A NOTE ON FINITE STRUVE TRANSFORMS

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During the last few years the theory of integral transforms (finite and infinite), subject to the choice of the kernels, occurring in the integrand as circular functions and Bessel functions, has been developed with the purpose of solving certain differential equations arising in problems of mathematical physics and other branches of applied sciences. The object of the present work is to study the finite Struve transform:

$$S_{a,\,b,\,\mu}[f(\xi_i)] = \int_a^b x\,H_\mu(x\xi_i)f(x)\,dx$$

under the assumption that ξ_i is a positive root of either of the transcendental equations

$$(a)\quad X\,J_\mu'(ax)+h\,J_\mu(ax) = 0$$

or

$$(b)\quad J_\mu(ax)G_\mu(bx)-J_\mu(bx)G_\mu(ax) = 0.$$

1. Introduction

Theory of Fourier-Bessel series has been utilized previously (Sneddon 1951) to determine the function $f(x)$, satisfying the integral equation

$$H_{0,\,a,\,\mu}[f(\xi_i)] = \int_0^a x\,J_\mu(x\xi_i)\,f(x)\,dx \quad \ldots \quad \ldots \quad \ldots \quad (1.1)$$

under the assumptions that ξ_i is a positive root of either of the transcendental equations

$$\text{(i)}\ \ J_\mu(aX) = 0; \quad \text{(ii)}\ \ X\,J_\mu'(aX)+h\,J_\mu(aX) = 0. \quad \ldots \quad \ldots \quad (1.2)$$

More generally the problem of determining the function $f(x)$, satisfying the equation

$$H_{a,\,b,\,\mu}[f(\xi_i)] = \int_a^b x\,F_{x,\,b,\,\mu}[\xi_i]f(x)\,dx \quad \ldots \quad \ldots \quad (1.3)$$

where ξ_i is a positive root of the transcendental equation

$$F_{a,\,b,\,\mu}(X) \equiv J_\mu(aX)G_\mu(bX)-J_\mu(bX)G_\mu(aX) = 0 \quad \ldots \quad \ldots \quad (1.4)$$

with the addition that

$$G_\mu(X) = \frac{\pi}{2\sin\mu\pi}\,[J_\mu(X)-e^{-i\mu\pi}J_{-\mu}(X)] \quad \ldots \quad \ldots \quad (1.5)$$

has also been solved (Sneddon 1951).

The object of the present paper is to obtain the function $f(x)$ connected with the finite Struve transform $S_{a,\,b,\,\mu}[f(\xi_i)]$ by the relation

$$S_{a,\,b,\,\mu}[f(\xi_i)] = \int_a^b xH_\mu(x\xi_i)\,f(x)\,dx \qquad \ldots \quad \ldots \quad \ldots \quad (1.6)$$

where $H_\mu(x\xi_i)$ is the Struve function of order μ (Bhowmick 1965); subject to the assumptions under eqns. (1.2) and (1.4).

In particular, for $\mu = \pm\frac{1}{2}$, eqn. (1.6) assumes the forms

$$S_{a,\,b,\,\frac{1}{2}}[f(\xi_i)] = \left(\frac{2}{\pi\xi_i}\right)^{\frac{1}{2}} \int_a^b \sqrt{x}f(x)\,dx - \left(\frac{2}{\pi\xi_i}\right)^{\frac{1}{2}} C_a^b\{\sqrt{\xi_i}f(\xi_i)\} \qquad \ldots \quad (1.7)$$

and

$$S_{a,\,b,\,-\frac{1}{2}}[f(\xi_i)] = \left(\frac{2}{\pi\xi_i}\right)^{\frac{1}{2}} S_a^b\{\sqrt{\xi_i}f(\xi_i)\} \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (1.8)$$

where $C_a^b\{\sqrt{\xi_i}f(\xi_i)\}$ and $S_a^b\{\sqrt{\xi_i}f(\xi_i)\}$ are finite cosine and sine transforms of $\sqrt{x}\,f(x)$ over the interval $(a \leqslant \xi_i \leqslant b)$.

It is remarkable to note, by virtue of the result (Baudox 1946)

$$H_\mu(x) = \frac{2}{\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{(x/2)^{\frac{1}{2}+k}}{\lfloor k\,(2k+1)} J_{\mu+k+\frac{1}{2}}(x). \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (1.9)$$

that

$$S_{a,\,b,\,\mu}[f(\xi_i)] = 2\pi^{-\frac{1}{2}} \sum_{k=0}^{\infty} \frac{(\xi_i/2)^{\frac{1}{2}+k}}{\lfloor k\,(2k+1)} H_{a,\,b,\,\frac{1}{2}+\mu+k}\left[\xi_i^{\frac{1}{2}+k} f(\xi_i)\right]. \quad \ldots \quad (1.10)$$

