ON ALMOST HERMITE SPACES—III

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In this paper, I have obtained some results concerning Kähler, almost Kähler and Tachibana spaces.

1. Introduction

We consider a $2n$-dimensional space M_{2n} of differentiability class C^{r+1} with a tensor F or f of the type (0, 2) or (1, 1) respectively, such that if early Latin letters $A, B, \ldots$ denote covariant vectors and later Latin letters X, Y, Z denote contravariant vectors,

$$\overline{X} \overset{\text{def}}{=} f(X) \qquad \ldots \quad (1.1)$$

$$f(X, g(Y)) = g(\overline{X}, Y) = F(X, Y) \qquad \ldots \quad (1.2a)$$

$$F(X, {}^{-1}g(A)) = {}^{-1}g(A, \overline{X}) = f(X, A) \qquad \ldots \quad (1.2b)$$

$${}^{(2)}F(X, Y) \overset{\text{def}}{=} f(X, F(Y)) = -g(X, Y) \qquad \ldots \quad (1.3a)$$

$${}^{(2)}f(X, A) \overset{\text{def}}{=} f(X, f(A)) = -A(X) \qquad \ldots \quad (1.3b)$$

g being the non-singular metric tensor of M_{2n}. The space M_{2n} is called an almost complex space, the almost complex structure being given by F or f.

From (1.1), we have

$$\overline{\overline{X}} = -X. \qquad \ldots \quad (1.4)$$

A space, for which (1.3) and

$$g(\overline{X}, \overline{Y}) = g(X, Y) = F(X, \overline{Y}) \qquad \ldots \quad (1.5)$$

are satisfied, is called an almost Hermite space. In an almost Hermite space

$$F(X, Y) = -F(Y, X) \qquad \ldots \quad (1.6)$$

$$F(\overline{X}. \overline{Y}) = g(\overline{X}, Y) = -g(X, \overline{Y}) = F(X, Y). \qquad \ldots \quad (1.7)$$

Let D_X be the connection with respect to g (Mishra 1965). Then an almost Hermite space, for which

$$D_X F = 0 \qquad \ldots \quad (1.8)$$

is satisfied, is called a Kähler space.

An almost Hermite space, for which

$$(D_X F)(Y, Z) + (D_Y F)(Z, X) + (D_Z F)(X, Y) = 0 \qquad \ldots \quad (1.9)$$

is satisfied, is called an almost Kähler space.

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An almost Hermite space, for which

$$(D_X f)Y+(D_Y f)X = 0 \quad \ldots \quad (1.10a)$$

or

$$(D_X F)(Y, Z)+(D_Y F)(X, Z) = 0 \quad \ldots \quad (1.10b)$$

is satisfied, is called an almost Tachibana space.

Nijenhuis tensor N of the type (1, 2) for f, of an almost Hermite space, is given by

$$N(X, Y) = -N(Y, X) = [\bar{X}, \bar{Y}]-[X, Y]-\overline{[X, \bar{Y}]}-\overline{[\bar{X}, Y]} \quad \ldots \quad (1.11a)$$

where

$$[X, Y] = D_X Y-D_Y X. \quad \ldots \quad (1.11b)$$

Let us put (Mishra 1967)

$$P(X, Y) \overset{\text{def}}{=} [\bar{X}, \bar{Y}]-\overline{[X, \bar{Y}]} \quad \ldots \quad (1.12)$$

$$Q(X, Y) \overset{\text{def}}{=} [\bar{X}, \bar{Y}]-\overline{[\bar{X}, Y]} \quad \ldots \quad (1.13)$$

$$T(X, Y) \overset{\text{def}}{=} [\bar{X}, \bar{Y}]-[X, Y]. \quad \ldots \quad (1.14)$$

