FOURIER SERIES FOR MEIJER'S G-FUNCTION AND MACROBERT'S E-FUNCTION

by C. L. Parihar, *Department of Mathematics, Holkar Science College, Indore* (*M.P.*)

(Communicated by P. L. Bhatnagar, F.N.I.)

(*Received May 19, 1967*)

Kesarwani (1966) and MacRobert (1961) have obtained the Fourier series for G-function and E-function respectively. In the present study we have generalized these Fourier series and from them we have deduced certain integrals involving G-function and E-function.

§ 1. MacRobert (1961) established the Fourier series for the E-function and in a recent paper Kesarwani (1966) has proved Fourier series for Meijer's G-function. We propose to give the following generalization of the Fourier series of Meijer's G-function in the present study:

$$\sum_{r=0}^{\infty} \frac{(n+r)!}{n!\,r!} \cdot G^{l+2;\,u+1}_{p+3;\,q+3}\left(z\left|\begin{matrix}1-r,\, a_1,\, \ldots,\, a_p,\, 1,\, n+r+2\,; \\ 1+\frac{n}{2},\, \frac{3}{2}+\frac{n}{2},\, b_1,\, \ldots,\, b_q,\, 1;\end{matrix}\right.\right) \sin\,(n+2r+1)\theta$$

$$= \frac{\sqrt{\pi}}{2} \sum_{k=0}^{n} \frac{\sin\theta(\cos\theta-1)^k}{k!\,(n-k)!} \cdot G^{l+2;\,u}_{p+2;\,q+2}\left(\frac{z}{\sin^2\theta}\left|\begin{matrix}a_1,\, \ldots,\, a_p,\, 1,\, \frac{3}{2}+k \\ 1+\frac{n+k}{2},\, \frac{3}{2}+\frac{n+k}{2},\, b_1,\, \ldots,\, b_q\end{matrix}\right.\right) \quad \ldots \quad (1.1)$$

where

$$0 \leqslant \theta \leqslant \pi\,; \quad |\arg z| < (l+u-\tfrac{1}{2}p-\tfrac{1}{2}q)\pi.$$

Meijer's G-function is defined (Bateman Project 1953) by a Mellin Barnes type integral

$$G^{l;\,u}_{p;\,q}\left(z\left|\begin{matrix}a_1,\, \ldots,\, a_p \\ b_1,\, \ldots,\, b_q\end{matrix}\right.\right) = \frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^{l} \Gamma(b_j-s) \prod_{j=1}^{u} \Gamma(1-a_j+s)\,.\,z^s}{\prod_{j=l+1}^{q} \Gamma(1-b_j+s) \prod_{j=u+1}^{p} \Gamma(a_j-s)}\, ds \qquad (1.2)$$

where l, u, p, q are integers with $q \geqslant 1$; $0 \leqslant u \leqslant p$; $0 \leqslant 1 \leqslant q$, the parameters a_j and b_j are such that no pole of $\Gamma(b_j-s)$; $j = 1, 2, \ldots, 1$ coincides with any pole of $\Gamma(1-a_j+s)$; $j = 1, 2, \ldots, u$. The poles of integrand must be simple and those of $\Gamma(b_j-s)$; $j = 1, 2, \ldots, 1$ lie on one side of the contour L and those of $\Gamma(1-a_j+s)$; $j, = 1, 2, \ldots, u$ must lie on the other side. The integral converges if $p+q < 2(l+u)$ and $|\arg z| < (l+u-\frac{1}{2}p-\frac{1}{2}q)\pi$.

From (1.1) we shall obtain the Fourier series for E-function

$$\sum_{r=0}^{\infty} \frac{(n+r)!}{r!\ n!} \sin (n+2r+1)\theta . z^{-r} . E\begin{pmatrix} 1+r+\frac{n}{2}, \frac{3}{2}+r+\frac{n}{2}, \alpha_1+r, \ldots, \alpha_1+r; & z \\ 1+r,\ n+2r+2,\ \rho_1+r, \ldots, \rho_q+r; & \end{pmatrix}$$

$$= \frac{\sqrt{\pi}}{2} \sum_{k=0}^{n} \frac{\sin\theta(\cos\theta-1)^k}{k!\ (n-k)!} E\begin{pmatrix} 1+\frac{n+k}{2}, \frac{3}{2}+\frac{n+k}{2}, \alpha_1, \ldots, \alpha_p; & \frac{z}{\sin^2\theta} \\ 1, \frac{3}{2}+k, \rho_1, \ldots, \rho_q; & \end{pmatrix} \tag{1.3}$$

where $0 \leqslant \theta \leqslant \pi$; $|\arg z| < \pi$, as a particular case.