2. Inversion Theorems of $S_{a,\,b,\,\mu}[f(\xi_i)]$

Theorem A: If $f(x)$ satisfies Dirichlet's conditions in the interval $(0 < a \leqslant x \leqslant b)$ and has its own finite Struve transform $S_{a,\,b,\,\mu}[f(\xi_i)]$, then at each point of (a, b) at which $f(x)$ is continuous

$$f(x) = 2\sum_i \frac{\xi_i^2 J^2_{\frac{1}{2}+\mu}(a\xi_i) F_{x,\,b,\,\frac{1}{2}+\mu}(\xi_i) H_{a,\,b,\,\frac{1}{2}+\mu}[f(\xi_i)]}{J^2_{\frac{1}{2}+\mu}(b\xi_i) - J^2_{\frac{1}{2}+\mu}(a\xi_i)} \qquad \ldots \quad (2.1)$$

where the summation is taken over all the positive roots of

$$F_{a,\,b,\,\frac{1}{2}+\mu}(X) = 0 \qquad \ldots \quad \ldots \quad \ldots \quad (2.2)$$

and

$$\begin{aligned} \xi_i^2 H_{a,\,b,\,\frac{1}{2}+\mu}[f(\xi_i)] = \frac{\sqrt{\pi}e^{-j(\frac{1}{2}+\mu)\pi}}{\sqrt{2}\sin(\frac{1}{2}+\mu)\pi} \Bigg[& \xi_i^{-\frac{1}{2}-\mu} J_{\frac{1}{2}+\mu}(b\xi_i) \int_0^{\xi_i} u^{\mu+2}(\xi_i^2 - u^2)^{-\frac{1}{2}} \\ & \times \{(2\mu+3)S_{a,\,b,\,-1-\mu}(\sqrt{u}f(u)) + u\,S_{a,\,b,\,-2-\mu}(u^{\frac{3}{2}}f(u))\}\,du \\ & + \xi_i^{\frac{1}{2}+\mu} J_{-\frac{1}{2}-\mu}(b\xi_i) \int_0^{\xi_i} u^{1-\mu}(\xi_i^2 - u^2)^{-\frac{1}{2}} \{(1-2\mu)S_{a,\,b,\,\mu}(f(u)/\sqrt{u}) \\ & + u\,S_{a,\,b,\,\mu-1}(\sqrt{u}f(u))\}\,du \Bigg] \qquad \ldots \quad (2.3) \end{aligned}$$

where $j = \sqrt{-1}$.

By virtue of (1.6), we find

$$I_1 = \xi_i^{-\mu-\frac{3}{2}} J_{\frac{1}{2}+\mu}(b\xi_i) \int_0^{\pi/2} \frac{\partial}{\partial \xi_i} [(\xi_i \sin\theta)^{2+\mu} S_{a,b,-1-\mu}\{\sqrt{\xi_i \sin\theta}\, f(\xi_i \sin\theta)\}]\, d\theta$$

$$= \xi_i^{-\mu-\frac{3}{2}} J_{\frac{1}{2}+\mu}(b\xi_i) \int_0^{\pi/2} d\theta \int_a^b \frac{\partial}{\partial \xi_i} \{(\xi_i \sin\theta)^{2+\mu} H_{-1-\mu}(\xi_i \sin\theta\, t)\} \sqrt{t}\, f(t)\, dt$$

$$= \frac{\sqrt{\pi}}{\sqrt{2}} J_{\frac{1}{2}+\mu}(b\xi_i) \int_a^b t\, J_{-\frac{1}{2}-\mu}(t\xi_i) f(t)\, dt$$

and on the same lines it can be shown that

$$I_2 = \xi_i^{\mu-\frac{1}{2}} J_{-\frac{1}{2}-\mu}(b\xi_i) \int_0^{\pi/2} \frac{\partial}{\partial \xi_i} [(\xi_i \sin\theta)^{1-\mu} S_{a,b,\mu}\{\sqrt{\xi_i \sin\theta}\, f(\xi_i \sin\theta)\}]\, d\theta$$

$$= \frac{\sqrt{\pi}}{\sqrt{2}} J_{-\frac{1}{2}-\mu}(b\xi_i) \int_a^b t\, J_{\frac{1}{2}+\mu}(t\xi_i) f(t)\, dt.$$

Thus,

$$I_1 - I_2 = \frac{\sqrt{\pi}}{\sqrt{2}} \int_a^b t[J_{\frac{1}{2}+\mu}(b\xi_i) J_{-\frac{1}{2}-\mu}(t\xi_i) - J_{\frac{1}{2}+\mu}(t\xi_i) J_{-\frac{1}{2}-\mu}(b\xi_i)] f(t)\, dt$$

$$= j \frac{\sqrt{2}}{\sqrt{\pi}} \cos \mu\pi\, e^{j\mu\pi} H_{a,b,\frac{1}{2}+\mu}[f(\xi_i)]$$

by virtue of (1.3) and (1.5).

Applying inversion theorem for $H_{a,b,\frac{1}{2}+\mu}[f(\xi_i)]$ (Sneddon 1955, p. 85) we obtain (2.1) subject to the condition (2.2).