Then

$$N(X, Y) = P(X, Y)-P(\bar{X}, \bar{Y}) \quad \ldots \quad (1.15)$$

$$N(X, Y) = Q(X, Y)-Q(\bar{X}, \bar{Y}) \quad \ldots \quad (1.16)$$

$$N(X, Y) = T(X, Y)+\overline{T(X, \bar{Y})}. \quad \ldots \quad (1.17)$$

If we put

$$n(X, Y, Z) \overset{\text{def}}{=} g(N(X, Y), Z) \quad \ldots \quad (1.18)$$

$$M(X, Y) \overset{\text{def}}{=} D_{\bar{X}}\bar{Y}-D_X Y-\overline{D_X \bar{Y}}-\overline{D_{\bar{X}} Y} \quad \ldots \quad (1.19a)$$

$$m(X, Y, Z) \overset{\text{def}}{=} g(M(X, Y), Z) \quad \ldots \quad (1.19b)$$

then

$$m(X, Y, Z) = (D_{\bar{X}}F)(Y, Z)+(D_X F)(Y, \bar{Z}) \quad \ldots \quad (1.20)$$

$$N(X, Y) = M(X, Y)-M(Y, X) \quad \ldots \quad (1.21a)$$

$$n(X, Y, Z) = m(X. Y, Z)-m(Y, X, Z). \quad \ldots \quad (1.21b)$$

From (1.20) it is clear that $M(X, Y, Z)$ is skew-symmetric in Y, Z.

We shall have occasion to use the following result:

Theorem (1.1). We have

$$(D_X F)(Y, \bar{Z}) = (D_X F)(\bar{Y}, Z) \quad \ldots \quad (1.22)$$

$$(D_X F)(\bar{Y}, \bar{Z}) = -(D_X F)(Y, Z). \quad \ldots \quad (1.23)$$

PROOF. In consequence of (1.1) and (1.3), we have

$$\begin{aligned}(D_X F)(\bar{Y}, Z) &= (D_X F)(f(Y), Z) = -F((D_X f)Y, Z)\\ &= (D_X F)(Y, f(Z)) = (D_X F)(Y, \bar{Z})\end{aligned}$$

which is the equation (1.22).

Barring Y in (1.22) and using (1.4), we obtain (1.23).

Note (1.1): It may be noted that all the equations given above or those which follow hold for arbitrary X, Y, Z. Hence barring X or Y or Z in an equation throughout does not affect the validity of the equation in that particular case.

2. Kähler Space

Theorem (2.1). The necessary and sufficient condition for a Hermite space M_{2n} to be the Kähler space is

$$D_X\bar{Y} = \overline{D_X Y} \qquad \text{(2.1}a\text{)}$$

equivalent to

$$D_X Y + \overline{D_X \bar{Y}} = 0 \qquad \text{(2.1}b\text{)}$$

or

$$D_{\bar{X}}\bar{Y} = \overline{D_{\bar{X}} Y}. \qquad \text{(2.1}c\text{)}$$

Proof. We have

$$D_X\bar{Y} = (D_X f)Y + \overline{D_X Y}.$$

Substituting from (1.8) in this equation, we obtain (2.1a). Barring Y and X respectively in (2.1a) and using (1.4), we obtain (2.1b, c).

Corollary (2.1): For a Kähler space, we have

$$[X, Y] = \overline{D_Y\bar{\bar{X}}} - \overline{D_X\bar{\bar{Y}}} \qquad \text{(2.2}a\text{)}$$

$$[\bar{X}, \bar{Y}] = \overline{D_{\bar{X}} Y} - \overline{D_{\bar{Y}} X} \qquad \text{(2.2}b\text{)}$$

$$[X, \bar{Y}] = \overline{D_{\bar{Y}}\bar{\bar{X}}} + \overline{D_X Y} \qquad \text{(2.2}c\text{)}$$

$$[\bar{X}, Y] = -\overline{D_Y X} - \overline{D_{\bar{X}}\bar{\bar{Y}}}. \qquad \text{(2.2}d\text{)}$$

Proof. Interchanging X, Y in (2.1b), subtracting the resulting equation from (2.1b) and using (1.11b), we obtain (2.2a). The equations (2.2b, c, d) follow from (2.2a) by barring X or Y and using (1.4).