§ 2. We shall use the following results (Richard Askey 1965) with $\lambda = 1-s$:

$$(\sin\theta)^{1-2s} P_n^{(1-s)}(\cos\theta) = \sum_{r=0}^{\infty} \frac{2^{2s}(n+r)!\ \Gamma(n+2-2s)\,\Gamma(r+s)}{\Gamma(1-s)\Gamma(s) r!\ n!\ \Gamma(n+r+2-s)} \sin (n+2r+1)\theta \tag{2.1}$$

where $s < 1$ and $0 \leqslant \theta \leqslant \pi$ and $P_n^{(\lambda)}(\cos\theta)$ is given by

$$(1-2r\cos\theta+r^2)^{-\lambda} = \sum_{n=0}^{\infty} P_n^{(\lambda)}(\cos\theta) . r^n$$

also (Rainville 1964)

$$P_n^{(v)}(z) = \sum_{k=0}^{n} \frac{(2r)_{n+k}\left(\frac{z-1}{2}\right)^k}{k!\ (n-k)!\ (r+\frac{1}{2})_k} \quad \ldots \quad \ldots \quad \ldots \tag{2.2}$$

and Legendre's duplication formula (Rainville 1964)

$$\sqrt{\pi}\,\Gamma(2z) = 2^{2z-1}\Gamma(z)\Gamma(z+\tfrac{1}{2}). \quad \ldots \quad \ldots \quad \ldots \tag{2.3}$$

The transformations (Bateman Project 1953)

$$z^{\sigma} G_{p;\,q}^{l;\,u}\left(z \,\middle|\, \begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix}\right) = G_{p;\,q}^{l;\,u}\left(z \,\middle|\, \begin{matrix} a_1+\sigma, \ldots, a_p+\sigma \\ b_1+\sigma, \ldots, b_q+\sigma \end{matrix}\right) \quad \ldots \tag{2.4}$$

and (Bateman Project 1953)

$$G_{q+1;\,p}^{p;\,1}\left(z \,\middle|\, \begin{matrix} 1, \beta_1, \ldots, \beta_q \\ \alpha_1, \ldots, \alpha_p \end{matrix}\right) = E\begin{pmatrix} \alpha_1, \ldots, \alpha_p; & z \\ \beta_1, \ldots, \beta_q; & \end{pmatrix}. \quad \ldots \quad \ldots \tag{2.5}$$

§ 3. *Proof*

On substitution in the left-hand side of (1.1) from (1.2) we get

$$\sum_{r=0}^{\infty} \frac{(n+r)!\sin(n+2r+1)\theta}{r!\ n!}$$

$$\times \frac{1}{2\pi i}\int_L \frac{\Gamma\left(1+\frac{n}{2}-s\right)\Gamma\left(\frac{3}{2}+\frac{n}{2}-s\right)\prod_{j=1}^{l}\Gamma(b_j-s)\Gamma(r+s)\prod_{j=1}^{u}\Gamma(1-a_j+s)}{\prod_{j=l+1}^{q}\Gamma(1-b_j+s)\Gamma(s)\prod_{j=u+1}^{p}\Gamma(a_j-s)\Gamma(1-s)\Gamma(n+r+2-s)} z^s\, ds.$$

The path of integration runs from $c-i\infty$ to $c+i\infty$. The conditions

$$0 < C < \frac{3}{2}+\frac{n}{2}$$

$$R(b_j) > C \qquad j = 1, 2, \ldots, 1$$

$$R(a_j) < C+1 \quad j = 1, 2, \ldots, u$$

are the required conditions in (1.1) that all the poles of $\Gamma\left(1+\frac{n}{2}-s\right)$; $\Gamma\left(\frac{3}{2}+\frac{n}{2}-s\right)$ and $\Gamma(b_j-s)$; $j = 1, 2, \ldots, 1$ lie on the right side of contour L. Those of $\Gamma(r+s)$ and $\Gamma(1-a_j+s)$; $j = 1, 2, \ldots, u$ lie to the left of contour L. The integral converges if $p+q < 2(1+u)$ and $|\arg z| < (1+u-\frac{1}{2}p-\frac{1}{2}q)\pi$.