Again, by virtue of the result (Watson 1958, p. 329)

$$\frac{d}{dx}\{x^{\mu} H_{\mu}(x)\} = x^{\mu} H_{\mu-1}(x) \quad \cdots \quad \cdots \quad \cdots \quad (2.4)$$

we find

$$I_1 = \xi_i^{-\mu-\frac{3}{2}} J_{\frac{1}{2}+\mu}(b\xi_i) \int_0^{\pi/2} \sin\theta\, d\theta \int_a^b t^{-\frac{1}{2}-\mu} f(t)\{(2\mu+3)(\xi_i t \sin\theta)^{1+\mu} H_{-1-\mu}(t\xi_i \sin\theta)$$

$$+ (\xi_i t \sin\theta)^{\mu+2} H_{-2-\mu}(t\xi_i \sin\theta)\}\, dt$$

$$= \xi_i^{-\frac{3}{2}-\mu} J_{\frac{1}{2}+\mu}(b\xi_i) \int_0^{\xi_i} u^{\mu+2}(\xi_i^2 - u^2)^{-\frac{1}{2}} \{(2\mu+3) S_{a,b,-1-\mu}(f(u)/\sqrt{u})$$

$$+ u\, S_{a,b,-\mu-2}(\sqrt{u}\, f(u))\}\, du$$

and similarly

$$I_2 = \xi_i^{\mu-\frac{1}{2}} J_{-\frac{1}{2}-\mu}(b\xi_i) \int_0^{\xi_i} \frac{u^{1-\mu}}{\sqrt{\xi_i^2 - u^2}} \left\{(1-2\mu) S_{a,b,\mu}(f(u)/u\sqrt{u}) + u\, S_{a,b,\mu-1}\left(\frac{f(u)}{\sqrt{u}}\right)\right\} du.$$

Hence (2.3) is proved.

2a. Inversion Theorems of $S_{0,b,\mu}[f(\xi_i)]$

Theorem A: If $f(x)$ satisfies Dirichlet's conditions in $(0, b)$ and has its own transform $S_{0,b,\mu}[f(\xi_i)]$, then at each point of $(0, b)$ at which $f(x)$ is continuous

$$f(x) = \sum_i \frac{\xi_i^{\frac{1}{2}+\mu} J_{\frac{1}{2}+\mu}(x\xi_i) H_{0,b,\frac{1}{2}+\mu}[f(\xi_i)]}{\left[h^2 - \left(\frac{\mu+\frac{1}{2}}{b}\right)^2 + \xi_i^2\right] J'^2_{\frac{1}{2}+\mu}(b\xi_i)} \quad \ldots \quad \ldots (2a.1)$$

where the summation is taken over all positive roots of the transcendental equation

$$xJ'_{\frac{1}{2}+\mu}(xb) + hJ_{\frac{1}{2}+\mu}(xb) = 0 \quad \ldots \quad \ldots \quad \ldots (2a.2)$$

where

$$H_{0,b,\frac{1}{2}\mu}[f(\xi_i)] = \sqrt{\frac{2}{\pi\xi_i}} \int_0^1 \frac{u^{1-\mu}}{\sqrt{1-u^2}} \left[(1-2\mu) S_{0,b,\mu}\left\{\frac{f(\xi_i u)}{\sqrt{\xi_i u}}\right\} + u\xi_i S_{0,b,\mu-1}\{\sqrt{\xi_i u}\, f(\xi_i u)\}\right] du. \quad \ldots \quad \ldots (2a.3)$$

Evidently,

$$\int_0^{\pi/2} \frac{\partial}{\partial \xi_i} [(\xi_i \sin\theta)^{1-\mu} S_{0,b,\mu}\{f(\xi_i \sin\theta)/\sqrt{\xi_i \sin\theta}\}]\, d\theta$$

$$= \sqrt{\frac{\pi}{2}}\, \xi_i^{\frac{1}{2}-\mu} H_{0,b,\frac{1}{2}+\mu}[f(\xi_i)] \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots (2a.4)$$

which on inverting (Sneddon 1955, p. 84) gives eqn. (2a.1), and eqn. (2a.3) follows obviously from (2.4).

Theorem B: If $f(x)$ satisfies Dirichlet's conditions in $(0, b)$ and has its own Struve transform $S_{0,b,\mu}[f(\xi_i)]$, then at each point of $(0, b)$ at which $f(x)$ is continuous

$$x^{\mu+1} f(x) = \frac{2^{1+\mu}\sqrt{\pi}}{b\Gamma(\frac{1}{2}-\mu)} \sum_i \sin(\xi_i x) S_0^b\{\xi_i^{\mu+1} f(\xi_i)\} \quad \ldots \quad \ldots (2a.5)$$

where

$$\xi_i^{\mu} S_0^b\{\xi_i^{\mu+1} f(\xi_i)\} = \int_0^1 u^{1+\mu}(1-u^2)^{-\frac{1}{2}-\mu} [(1-2\mu) S_{0,b,\mu}\{f(u\xi_i)/\sqrt{\xi_i u}\} + u\xi_i\, S_{0,b,\mu-1}\{f(\xi_i u)\}]\, du. \quad \ldots (2a.6)$$

Evidently,

$$I_3 = \int_0^{\pi/2} \sin^{\mu+1}\theta \cos^{-2\mu}\theta \frac{\partial}{\partial \xi_i}[\xi_i^{1-\mu} S_{0,b,\mu}\{f(\xi_i \sin\theta)\}]\, d\theta$$

$$= \int_0^{\pi/2} \sin^{\mu+1}\theta \cos^{-2\mu}\theta\, d\theta \int_0^b t \frac{\partial}{\partial \xi_i}[\xi_i^{1-\mu} H_\mu(t\xi_i \sin\theta)] f(t)\, dt$$

$$= 2^{-\mu}\pi^{-\frac{1}{2}}\Gamma(\tfrac{1}{2}-\mu) S_0^b\{\xi_i^{\mu+1} f(\xi_i)\}. \quad \ldots \quad \ldots \quad \ldots \quad \ldots (2a.7)$$

Inverting eqn. ($2a.7$) and using eqn. (2.4) we arrive at eqns. ($2a.5$) and ($2a.7$).