Corollary (2.2): For a Kähler space

$$P(X, Y) = P(\bar{X}, \bar{Y}) \qquad \text{(2.3}a\text{)}$$

$$Q(X, Y) = Q(\bar{X}, \bar{Y}) \qquad \text{(2.3}b\text{)}$$

$$-T(X, Y) = \overline{T(X, \bar{Y})} = \overline{T(\bar{X}, Y)} \qquad \text{(2.3}c\text{)}$$

$$M(X, Y) = 0. \qquad \text{(2.4)}$$

Proof. Subtracting (2.2a) from (2.2b) and using (1.14) and (1.11b), we obtain (2.3c). The equations (2.3a, b) can be obtained similarly.

Substituting from (1.8) in (1.20) we obtain

$$m(X, Y, Z) = 0$$

which implies (2.4).

Note (2.1): Equations (2.3) and (2.4) are statements of the fact that for a Kähler space, Nijenhuis tensor vanishes, a result already known.

3. Almost Tachibana Space

Theorem (3.1). The necessary and sufficient condition for an almost Hermite space to be an almost Tachibana space is

$$(D_{\bar{X}}F)(Y, Z)+(D_{Y}F)(X, \bar{Z}) = 0 \qquad (3.1a)$$

or

$$(D_{\bar{X}}F)(\bar{Y}, Z)+(D_{\bar{Y}}F)(X, \bar{Z}) = 0 \qquad (3.1b)$$

or

$$(D_{\bar{X}}F)(\bar{Y}, Z) = (D_{Y}F)(X, Z) \qquad (3.1c)$$

or

$$(D_{\bar{X}}F)(Y, Z)+(D_{\bar{Y}}F)(X, Z) = 0. \qquad (3.1d)$$

Proof. Barring X in

$$(D_{X}F)(Y, Z)+(D_{Y}F)(X, Z) = 0 \qquad (3.1e)$$

and using (1.22), we obtain (3.1*a*). Barring Y in (3.1*a*), we obtain (3.1*b*). Barring Z in (3.1*a*) and using (1.22), we obtain (3.1*c*). Barring Y in (3.1*c*) and using (1.4), we obtain (3.1*d*).

Note (3.1): Some more relations of the type (3.1) can be given by barring X or Y or Z in (3.1*e*), but they reduce to (3.1*a*, *b*, *c*, *d*).

Theorem (3.2). The necessary and sufficient condition for an almost Hermite space to be an almost Tachibana space is

$$D_{X}\bar{Y}+D_{Y}\bar{X} = \overline{D_{X}Y}+\overline{D_{Y}X} \qquad (3.2a)$$

equivalent to

$$D_{\bar{X}}\bar{Y}-D_{Y}X = \overline{D_{\bar{X}}Y}+\overline{D_{Y}\bar{X}} \qquad (3.2b)$$

or

$$-D_{\bar{X}}Y-D_{\bar{Y}}X = \overline{D_{\bar{X}}\bar{Y}}+\overline{D_{\bar{Y}}\bar{X}}. \qquad (3.2c)$$

Proof. We have

$$D_{X}\bar{Y}+D_{Y}\bar{X} = (D_{X}f)Y+(D_{Y}f)X+\overline{D_{X}Y}+\overline{D_{Y}X}.$$

Substituting from (1.10) in this equation, we obtain (3.2*a*). Barring X in (3.2*a*) and using (1.4), we obtain (3.2*b*). (3.2*c*) follows from (3.2*b*) by barring Y and using (1.4).

Theorem (3.3). The necessary and sufficient condition for an almost Hermite space to be an almost Tachibana space is

$$M(X, Y) = \overline{D_{Y}\bar{X}}-\overline{D_{X}\bar{Y}}-[X, Y] \qquad (3.3)$$

equivalent to

$$N(X, Y) = 2\overline{D_{Y}\bar{X}}-2\overline{D_{X}\bar{Y}}-2[X, Y]. \qquad (3.4)$$

Consequently, for an almost Tachibana space $M(X, Y)$ is skew-symmetric.

Also the condition that an almost Tachibana space be a Kähler space is that Nijenhuis tensor vanishes.

