EIGENFUNCTION EXPANSIONS ASSOCIATED WITH A PAIR OF SECOND ORDER DIFFERENTIAL EQUATIONS

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The expansions of a vector having two components in terms of the solutions of a matrix differential equation of the second order with suitable boundary conditions at the end points of a finite interval have been considered. The existence of eigenvalues and eigenvectors has been established and their nature discussed. Explicit expression for Green's matrix has been derived. The function $\Phi(x, \lambda)$ has been defined; some of its properties have been proved and using these properties the expansion theorem and Parseval's relation have been deduced.

§ 1. Let L denote the matrix operator

$$L \equiv \begin{pmatrix} \frac{d}{dx}\left(p_0(x)\frac{d}{dx}\right)+p_1(x) & r(x) \\ r(x) & \frac{d}{dx}\left(q_0(x)\frac{d}{dx}\right)+q_1(x) \end{pmatrix}. \quad \ldots \quad (1.1)$$

F the symmetric matrix

$$F \equiv (F_{ij}) \equiv \begin{pmatrix} F_{11}(x) & F_{12}(x) \\ F_{21}(x) & F_{22}(x) \end{pmatrix} \quad \ldots \quad \ldots \quad \ldots \quad (1.2)$$

and $\phi \equiv \phi(x)$ a vector having two components $u \equiv u(x)$ and $v \equiv v(x)$ represented as a column matrix

$$\phi = \begin{pmatrix} u \\ v \end{pmatrix}.$$

Consider the homogeneous system

$$(L-\lambda F)\phi = 0, \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (1.3)$$

where λ is a parameter, real or complex.

The corresponding non-homogeneous system is given by

$$(L-\lambda F)\phi = -Ff \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (1.4)$$

where $f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$, f_1 and f_2 being real valued functions of x in $a \leqslant x \leqslant b$.

The system considered here is more general than the one considered by Chakravarty (1965). Though the differential operator is a special case of that of Kodaira (1950), Coddington and Levinson (1955), Neumark (1960) and Hilbert (1912), the results and methods are different from theirs. The work is based on Chapters I and II of Titchmarsh's book (1962).

We assume the following conditions to be satisfied:

(i) $p_0(x)$, $q_0(x)$ are real valued and possess continuous derivatives of the first order in $a \leqslant x \leqslant b$;

(ii) $p_1(x)$, $q_1(x)$ and $r(x)$ are real valued and continuous in $a \leqslant x \leqslant b$;

(iii) $p_0(x), q_0(x) > 0$ for $a \leqslant x \leqslant b$;

(iv) the symmetric matrix F is real valued, continuous and positive definite for $a \leqslant x \leqslant b$.

We define the boundary conditions to be satisfied by the vector $\phi = \begin{pmatrix} u \\ v \end{pmatrix}$ at $x = a$ and $x = b$ respectively by

$$M(a, \phi) \equiv p_o(a)[a_{j1}u(a)+a_{j2}u'(a)]+q_o(a)[a_{j3}v(a)+a_{j4}v'(a)] = 0 \qquad j = 1, 2, \qquad (1.5)$$

$$N(b, \phi) \equiv p_o(b)[b_{j1}u(b)+b_{j2}u'(b)]+q_o(b)[b_{j3}v(b)+b_{j4}v'(b)] = 0 \qquad j = 1, 2, \qquad (1.6)$$

where

(*a*) a_{jk} and b_{jk} ($j = 1, 2$; $k = 1, 2, 3, 4$) are real-valued constants;

(*b*) the trivial case of $a_{jk} = 0$ ($j = 1, 2$; $k = 1, 2, 3, 4$) and $b_{jk} = 0$ ($j = 1, 2$; $k = 1, 2, 3, 4$) is excluded;

(*c*) the set $\{a_{1k};\ k = 1, 2, 3, 4\}$ is linearly independent of the set $\{a_{2k};\ k = 1, 2, 3, 4\}$ and the set $\{b_{1k};\ k = 1, 2, 3, 4\}$ is linearly independent of the set $\{b_{2k};\ k = 1, 2, 3, 4\}$;

(*d*)
$$q_o(a)(a_{14}a_{23}-a_{24}a_{13})+p_o(a)(a_{12}a_{21}-a_{11}a_{22}) = 0, \qquad (1.7)$$
$$q_o(b)(b_{14}b_{23}-b_{24}b_{13})+p_o(b)(b_{12}b_{21}-b_{11}b_{22}) = 0. \qquad (1.8)$$

The value of λ for which the system (1.3) has a non-trivial solution satisfying (1.5) and (1.6) is an eigenvalue and the corresponding vector solution is an eigenvector.