2b. Special Cases

(i) On putting $\mu = 0$, Theorem A under § 2 may be expressed as follows:

Theorem A: If $f(x)$ satisfies Dirichlet's conditions in $(0 < a \leqslant x \leqslant b)$ and has its own Struve transform $S_{a,b,0}[f(\xi_i)]$, then at each point of the interval at which $f(x)$ is continuous

$$\sqrt{x}f(x) = \frac{2j\sqrt{b}}{a-b}\sum_{n=0}^{\infty}\frac{\sin^2(an\pi/a-b)H_{a,b,\frac{1}{2}}\{f(n\pi/a-b)\}\sin\{n\pi(x-b)/a-b\}}{a\sin^2(bn\pi/a-b)-b\sin^2(an\pi/a-b)} \quad \ldots (2b.1)$$

where

$$\begin{aligned}H_{a,b,\frac{1}{2}}[f(\xi_i)] = &-\frac{j}{\sqrt{b\xi_i}}\{\tfrac{1}{2}\xi_i\cos b\xi_i+\sin b\xi_i\}\int_a^b t^{3/2}f(t)\,dt\\&+\frac{j}{\sqrt{b\xi_i^2}}\int_0^{\xi_i}\frac{u^2}{\sqrt{\xi_i^2-u^2}}\left\{\cos b\xi_i+\frac{\sin b\xi_i}{\xi_i}S_{a,b,1}(\sqrt{u}f(u))\right\}\\&-\frac{j}{\sqrt{b\xi_i^2}}\int_0^{\xi_i}\frac{u}{\sqrt{\xi_i^2-u^2}}\left\{\cos b\xi_i S_{a,b,0}(f(u)/\sqrt{u})\right.\\&\left.-\frac{u^2}{\xi_i}\sin b\xi_i S_{a,b,0}(u^{\frac{3}{2}}f(u))\right\}du\end{aligned} \quad \ldots\ \ldots\ \ldots (2b.2)$$

where

$$\xi_i = n\pi/a-b, \quad j = \sqrt{-1}. \quad \ldots\ \ldots\ \ldots (2b.3)$$

Obviously, for

$$\mu = 0, \quad F_{x,b,\frac{1}{2}}(\xi_i) = \frac{j\sin(x-b)\xi_i}{\sqrt{ab}\,\xi_i}$$

where ξ_i is a root of $F_{a,b,\frac{1}{2}}(X) = 0$, which furnishes solutions in the form $X = n\pi/a-b$, as indicated by ($2b.3$). Again, for $\mu = 0$, we observe that ($2b.2$) reduces to the form

$$\begin{aligned}\xi_i^2 H_{a,b,\frac{1}{2}}[f(\xi_i)] = -j\frac{\sqrt{\pi}}{\sqrt{2}}\Big[&\xi_i^{-\frac{1}{2}}J_{\frac{1}{2}}(b\xi_i)\int_0^{\xi_i}u^2(\xi_i^2-u^2)^{-\frac{1}{2}}\{3S_{a,b,-1}(\sqrt{u}f(u))\\&+uS_{a,b,-2}(u^{\frac{3}{2}}f(u))\}du+\xi_i^{\frac{1}{2}}J_{-\frac{1}{2}}(b\xi_i)\int_0^{\xi_i}u(\xi_i^2-u^2)^{-\frac{1}{2}}\{S_{a,b,0}(f(u)/\sqrt{u})\\&+uS_{a,b,-1}(\sqrt{u}f(u))\}du\Big]\end{aligned}$$

or

$$\begin{aligned}j\sqrt{b}\xi_i^2 H_{a,b,\frac{1}{2}}[f(\xi_i)] = &\left(\cos b\xi_i+\frac{3}{\xi_i}\sin b\xi_i\right)\int_0^{\xi_i}\frac{u^2}{\sqrt{\xi_i^2-u^2}}S_{a,b,-1}\{\sqrt{u}f(u)\}du\\&+\int_0^{\xi_i}\frac{u}{\sqrt{\xi_i^2-u^2}}\left[\cos b\xi_i S_{a,b,0}(f(u)/\sqrt{u})+\frac{u^2}{\xi_i}\sin b\xi_i S_{a,b,-2}(u^{\frac{3}{2}}f(u))\right]du\end{aligned}$$

using the result (Watson 1958)

$$\frac{2\mu}{x} H_\mu(x) = H_{\mu-1}(x) + H_{\mu+1}(x) - \frac{(x/2)^\mu}{\sqrt{\pi}\Gamma(\mu+\frac{3}{2})} \quad \ldots \quad \ldots (2b.4)$$

we get

$$S_{a,b,-1}\{f(u)\} = \frac{2}{\pi}\int_a^b t f(t)\,dt - S_{a,b,1}\{f(u)\} \quad \ldots \quad \ldots \quad \ldots (2b.5)$$