PROOF. The eqn. (3.2*b*) implies

$$D_{\overline{X}}\overline{Y}-\overline{D_{\overline{X}}Y}-D_XY-\overline{D_X\overline{Y}} = D_YX+\overline{D_Y\overline{X}}-D_XY-\overline{D_X\overline{Y}}$$

which in consequence of (1.19*a*) and (1.11*b*) implies

$$M(X, Y) = \overline{D_Y\overline{X}}-\overline{D_X\overline{Y}}-[X, Y].$$

This is the equation (3.3). The eqn. (3.4) follows from (3.3) and (1.21*a*).

Theorem (3.4). For an almost Tachibana space, $M(X, Y, Z)$ is completely skew-symmetric.

PROOF. It is known that $m(X, Y, Z)$ is skew-symmetric in Y, Z. Also, in consequence of Theorem (3.3), $m(X, Y, Z)$ is skew-symmetric in X, Y for an almost Tachibana space. Hence it is completely skew-symmetric.

Note (3.2): In consequence of (1.21*b*) and Theorem (3.4), $n(X, Y, Z)$ is skew-symmetric in all the vectors X, Y, Z, a result already known (Yano 1965).

Theorem (3.5). For an almost Tachibana space $m(X, Y, \overline{Z})$ is skew-symmetric in all the vectors X, Y, Z:

$$m(X, Y, \overline{Z}) = -m(Y, X, \overline{Z}) = -m(Z, Y, \overline{X}) = -m(X, Z, \overline{Y}). \qquad (3.5)$$

PROOF. In view of Theorem (3.4) and the fact (Mishra 1967)

$$m(X, Y, \overline{Z}) = m(X, \overline{Y}, Z) = m(\overline{X}, Y, Z) \quad .. \quad .. \quad (3.6a)$$

we have

$$m(X, Y, \overline{Z}) = -m(Y, X, \overline{Z})$$

$$m(X, Y, \overline{Z}) = -m(\overline{Z}, Y, X) = -m(Z, Y, \overline{X})$$

$$m(X, Y, \overline{Z}) = -m(X, \overline{Z}, Y) = -m(X, Z, \overline{Y}).$$

Note (3.3): In consequence of (1.21*b*) and Theorem (3.5), $n(X, Y, \overline{Z})$ is skew-symmetric in all the vectors X, Y, Z, a result already known (Yano 1965).

Theorem (3.6). For an almost Tachibana space $m(X, \overline{Y}, \overline{Z})$, $m(\overline{X}, Y, \overline{Z})$, $m(\overline{X}, \overline{Y}, Z)$ and consequently $n(X, \overline{Y}, \overline{Z})$, $n(\overline{X}, Y, \overline{Z})$, $n(\overline{X}, \overline{Y}, Z)$ are skew-symmetric in all the vectors X, Y, Z.

PROOF. The first part of the statement follows from Theorem (3.4) and the fact (Mishra 1967)

$$m(X, \overline{Y}, \overline{Z}) = m(\overline{X}, Y, \overline{Z}) = m(\overline{X}, \overline{Y}, Z) = -m(X, Y, Z). \qquad (3.6b)$$

Second part of the statement follows from (1.21*b*) and Note (3.2).

4. Almost Kähler Space

Theorem (4.1). The necessary and sufficient condition for an almost Hermite space to be an almost Kähler space is

$$(D_{\overline{X}}F)(Y, Z)+(D_YF)(Z, \overline{X})+(D_ZF)(\overline{X}, Y) = 0 \quad .. \quad .. \quad (4.1)$$

or

$$(D_{\overline{X}}F)(\overline{Y}, Z)+(D_{\overline{Y}}F)(Z, \overline{X})-(D_{Z}F)(X, Y) = 0 \quad .. \quad .. \quad (4.2)$$

or

$$(D_{\overline{X}}F)(Y, Z)+(D_{\overline{Y}}F)(Z, X)+(D_{\overline{Z}}F)(X, Y) = 0. \quad .. \quad (4.3)$$

PROOF. Barring X in (1.9) we obtain (4.1). Similarly, barring Y in (4.1) and Z in (4.2) and using (1.23), we obtain (4.2) and (4.3) respectively.

