,_3}{4C} + \frac{B,_1C,_1}{4A} - \frac{B,_4C,_4}{4D},$$

$$R_{2324} = \frac{B,_{34}}{2} - \frac{B,_3B,_4}{4B} - \frac{B,_3C,_4}{4C} - \frac{D,_3B,_4}{4D},$$

$$R_{2334} = -\frac{C,_{24}}{2}+\frac{C,_2C,_4}{4C}+\frac{C,_2B,_4}{4B}+\frac{D,_2C,_4}{4D},$$

$$R_{2424} = \frac{B,_{44}-D,_{22}}{2}-\frac{(B,_4)^2-B,_2D,_2}{4B}+\frac{(D,_2)^2-B,_4D,_4}{4D}-\frac{B,_3D,_3}{4C}-\frac{B,_1D,_1}{4A},$$

$$R_{2434} = -\frac{D,_{23}}{2}+\frac{D,_2B,_3}{4B}+\frac{D,_3C,_2}{4C}+\frac{D,_2D,_3}{4D},$$

$$R_{3434} = \frac{C,_{44}-D,_{33}}{2}-\frac{(C,_4)^2-C,_3D,_3}{4C}+\frac{(D,_3)^2-C,_4D,_4}{4D}-\frac{C,_2D,_2}{4B}-\frac{C,_1D,_1}{4A}.$$

A comma followed by the lower suffixes denotes partial differentiation with respect to the corresponding variables.

The components of R_{ij}:

$$R_{11} = \frac{A,_{22}+B,_{11}}{2B}+\frac{A,_{33}+C,_{11}}{2C}+\frac{D,_{11}-A,_{44}}{2D}-\frac{(B,_1)^2+A,_2B,_2}{4B^2}$$
$$-\frac{(C,_1)^2+A,_3C,_3}{4C^2}+\frac{A,_4D,_4-(D,_1)^2}{4D^2}-\frac{(A,_2)^2+A,_1B,_1}{4AB}-\frac{(A,_3)^2+A,_1C,_1}{4AC}$$
$$+\frac{(A,_4)^2-A,_1D,_1}{4AD}+\frac{A,_3B,_3+A,_2C,_2}{4BC}+\frac{A,_3D,_3-A,_4C,_4}{4CD}$$
$$+\frac{A,_2D,_2-A,_4B,_4}{4BD},$$

$$R_{22} = \frac{A,_{22}+B,_{11}}{2A}+\frac{B,_{33}+C,_{22}}{2C}+\frac{D,_{22}-B,_{44}}{2D}-\frac{(A,_2)^2+A,_1B,_1}{4A^2}-\frac{(C,_2)^2+B,_3C,_3}{4C^2}$$
$$+\frac{B,_4D,_4-(D,_2)^2}{4D^2}-\frac{(B,_1)^2+A,_2B,_2}{4AB}-\frac{(B,_3)^2+B,_2C,_2}{4BC}+\frac{(B,_4)^2-B,_2D,_2}{4BD}$$
$$+\frac{A,_3B,_3+B,_1C,_1}{4AC}+\frac{B,_3D,_3-B,_4C,_4}{4CD}+\frac{B,_1D,_1-A,_4B,_4}{4AD},$$

$$R_{33} = \frac{A,_{33}+C,_{11}}{2A}+\frac{B,_{33}+C,_{22}}{2B}+\frac{D,_{33}-C,_{44}}{2D}-\frac{(A,_3)^2+A,_1C,_1}{4A^2}$$
$$-\frac{(B,_3)^2+B,_2C,_2}{4B^2}+\frac{C,_4D,_4-(D,_3)^2}{4D^2}-\frac{(C,_1)^2+A,_3C,_3}{4AC}-\frac{(C,_2)^2+B,_3C,_3}{4BC}$$
$$+\frac{(C,_4)^2-C,_3D,_3}{4CD}+\frac{A,_2C,_2+B,_1C,_1}{4AB}+\frac{C,_2D,_2-B,_4C,_4}{4BD}+\frac{C,_1D,_1-A,_4C,_4}{4DA},$$

$$R_{44} = \frac{A,_{44}-D,_{11}}{2A}+\frac{B,_{44}-D,_{22}}{2B}+\frac{C,_{44}-D,_{33}}{2C}+\frac{A,_1D,_1-(A,_4)^2}{4A^2}+\frac{B,_2D,_2-(B,_4)^2}{4B^2}$$
$$+\frac{C,_3D,_3-(C,_4)^2}{4C^2}+\frac{(D,_1)^2-A,_4D,_4}{4DA}+\frac{(D,_2)^2-B,_4D,_4}{4DB}+\frac{(D,_3)^2-C,_4D,_4}{4DC}$$
$$-\frac{A,_2D,_2+B,_1D,_1}{4AB}-\frac{B,_3D,_3+C,_2D,_2}{4BC}-\frac{A,_3D,_3+C,_1D,_1}{4CA},$$

$$R_{12} = \frac{C,_{12}}{2C} + \frac{D,_{12}}{2D} - \frac{C,_1C,_2}{4C^2} - \frac{D,_1D,_2}{4D_2} - \frac{C,_1A,_2}{4CA} - \frac{C,_2B,_1}{4CB} - \frac{D,_1A,_2}{4DA} - \frac{D,_2B,_1}{4DB},$$

$$R_{13} = \frac{B,_{13}}{2B} + \frac{D,_{13}}{2D} - \frac{B,_1B,_3}{4B^2} - \frac{D,_1D,_3}{4D^2} - \frac{B,_1A,_3}{4BA} - \frac{B,_3C,_1}{4BC} - \frac{D,_1A,_3}{4DA} - \frac{D,_3C,_1}{4DC},$$

$$R_{14} = \frac{B,_{14}}{2B} + \frac{C,_{14}}{2C} - \frac{B,_1B,_4}{4B^2} - \frac{C,_1C,_4}{4C^2} - \frac{B,_1A,_4}{4BA} - \frac{B,_4D,_1}{4BD} - \frac{C,_1A,_4}{4CA} - \frac{C,_4D,_1}{4CD},$$

$$R_{23} = \frac{A,_{23}}{2A} + \frac{D,_{23}}{2D} - \frac{A,_2A,_3}{4A^2} - \frac{D,_2D,_3}{4D^2} - \frac{A,_2B,_3}{4AB} - \frac{A,_3C,_2}{4AC} - \frac{D,_2B,_3}{4DB} - \frac{D,_3C,_2}{4DC},$$

$$R_{24} = \frac{A,_{24}}{2A} + \frac{C,_{24}}{2C} - \frac{A,_2A,_4}{4A^2} - \frac{C,_2C,_4}{4C^2} - \frac{A,_2B,_4}{4AB} - \frac{A,_4D,_2}{4AD} - \frac{C,_2B,_4}{4CB} - \frac{C,_4D,_2}{4CD},$$

$$R_{34} = \frac{A,_{34}}{2A} + \frac{B,_{34}}{2B} - \frac{A,_3A,_4}{4A^2} - \frac{B,_3B,_4}{4B^2} - \frac{A,_3C,_4}{4AC} - \frac{A,_4D,_3}{4AD} - \frac{B,_3C,_4}{4BC} - \frac{B,_4D,_3}{4BD}.$$

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