and

$$S_{a,b,-2}\{f(u)\} = \frac{2}{\pi u}\int_a^b f(t)\,dt - \frac{2}{u} S_{a,b,-1}\{f(u)/u\} - S_{a,b,0}\{f(u)\} \quad \ldots (2b.6)$$

and thus we obtain

$$j\sqrt{b}\xi_i^2 H_{a,b,\frac{1}{2}}[f(\xi_i)] = \int_0^{\xi_i} \frac{u^2}{\sqrt{\xi_i^2-u^2}} \left\{\cos b\xi_i + \frac{\sin b\xi_i}{\xi_i}\right\}$$

$$\times \left\{\frac{2}{\pi}\int_a^b t^{\frac{3}{2}} f(t)\,dt - S_{a,b,1}\left(\sqrt{u}\, f(u)\right)\right\} du$$

$$+ \int_0^{\xi_i} \frac{u}{\sqrt{\xi_i^2-u^2}} \left\{\cos b\xi_i\, S_{a,b,0}\left(f(u)/\sqrt{u}\right)\right.$$

$$\left. - \frac{u^2}{\xi_i} \sin b\xi_i\, S_{a,b,0}(u^{\frac{3}{2}} f(u))\right\} du + \tfrac{1}{2}\xi_i \sin b\xi_i \int_a^b t^{\frac{3}{2}} f(t)\,dt$$

$$= \left\{\tfrac{1}{2}\xi_i^2 \cos b\xi_i + \xi_i \sin b\xi_i\right\} \int_a^b t^{\frac{3}{2}} f(t)\,dt + \int_0^{\xi_i} \frac{u}{\sqrt{\xi_i^2-u^2}}$$

$$\times \left\{\cos b\xi_i\, S_{a,b,0}\left(f(u)/\sqrt{u}\right) - \frac{u^2}{\xi_i} \sin b\xi_i\, S_{a,b,0}(u^{\frac{3}{2}} f(u))\right\} du$$

$$- \int_0^{\xi_i} \frac{u^2}{\sqrt{\xi_i^2-u^2}} \left\{\cos b\xi_i + \frac{\sin b\xi_i}{\xi_i}\right\} S_{a,b,1}\left(\sqrt{u}\, f(u)\right) du,$$

which is the result required.

(ii) On putting $\mu = 0, -1$, Theorem A under § $2a$ assumes the form:

THEOREM A: If $f(x)$ satisfies Dirichlet's conditions in $(0, b)$ and has its own Struve transform $S_{0,b,0}\{f(\xi_i)\}$, then at each point of $(0, b)$ at which $f(x)$ is continuous

$$\sqrt{x} f(x) = 2\sqrt{b} \sum_i \frac{\xi_i^{\frac{3}{2}} \sin x\xi_i\, H_{0,b,\frac{1}{2}}[f(\xi_i)]}{\left(h^2 - \frac{1}{4b^2} + \xi_i^2\right)(2b\xi_i \cos b\xi_i - \sin b\xi_i)} \quad \ldots (2b.7)$$

where

$$H_{0,b,\frac{1}{2}}[f(\xi_i)] = \sqrt{\frac{2}{\pi\xi_i}} \int_0^1 \frac{u}{\sqrt{1-u^2}} \left[S_{0,b,0}\{f(\xi_i u)/\sqrt{\xi_i u}\} - u\xi_i\, S_{0,b,1}\{\sqrt{\xi_i u}\, f(\xi_i u)\}\right] du$$

$$+ \frac{1}{2\xi_i}\left(\frac{2}{\pi\xi_i}\right)^{\frac{1}{2}} \int_0^b t f(t)\,dt \quad \ldots (2b.8)$$

where ξ_i is a root of the equation

$$\tan Xb = \frac{2Xb}{1-2h}. \quad \ldots \quad (2b.9)$$

THEOREM B: If $f(x)$ satisfies Dirichlet's conditions in $(0, b)$ and has its own transform $S_{0,b,1}[f(\xi_i)]$, then at each point of $(0, b)$ at which $f(x)$ is continuous

$$\sqrt{x}\,f(x) = -2\sqrt{b}\sum_i \frac{\sqrt{\xi_i}\,H_{0,b,-\frac{1}{2}}[f(\xi_i)]\cos x\xi_i}{\left(h^2-\frac{1}{4b^2}+\xi_i^2\right)(\cos b\xi_i+b\xi_i\sin b\xi_i)} \quad (2b.10)$$

where

$$H_{0,b,-\frac{1}{2}}[f(\xi_i)] = (2/\pi\xi_i)^{\frac{3}{2}}\int_0^b f(t)\left\{\left(\frac{3\pi}{4}-\frac{4}{3}\right)\frac{t}{\xi_i}+\frac{2\xi_i}{3}\right\}dt-\left(\frac{2}{\pi\xi_i}\right)^{\frac{1}{2}}\int_0^1\frac{u^2}{\sqrt{1-u^2}}$$

$$\times\left[(3-2u)\,S_{0,b,1}\{f(\xi_i u)/\sqrt{\xi_i u}\}+u\xi_i\,S_{0,b,0}\{\sqrt{\xi_i u}\,f(\xi_i u)\}\right]du \quad \ldots (2b.11)$$

with the addition that ξ_i is a root of the equation $\tan Xb = \dfrac{2h-1}{2Xb}$.

These results are obviously true by virtue of (2b.4), (2b.5) and (2b.6).