Theorem (4.2). The necessary and sufficient condition for a Hermite space to be an almost Kähler space is

$$Z(F(X, Y))+X(F(Y, Z))+Y(F(Z, X))$$
$$= F([X, Y], Z)+F([Y, Z], X)+F([Z, X], Y). \quad .. \quad (4.4)$$

PROOF. We have

$$Z(F(X, Y)) = (D_ZF)(X, Y)+F(D_ZX, Y)+F(X, D_ZY).$$

Writing two other equations by cyclic permutation of X, Y, Z, adding the three equations thus obtained and using (1.9), we obtain

$$Z(F(X, Y))+Y(F(Z, X))+X(F(Y, Z))$$
$$= F(D_ZX, Y)+F(D_XY, Z)+F(D_YZ, X)$$
$$+F(X, D_ZY)+F(Y, D_XZ)+F(Z, D_YX).$$

Using the skew-symmetry of F and (1.11*b*) in this equation, we obtain (4.4).

Corollary (4.1): For an almost Kähler space we have

$$(D_XF)(Y, \overline{Z})+(D_YF)(Z, \overline{X})+(D_ZF)(X, \overline{Y}) = 0 \quad .. \quad (4.5a)$$

or

$$(D_{\overline{X}}F)(\overline{Y}, Z)+(D_{\overline{Y}}F)(\overline{Z}, X)+(D_{\overline{Z}}F)(\overline{X}, Y) = 0. \quad .. \quad (4.5b)$$

PROOF. Writing two other equations by cyclic permutation of X, Y, Z in (4.1), adding these equations to (4.1) and making use of (4.3) and (1.22), we obtain (4.5*a*). Similarly writing (4.2) by cyclic permutation of X, Y, Z, adding the three equations and using (1.9), we obtain (4.5*b*).

Theorem (4.3). For an almost Kähler space we have

$$m(\overline{X}, Y, Z)+m(\overline{Y}, Z, X)+m(\overline{Z}, X, Y) = 0. \quad .. \quad (4.6)$$

PROOF. Barring X in (1.20) and using (1.4), we get

$$m(\overline{X}, Y, Z) = -(D_XF)(Y, Z)+(D_{\overline{X}}F)(\overline{Y}, Z).$$

Adding to this equation two other equations obtained by cyclic permutation of X, Y, Z and using (1.9) and (4.5*b*), we obtain (4.6).

Corollary (4.2): The equation (4.6) is equivalent to

$$m(X, Y, Z)+m(Y, Z, X)+m(Z, X, Y) = 0 \quad .. \quad .. \quad (4.7a)$$

or

$$n(X,\ Y,\ Z)+n(Y,\ Z,\ X)+n(Z,\ X,\ Y)=0. \qquad \text{(4.7}b\text{)}$$

PROOF. Barring X, Y, Z in (4.5) and using (3.6) and (1.4), we obtain (4.7a). This equation can also be obtained directly from (1.20), (4.3) and (4.5a). Using (1.21b) in this equation, we obtain (4.7b).

Theorem (4.4). For an almost Kähler space, we have

$$n(X,\ Y,\ \bar{Z})+n(Y,\ Z,\ \bar{X})+n(Z,\ X,\ \bar{Y})=0 \qquad \text{(4.8}a\text{)}$$

$$T(X,\ \bar{Y},\ Z)+T(Y,\ \bar{Z},\ X)+T(Z,\ \bar{X},\ Y)=0 \qquad \text{(4.8}b\text{)}$$

$$T(\bar{X},\ Y,\ Z)+T(\bar{Y},\ Z,\ X)+T(\bar{Z}.\ X,\ Y)=0 \qquad \text{(4.8}c\text{)}$$

equivalent to

$$\left.\begin{aligned} T(X,\ Y,\ Z)+T(Z,\ X,\ Y) &= T(Y,\ \bar{Z},\ \bar{X}) \\ T(Y,\ Z,\ X)+T(X,\ Y,\ Z) &= T(Z,\ \bar{X},\ \bar{Y}) \\ T(Z,\ X,\ Y)+T(Y,\ Z,\ X) &= T(X,\ \bar{Y},\ \bar{Z}) \end{aligned}\right\} \qquad \text{(4.9}a\text{)}$$

$$T(X,\ Y,\ \bar{Z})+T(Y,\ Z,\ \bar{X})+T(Z,\ X,\ \bar{Y})=0. \qquad \text{(4.9}b\text{)}$$