Hence, on changing the order of integration and summation which is permissible, the above expression becomes

$$\frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^{l}\Gamma(b_j-s)\prod_{j=1}^{u}\Gamma(1-a_j+s)}{\prod_{j=l+1}^{q}\Gamma(1-b_j+s)\prod_{j=u+1}^{b}\Gamma(a_j-s)}$$

$$\times\left\{\sum_{r=0}^{\infty}\frac{(n+r)!\,\Gamma\left(1+\frac{n}{2}-s\right)\Gamma\left(\frac{3}{2}+\frac{n}{2}-s\right)\Gamma(r+s)}{r!\,n!\,\Gamma(s)\Gamma(1-s)\Gamma(n+r+2-s)}\sin(n+2r+1)\theta\right\}z^s\,ds.$$

Using (2.3) and (2.1) we have

$$\frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^{l}\Gamma(b_j-s)\prod_{j=1}^{u}\Gamma(1-a_j+s)}{\prod_{j=l+1}^{q}\Gamma(1-b_j+s)\prod_{j=u+1}^{p}\Gamma(a_j-s)}\left\{\frac{\sqrt{\pi}}{2^{n+1}}(\sin\theta)^{1-2s}\,.\,P_n^{(1-s)}(\cos\theta)\right\}z^s\,ds.$$

Substituting the value of $P_n^{(1-s)}(\cos\theta)$ from (2.2) we get

$$\frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^{l}\Gamma(b_j-s)\prod_{j=1}^{u}\Gamma(1-a_j+s)}{\prod_{j=l+1}^{q}\Gamma(1-b_j+s)\prod_{j=u+1}^{p}\Gamma(a_j-s)}\left\{\frac{\sqrt{\pi}}{2^{n+1}}\sum_{k=0}^{n}\frac{(\sin\theta)^{1-2s}(2-2s)_{n+k}}{k!\,(n-k)!\,(\frac{3}{2}-s)_k}\left(\frac{\cos\theta-1}{2}\right)^k\right\}z^s\,ds.$$

On simplification we get

$$\sum_{k=0}^{n}\frac{\sqrt{\pi}}{2}\frac{\sin\theta\,(\cos\theta-1)^k}{k!\,(n-k)!}$$

$$\times\frac{1}{2\pi i}\int_L \frac{\Gamma\left(1+\frac{n+k}{2}-s\right)\Gamma\left(\frac{3}{2}+\frac{n+k}{2}-s\right)\prod_{j=1}^{l}\Gamma(b_j-s)\prod_{j=1}^{u}\Gamma(1-a_j+s)}{\Gamma(1-s)\Gamma(\frac{3}{2}+k-s)\prod_{j=l+1}^{q}\Gamma(1-b_j+s)\prod_{j=u+1}^{p}\Gamma(a_j-s)}\left(\frac{z}{\sin^2\theta}\right)^s ds.$$

Hence, we have the right-hand side of (1.1).

Using the transformation (2.4) and then substituting $a_1 = 1$, (1.1) reduces to

$$\sum_{r=0}^{\infty} \frac{(n+r)!}{n!\,r!} \sin(n+2r+1)\theta \,.\, z^{-r} \,.\, G^{l+2;\,u}_{p+2;\,q+2}\left(z \,\middle|\, \begin{matrix} 1, a_2+r, \ldots, a_p+r, 1+r, n+2r+2 \\ 1+r+\frac{n}{2};\, \frac{3}{2}+r+\frac{n}{2},\, b_1+r, \ldots, b_q+r \end{matrix}\right)$$

$$= \frac{\sqrt{\pi}}{2} \sum_{k=0}^{n} \frac{\sin\theta(\cos\theta-1)^k}{k!\,(n-k)!} G^{l+2,\,u}_{p+2,\,q+2}\left(\frac{z}{\sin^2\theta} \,\middle|\, \begin{matrix} 1, a_2, \ldots, a_p, 1, \frac{3}{2}+k \\ 1+\frac{n+k}{2}, \frac{3}{2}+\frac{n+k}{2}, b_1, \ldots, b_q \end{matrix}\right) \tag{3.1}$$

Now, by changing the G-function to E-function by the formula (2.5), i.e. on replacing in (3.1), p by $aq+1$, q by p, putting $u = l$, $l = p$ and changing a_{s+1} to l_s ($s = 1, 2, \ldots, q$), b_r to α_r ($r = 1, 2, \ldots, p$), we obtain the series (1.3).