(iii) On putting $\mu = \mp\frac{1}{2}$, Theorem A under § 2a reduces to the forms:

THEOREM A: If $f(x)$ satisfies Dirichlet's conditions in $(0, b)$ and has its own sine transform in the same interval, then at each point of the interval at which $f(x)$ is continuous

$$f(x) = \sum_i \frac{J_0(x\xi_i)\,H_{0,b,0}[f(\xi_i)]}{(h^2+\xi_i^2)\,J_1^2(b\xi_i)} \quad \ldots \quad \ldots \quad \ldots (2b.12)$$

where ξ_i is a positive root of the transcendental equation

$$X\,J_1(Xb) = h\,J_0(Xb) \quad \ldots \quad \ldots \quad \ldots (2b.13)$$

and

$$\xi_i\,H_{0,b,0}[f(\xi_i)] = \sqrt{\frac{2}{\pi}}\int_0^1\frac{u}{\sqrt{1-u^2}}\left[S_0^b\{f(\xi_i u)\}+u\xi_i\,C_b^a\{\xi_i u f(\xi_i u)\}\right]du. \quad (2b.14)$$

THEOREM B: If $\sqrt{x}f(x)$ satisfies Dirichlet's conditions in $(0, b)$ and has its own cosine transform in the same interval, then at each point of the interval at which $f(x)$ is continuous

$$f(x) = \sum_i \frac{\xi_i\,J_1(x\xi_i)\,H_{0,b,1}[f(\xi_i)]}{\left(h^2-\frac{1}{b^2}+\xi_i^2\right)J_1'^2(b\xi_i)} \quad \ldots \quad \ldots \quad \ldots (2b.15)$$

where ξ_i is a positive root of the transcendental equation

$$X\,J_0(bX)+\left(h-\frac{1}{b}\right)J_1(bX) = 0$$

and

$$H_{0,b,1}[f(\xi_i)] = \sqrt{\frac{2}{\pi}} \int_0^1 \frac{u}{\sqrt{1-u^2}} S_0^b\{\xi_i u f(\xi_i u)\}\, dx. \qquad .. \ (2b.16)$$

The results are obviously true by virtue of (1.7), (1.8) and (2*b*.4).

2*c*. Determination of Functions $f(x)$ and $S_{a\,b,\mu}[f(x)]$ under certain Conditions

In the interpretation of the above results we shall now obtain the finite Struve transform $S_{a,b,\mu}[f(x)]$ of certain special functions $f(x)$, which are generally known to be the transform pairs under the transformation (1.6).

Result I:

$$x\, S_{0,b,\mu}(x^\mu) = b^{\mu+1} H_{\mu+1}(bx) \qquad .. \quad .. \quad .. \ (2c.1)$$

which is a trivial consequence of the result (2.4).

Result II:

$$S_{0,b,0}\{b^2-x^2\} = \frac{4b}{x^3} H_1(bx) - \frac{2b}{x^2} H_0(bx) + \frac{4b^3}{3\pi x} \qquad .. \quad .. \ (2c.2)$$

which is obviously true by virtue of (2.4) and (2*b*.4).

Result III:

$$\begin{aligned} c^\mu(x^2-c^2)\, S_{a,b,\mu}[J_\mu(cx)] &= a[x\, J_\mu(ac) H_{\mu-1}(ax) - c\, J_{\mu-1}(ac) H_\mu(ax)] \\ &\quad - b[x\, J_\mu(ac) H_{\mu-1}(bx) - c\, J_{\mu-1}(ac) H_\mu(bx)] \\ &\quad + c(x/c)^{\mu+1}[b\{J_\mu(bc) H_{\mu-1}(bc) - J_{\mu-1}(bc) H_\mu(bc)\} \\ &\quad - a\{J_\mu(ac) H_{\mu-1}(ac) - J_{\mu-1}(ac) H_\mu(ac)\}]. \qquad .. \ (2c.3) \end{aligned}$$

Evidently, by virtue of the differential equations (Watson 1958, p. 329)

$$x^2u'' + xu' + (\alpha^2x^2 - \mu^2)u = \frac{4\pi^{-\frac{1}{2}}(\alpha x/2)^{1+\mu}}{\Gamma(\frac{1}{2}+\mu)} \qquad .. \quad .. \ (2c.4)$$

and

$$x^2v'' + xv' + (\beta^2x^2 - \mu^2)v = 0 \qquad .. \quad .. \quad .. \ (2c.5)$$

satisfied by the Struve function $u = H_\mu(dx)$ and the Bessel function $v = J_\mu(\beta x)$, we find

$$x(u''v - uv'') + (u'v - uv') + (\alpha^2 - \beta^2)xuv = \frac{4\pi^{-\frac{1}{2}}(\alpha/2)^{1+\mu}x^\mu}{\Gamma(\frac{1}{2}+\mu)} J_\mu(\beta x)$$

or

$$\frac{d}{dx}[x(u'v - uv')] + (\alpha^2 - \beta^2)xuv = \frac{4\pi^{-\frac{1}{2}}(\alpha/2)^{1+\mu}}{\Gamma(\frac{1}{2}+\mu)} x^\mu J_\mu(\beta x).$$