PROOF. In consequence of (4.4), (1.7) and (1.11), we have

$$\begin{aligned} &F(N(X,\ Y),\ Z)+F(N(Y,\ Z),\ X)+F(N(Z,\ X),\ Y) \\ &= -F(\overline{[X,\ \bar{Y}]},\ Z)-F(\overline{[Y,\ \bar{Z}]},\ X)-F(\overline{[Z,\ \bar{X}]},\ Y) \\ &\quad -F(\overline{[\bar{X},\ Y]},\ Z)-F(\overline{[\bar{Y},\ Z]},\ X)-F(\overline{[\bar{Z},\ X]},\ Y). \end{aligned}$$

Using eqns. (1.7), (1.14) and (1.5) in this equation, we get

$$\begin{aligned} &n(X,\ Y,\ \bar{Z})+n(Y,\ Z,\ \bar{X})+n(Z,\ X,\ \bar{Y}) \\ &= T(X,\ \bar{Y},\ Z)+T(Y,\ \bar{Z},\ X)+T(Z,\ \bar{X},\ Y) \\ &\quad +T(\bar{X},\ Y,\ Z)+T(\bar{Y},\ Z,\ X)+T(\bar{Z},\ X,\ Y). \end{aligned} \qquad \text{(4.10)}$$

Barring Z in (4.7b) and using (Mishra 1967)

$$n(X,\ Y,\ \bar{Z})=n(\bar{X},\ Y,\ Z)=n(X,\ \bar{Y},\ Z) \qquad \text{(4.11)}$$

we obtain (4.8a). Equations (4.8b, c) follow from eqns. (4.10) and (4.8a). Barring Z in (4.8b) and using (1.4) and (1.14), we get

$$T(X,\ \bar{Y},\ \bar{Z})=T(Y,\ Z,\ X)+T(Z,\ X,\ Y).$$

This proves one of the equations (4.9a). The other equations (4.9a) can be proved similarly. The equation (4.9b) follows from eqns. (4.8b, c) and (4.9a).

Theorem (4.5). For an almost Kähler space we have

$$T(X,\ Y,\ Z)+T(Y,\ Z,\ X)+T(Z,\ X,\ Y)=0 \qquad \text{(4.12)}$$

$$T(X,\ \bar{Y},\ \bar{Z})+T(Y,\ \bar{Z},\ \bar{X})+T(Z,\ \bar{X},\ \bar{Y})=0. \qquad \text{(4.13)}$$

PROOF. Adding the three equations (4.9a) we get

$$2T(X, Y, Z)+2T(Y, Z, X)+2T(Z, X, Y)$$
$$= T(X, \overline{Y}, \overline{Z})+T(Y, \overline{Z}, \overline{X})+T(Z, \overline{X}, \overline{Y}). \quad \ldots \quad \ldots \quad (4.14)$$

Also from (1.17) and (4.7b), we get

$$n(X, Y, Z)+n(Y, Z, X)+n(Z, X, Y) = 0 = T(X, Y, Z)+T(Y, Z, X)$$
$$+T(Z, X, Y)-T(X, \overline{Y}, \overline{Z})-T(Y, \overline{Z}, \overline{X})-T(Z, \overline{X}, \overline{Y}). \quad \ldots \quad (4.15)$$

From (4.14) and (4.15) we get (4.12) and (4.13).

Theorem (4.6). The necessary and sufficient condition that an almost Kähler space be an almost Tachibana space is

$$m(X, Y, Z)+m(Y, X, Z) = 0 \quad \ldots \quad \ldots \quad \ldots \quad (4.16)$$

that is, m is completely skew-symmetric.