§ 4. *Particular Cases*

If $n = 0$, (1.1) will be reduced to the following result of Kesarwani (1966):

$$\sum_{r=0}^{\infty} G^{l+1;\,u+1}_{p+2;\,q+2}\left(z \,\middle|\, \begin{matrix} 1-r, a_1, \ldots, a_p, 2+r \\ \frac{3}{2}, b_1, \ldots, b_q, 1 \end{matrix}\right) = \frac{\sqrt{\pi}}{2} \sin\theta\, G^{l,\,u}_{p,\,q}\left(\frac{z}{\sin^2\theta} \,\middle|\, \begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix}\right)$$

Similarly, if $n = 0$, (1.3) will be reduced to the Fourier series due to MacRobert (1961),

$$\sum_{r=0}^{\infty} z^{-r} E\left(\begin{matrix} \frac{3}{2}+r, \alpha_1+r, \ldots, \alpha_p+r; \\ 2r+2, \rho_1+r, \ldots, \rho_q+r; \end{matrix}\, z\right) \sin(2r+1)\theta = \frac{\sqrt{\pi}}{2} \sin\theta\, E\left(\begin{matrix} \alpha_1, \ldots, \alpha_p; \\ \beta_1, \ldots, \beta_q; \end{matrix}\, z\right)$$

§ 5. *Some Deductions*

From (1.1) and (1.3) we can deduce the following integrals:

$$\int_0^{\pi} \sum_{r=0}^{n} \frac{\sin\theta \sin(n+2r+1)\theta(\cos\theta-1)^k}{k!\,(n-k)!} G^{l+2;\,u}_{p+2;\,q+2}\left(\frac{z}{\sin^2\theta} \,\middle|\, \begin{matrix} a_1, \ldots, a_p, 1, \frac{3}{2}+k \\ 1+\frac{n+k}{2}, \frac{3}{2}+\frac{n+k}{2}, b_1, \ldots, b_q \end{matrix}\right) d\theta$$

$$= \frac{\sqrt{\pi}\,(n+r)!}{n!\,r!} G^{l+2;\,u+1}_{p+3;\,q+3}\left(z \,\middle|\, \begin{matrix} 1-r, a_1, \ldots, a_p, 1, n+r+2 \\ 1+\frac{n}{2}, \frac{3}{2}+\frac{n}{2}, b_1, \ldots, b_q, 1 \end{matrix}\right) \quad \ldots \quad \ldots \tag{5.1}$$

where $p+q < 2(l+u)$; $|\arg z| < (l+u-\frac{1}{2}p-\frac{1}{2}q)\pi$; $r = 1, 2, \ldots$ and

$$\int_0^{\pi} \sum_{k=0}^{n} \frac{\sin\theta \sin(n+2r+1)\theta(\cos\theta-1)^k}{k!\,(n-k)!} E\left(\begin{matrix} 1+\frac{n+k}{2}, \frac{3}{2}+\frac{n+k}{2}, \alpha_1, \ldots, \alpha_p; \\ 1, \frac{3}{2}+k, \rho_1, \ldots, \rho_q; \end{matrix}\, \frac{z}{\sin^2\theta}\right)$$

$$= \frac{\sqrt{\pi}\,(n+r)!}{n!\,r!} z^{-r} E\left(\begin{matrix} 1+r+\frac{n}{2}, \frac{3}{2}+r+\frac{n}{2}, \alpha_1+r, \ldots, \alpha_p+r; \\ 1+r, n+2r+2, \rho_1+r, \ldots, \rho_q+r; \end{matrix}\, z\right) \quad \ldots \tag{5.2}$$

on substitution $n = 0$ (5.1) and (5.2) will be reduced to the results of Kesarwani (1966) and MacRobert (1961).

ACKNOWLEDGEMENTS

The author wishes to express his sincere thanks to Dr. R. N. Jain and Prof. B. M. Agarwal for kind supervision in the preparation of this paper and also Prof. P. L. Bhatnagar for communicating it.

REFERENCES

Bateman Project (1953). Higher Transcendental Functions. Vol. 1. McGraw-Hill Book Co. Inc., New York, pp. 207, 209, 215.

Kesarwani, R. N. (1966). Fourier series for Meijer's G-function. *Compositio math.*, **17**, 149–151.

MacRobert, T. M. (1961). Fourier series for E-function. *Math. Z.*, **75**, 79–82.

——— (1966). Functions of a Complex Variable. MacMillan & Co., London.

Rainville, E. D. (1964). Special Functions. MacMillan & Co., New York, pp. 24, 279.

Richard Askey (1965). Orthogonal expansions with positive coefficients. *Proc. Am. Math. Soc.*, **16**, 1191–1194.