Integrating with respect to x between the limits (a, b) by aid of the results (2.4) and $\frac{d}{dx}\{x^\mu J_\mu(x)\} = x^\mu J_{\mu-1}(x)$, we obtain

$$b[\alpha J_\mu(b\beta)H_{\mu-1}(\alpha b)-\beta J_{\mu-1}(b\beta)H_\mu(\alpha b)]-a[\alpha J_\mu(a\beta)H_{\mu-1}(a\alpha)-\beta J_{\mu-1}(a\beta)H_\mu(a\alpha)$$
$$+(\alpha^2-\beta^2)\, S_{a,\,b,\,\mu}[J_\mu(\alpha\beta)] = \frac{4\pi^{-\frac{1}{2}}(\alpha/2\beta)^{1+\mu}}{\Gamma(\frac{1}{2}+\mu)}\int_{a\beta}^{b\beta} x^\mu J_\mu(x)\, dx.$$

Again, using the result (Bateman 1953)

$$\int^x x^\mu J_\mu(x)\, dx = \sqrt{\pi}\, 2^{\mu-1}\Gamma(\tfrac{1}{2}+\mu)\, x[J_\mu(x)H_{\mu-1}(x)-J_{\mu-1}(x)H_\mu(x)]$$

we arrive at (2*c*.3).

$2c_1$. *Particular Case*

If a and b are two distinct roots of the equation $J_\mu(cx)=0$, (2*c*.3) assumes the form

$$c^\mu(x^2-c^2)\, S_{a,\,b,\,\mu}[J_\mu(cx)] = a\, J_{\mu-1}(ac)\{x^{\mu+1}\, H_\mu(ac)-c^{\mu+1}\, H_\mu(ax)\}$$
$$-b\, J_{\mu-1}(bc)\{x^{\mu+1}\, H_\mu(bc)-c^{\mu+1}\, H_\mu(bx)\} \quad .. \quad (2c.6)$$

Result IV:

$$S_{a,\,b,\,\mu}\left(\frac{df}{dx}\right) = \frac{x}{2}\left[\left(1+\frac{1}{\mu}\right)S_{a,\,b,\,\mu-1}\{f(x)\}+\left(1-\frac{1}{\mu}\right)S_{a,\,b,\,\mu+1}\{f(x)\}\right]$$
$$+\frac{\left(1-\frac{1}{\mu}\right)(x/2)^{1+\mu}}{\sqrt{\pi}\,\Gamma(\mu+\frac{3}{2})}\int_0^b t^{\mu+1}f(t)\, dt. \quad .. \quad .. \quad .. \quad (2c.7)$$

Integrating by parts, we get

$$S_{a,\,b,\,\mu}(f') = \int_a^b t\, H_\mu(xt) f'(t)\, dt = b\, H_\mu(bx) f(b)-a\, H_\mu(ax) f(a)+\int_a^b f(t)\, \frac{d}{dt}\, [t\, H_\mu(xt)]\, dt$$

which, on using (2.4) and (2*b*.4), furnishes the result (2*c*.6).

$2c_2$. *Particular Case*

For $\mu = 1$, (2*c*.7) reduces to the form

$$S_{a,\,b,\,1}(f') = x\, S_{a,\,b,\,0}(f). \quad .. \quad .. \quad .. \quad (2c.8)$$

Result V:

$$2\mu\, S_{a,\,b,\,\mu}(f/x) = x[S_{a,\,b,\,\mu+1}(f)+S_{a,\,b,\,\mu-1}(f)]-\frac{2^{-\mu}\pi^{-\frac{1}{2}}x^{1+\mu}}{\Gamma(\mu+\frac{3}{2})}\int_a^b t^{1+\mu} f(t)\, dt \quad .. \quad (2c.9)$$

which is obviously true by virtue of (2*b*.6).

Result VI:

$$S_{a,\,b,\,\mu}\{\psi f\}+x^2\, S_{a,\,b,\,\mu}(f) = H_\mu(xb)\{b f'(b)+\mu f(b)\}-H_\mu(xa)\{a f'(a)+\mu f(a)\}$$
$$+x\{b f(b) H_{\mu-1}(bx)-a f(a) H_{\mu-1}(ax)\}+\frac{2\pi^{-\frac{1}{2}}(x/2)^\mu}{\Gamma(\frac{1}{2}+\mu)}\int_a^b t^\mu f(t)\, dt \quad .. \quad (2c.10)$$

where

$$\psi \equiv \left(\frac{d^2}{dx^2}+\frac{1}{x}\frac{d}{dx}-\frac{\mu^2}{x^2}\right).$$

Integrating by parts, we get

$$S_{a,b,\mu}\left(\frac{d^2f}{dx^2}\right) = \int_a^b t\,H_\mu(xt)\frac{d^2f}{dt^2}\,dt = \left[t\,H_\mu(xt)\frac{df}{dt}\right]_a^b - \int_a^b \frac{df}{dt}\frac{d}{dt}[t\,H_\mu(xt)]\,dt$$

$$= b\,H_\mu(xb)\,f'(b)-a\,H_\mu(xa)\,f'(a)-\int_a^b f'(t)\left\{xt\,H_\mu'(xt)+H_\mu(xt)\right\}dt$$

$$= b\,H_\mu(xb)\,f'(b)-a\,H_\mu(xa)\,f'(a)-x\int_a^b t\,f'(t)H_\mu'(xt)\,dt - S_{a,b,\mu}\left\{\frac{1}{x}\frac{df}{dx}\right\}$$

or

$$S_{a,b,\mu}\left\{\frac{d^2f}{dx^2}+\frac{1}{x}\frac{df}{dx}\right\} = b\,H_\mu(xb)\,f'(b)-a\,H_\mu(xa)\,f'(a)-x\left\{b\,f(b)H_\mu'(xb)-a\,f(a)H_\mu'(xa)\right\}$$
$$+x\int_a^b f(t)\frac{d}{dt}\left[t\,H_\mu'(xt)\right]dt$$

which, on using (2*c*.4), furnishes (2*c*.10).