PROOF. From (1.20) we have

$$m(X, Y, Z)+m(Y, X, Z) = -(D_{\overline{X}}F)(Y, Z)+(D_XF)(Y, \overline{Z})+(D_{\overline{Y}}F)(X, Z)$$
$$+(D_YF)(X, \overline{Z}). \quad \ldots \quad \ldots \quad \ldots \quad (4.17)$$

Substituting from (4.1) in this equation and using (1.22) and skew-symmetry of F, we obtain

$$m(X, Y, Z)+m(Y, X, Z) = -(D_YF)(Z, \overline{X})-(D_ZF)(\overline{X}, Y)+(D_XF)(Y, \overline{Z})$$
$$-(D_XF)(Z, \overline{Y})-(D_ZF)(\overline{Y}, X)+(D_YF)(X, \overline{Z})$$
$$= 2(D_XF)(Y, \overline{Z})+2(D_YF)(X, \overline{Z}).$$

The statement follows from this equation and (3.1e).

Theorem (4.7). The necessary and sufficient condition that an almost Kähler space be an almost Tachibana space is

$$m(X, Y, \overline{Z}) = -m(\overline{Z}YX) = -m(Z\overline{Y}X) = -m(ZY\overline{X}) \quad \ldots \quad (4.18)$$

or

$$n(Y, Z, \overline{X}) = -n(Y, \overline{X}, Z) = -m(\overline{Y}, X, Z) = -n(Y, X, \overline{Z}). \quad (4.19)$$

PROOF. Barring Z in (4.16) and using (3.6a) we obtain (4.18). The equations (4.19) follow from (4.18) and (1.21b).

Theorem (4.8). If an almost Hermite space has any two of the following properties, it has the third also:

(a) it is an almost Kähler space,
(b) it is an almost Tachibana space,
(c) it is a space for which

$$2(D_XF)(Y, Z)+(D_ZF)(X, Y) = 0 \quad \ldots \quad \ldots \quad (4.20)$$

is satisfied.

PROOF. Let us put

$$G(X, Y, Z) = (D_X F)(Y, Z) + (D_Y F)(X, Z) \quad \ldots \quad \ldots \quad \ldots \quad (4.21a)$$

$$'F(X, Y, Z) = (D_X F)(Y, Z) + (D_Y F)(Z, X) + (D_Z F)(X, Y). \quad \ldots \quad (4.21b)$$

Then we have from the skew-symmetry of F

$$G(X, Y, Z) + 'F(X, Y, Z) = 2(D_X F)(Y, Z) + (D_Z F)(X, Y).$$

This equation proves the statement.

Corollary (4.3): The equation (4.20) is equivalent to

$$2(D_X F)(Y, Z) + (D_Z F)(\bar{X}, Y) = 0 \quad \ldots \quad \ldots \quad \ldots \quad (4.22a)$$

or

$$2(D_X F)(\bar{Y}, Z) + (D_Z F)(X, \bar{Y}) = 0 \quad \ldots \quad \ldots \quad \ldots \quad (4.22b)$$

or

$$2(D_X F)(Y, \bar{Z}) + (D_{\bar{Z}} F)(X, Y) = 0 \quad \ldots \quad \ldots \quad \ldots \quad (4.22c)$$

or

$$2(D_{\bar{X}} F)(\bar{Y}, Z) = (D_Z F)(X, Y) \quad \ldots \quad \ldots \quad \ldots \quad (4.22d)$$

or

$$2(D_{\bar{X}} F)(Y, \bar{Z}) + (D_{\bar{Z}} F)(\bar{X}, Y) = 0 \quad \ldots \quad \ldots \quad \ldots \quad (4.22e)$$

or

$$2(D_{\bar{X}} F)(Y, Z) + (D_{\bar{Z}} F)(X, Y) = 0 \quad \ldots \quad \ldots \quad \ldots \quad (4.22f)$$

or

$$2(D_X F)(Y, Z) = (D_{\bar{Z}} F)(X, \bar{Y}). \quad \ldots \quad \ldots \quad \ldots \quad (4.22g)$$

PROOF. Equations (4.22) follow from eqn. (4.20) by barring X, Y or Z and using eqns. (1.22) and (1.23).

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