$2c_3$. *Particular Cases*

(i) If a and b are distinct roots of the equation

$$x\,J_\nu'(x)+\mu\,J_\nu(x) = 0$$

then

$$\begin{aligned} S_{a,b,\mu}\{\psi\,J_\nu(k)\}+x^2 S_{a,b,\mu}\{J\;\;(x)\} &= \{b\,J_\nu(b)H_{\mu-1}(bx)-a\,J_\nu(a)H_{\mu-1}(ax)\} \\ &+\frac{2\pi^{\frac{1}{2}}(x/2)^\mu}{\Gamma(\frac{1}{2}+\mu)}[(\mu+\nu-1)\{b\,J_\nu(b)\,S_{\mu-1,\nu-1}(b)-a\,J_{\nu-1}(a)\,S_{\mu-1,\nu-1}(a)\} \\ &-b\,J_{\nu-1}(b)\,S_{\mu,\nu}(b)+a\,J_{\nu-1}(a)\,S_{\mu,\nu}(a)] \quad \ldots \quad (2c.11)\end{aligned}$$

in accordance with the result

$$\int^z z^\mu J_\nu(z)\,dz = (\mu+\nu-1)z\,J_\nu(z)\,S_{\mu-1,\nu-1}(z)-z\,J_{\nu-1}(z)\,S_{\mu,\nu}(z)$$

where $S_{\mu,\nu}(z)$ is Lommel's function (Watson 1958, p. 350).

(ii) If $a = 0 = \nu$, $\mu = 1$, then (2*c*.11) reduces to the form

$$S_{0,b,1}\{\psi\,J_0(x)\}+x^2 S_{0,b,1}\{J_0(x)\} = xb\,J_0(b)H_0(xb)+\frac{2bx}{\pi}J_1(b)\,S_{1,0}(b) \quad (2c.12)$$

provided that b is a root of the equation

$$x\,J_0'(x)+J_0(x) = 0.$$

Summary and Conclusions

Summing up the above results it may be concluded as follows:

(i) An arbitrary function $f(x)$ satisfying (1.1) may be expressed in the form given by (2.1), where the summation is taken over all positive roots of the equation (2.2).

(ii) If a function $f(x)$ be expressed in the form

$$f(x) = \sum_k A_k \sin kx$$

then A_k may be evaluated by means of an integral involving the Struve transforms $S_{0,b,\mu}\{f(ku)/\sqrt{ku}\}$ and $S_{0,b,\mu-1}\{f(ku)\}$, as given by (2a.6).

(iii) If $f(x)$ be expressed in the form

$$f(x) = \sum_k B_k \sin k(x-b), \quad k = n\pi/a-b$$

($n \equiv$ zero or a positive integer), then B_k may be determined by means of integrals involving the Struve transforms

$$S_{a,b,0}\{f(u)/\sqrt{u}\}, \quad S_{a,b,0}\{u^{\frac{1}{2}}f(u)\} \quad \text{and} \quad S_{a,b,1}\{\sqrt{u}\, f(u)\}$$

as given by (2b.2).

(iv) If $f(x)$ be expressed in the form

$$f(x) = \sum_k C_k \sin xk$$

where k is a positive root of the equation $\tan bX = \dfrac{2Xb}{1-2h}$, then C_k may be expressed in the form of an integral involving the Struve transforms $S_{0,b,0}[f(ku)/ku]$ and $S_{0,b,1}[f(ku)]$, as given by (2b.8).

(v) If $f(x)$ be expressed in the form

$$f(x) = \sum_k D_k \cos xk$$

where k is a positive root of the equation $\tan bX = \dfrac{2h-1}{2bX}$, then D_k may be expressed in the form of an integral involving the Struve transforms $S_{0,b,1}[f(ku)/ku]$ and $S_{0,b,0}[f(ku)]$, as given by (2b.11).

(vi) If $f(x)$ be expressed in the form

$$f(x) = \sum_k E_k J_0(x_k)$$

where k is a positive root of the equation $X J_1(Xb) = h J_0(Xb)$, then E_k may be expressed in the form of an integral involving the finite sine and cosine transforms $S_0^b\{f(ku)\}$ and $C_0^b\{ku\, f(ku)\}$, as given by (2b.14).

(vii) If $f(x)$ be expressed in the form

$$f(x) = \sum_k F_k J_1(xk)$$

where k is a positive root of the equation $X J_0(bX) + \left(h - \dfrac{1}{b}\right) J_1(bX) = 0$, then F_k may be expressed in the form of an integral involving the finite sine transform $S_0^b\{ku\, f(ku)\}$, as given by (2b.16).

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