§ 2. The following existence theorem holds:

Theorem 2.1. Let $p_0(x)$, $q_0(x)$, $p_1(x)$, $q_1(x)$, $r(x)$ and $F_{ij}(i, j = 1, 2)$ satisfy the conditions of § 1 and let A, B, C, D be four constants not all vanishing simultaneously, then the system (1.3) has a solution $\phi(x) = \begin{pmatrix} u(x) \\ v(x) \end{pmatrix}$ which is unique and which satisfies

$$u(\alpha) = A,\ u'(\alpha) = B,\ v(\alpha) = C,\ v'(\alpha) = D, \qquad (2.1)$$

where $a \leqslant \alpha \leqslant b$, the accents denoting differentiation with respect to x. Also for each x in the closed interval $[a, b]$, $u(x)$, $v(x)$, $u'(x)$, $v'(x)$, $u''(x)$, $v''(x)$ are all integral functions of the complex variable λ and take real values when λ is real.

A vector $\phi(x, \lambda) = \begin{pmatrix} u(x, \lambda) \\ v(x, \lambda) \end{pmatrix}$ $(a \leqslant x \leqslant b)$ which is such that $u(\alpha, \lambda) = A$, $u'(\alpha, \lambda) = B$, $v(\alpha, \lambda) = C$ and $v'(\alpha, \lambda) = D$ $(a \leqslant \alpha \leqslant b)$, where A, B, C, D are constants not vanishing simultaneously will, after Chakravarty (1965), be represented by the symbol

$$\phi(\alpha | x, \lambda) = \begin{pmatrix} u(\alpha | x, \lambda) \\ v(\alpha | x, \lambda) \end{pmatrix}. \qquad \ldots \quad \ldots \quad \ldots \quad (2.2)$$

§ 3. Let

$$\phi_j = \begin{pmatrix} u_j \\ v_j \end{pmatrix} \text{ and } \phi_k = \begin{pmatrix} u_k \\ v_k \end{pmatrix},$$

where u_j, v_j, etc., are functions of x, be two given vectors. Then the *bilinear concomitant* of the two vectors ϕ_j and ϕ_k is defined by

$$p_0 \begin{vmatrix} u_j' & u_k' \\ u_j & u_k \end{vmatrix} + q_0 \begin{vmatrix} v_j' & v_k' \\ v_j & v_k \end{vmatrix} = p_0(u_j' u_k - u_j u_k') + q_0(v_j' v_k - v_j v_k'). \qquad \ldots \quad (3.1)$$

where p_0, q_0 are functions of x of § 1, and is represented by $[\phi_j \phi_k]$.

The following important properties of the bilinear concomitant hold:

(*a*) $[\phi_j \phi_j] = [\phi_k \phi_k] = 0$;

(*b*) $[\phi_j \phi_k] = -[\phi_k \phi_j]$;

(*c*) $[\phi_j(k_1 \phi_l + k_2 \phi_m)] = k_1[\phi_j \phi_l] + k_2[\phi_j \phi_m]$, where k_1 and k_2 are constants;

(*d*) if ϕ_j and ϕ_k are the solutions of (1.3) for the same value of λ, $[\phi_j \phi_k]$ is independent of x and is a function of λ alone;

(*e*) if $\phi_j(\alpha_1 | x, \lambda)$ and $\phi_k(\alpha_2 | x, \lambda)$ are the solutions of (1.3) satisfying the conditions of Thm. (2.1), $[\phi_j(\alpha_1 | x, \lambda) \phi_k(\alpha_2 | x, \lambda)]$ is an integral function of λ independent of x and also is real valued for real λ.

The boundary conditions (1.5) and (1.6) can be expressed in the alternative 'Kodaira form' [Kodaira (1950)].

$$M(a, \phi) \equiv [\phi \phi_j](a) = 0 \quad (j = 1, 2) \qquad \ldots \quad \ldots \quad \ldots \quad (3.2)$$

and

$$N(b, \phi) \equiv [\phi \phi_k](b) = 0, \quad (k = 3, 4) \qquad \ldots \quad \ldots \quad \ldots \quad (3.3)$$

where

$$\phi_j(a | x, \lambda) = \begin{pmatrix} u_j(a | x, \lambda) \\ v_j(a | x, \lambda) \end{pmatrix} \quad (j = 1, 2)$$

and

$$\phi_k(b | x, \lambda) = \begin{pmatrix} u_k(b | x, \lambda) \\ v_k(b | x, \lambda) \end{pmatrix} \quad (k = 3, 4)$$

are solutions of (1.3) and are such that

$$\left. \begin{aligned} u_j(a | a, \lambda) = a_{j2}, \; u_j'(a | a, \lambda) = -a_{j1}, \\ v_j(a | a, \lambda) = a_{j4}, \; v_j'(a | a, \lambda) = -a_{j3}, \end{aligned} \right\} \quad (j = 1, 2)$$

$$\left.\begin{aligned} u_k(b\,|\,b,\lambda) &= b_{l2},\ u_k'(b\,|\,b,\lambda) = -b_{l1}, \\ v_k(b\,|\,b,\lambda) &= b_{l4},\ v_k'(b\,|\,b,\lambda) = -b_{l3}, \end{aligned}\right\} \begin{aligned} &\text{when } k = 3,\ l = 1; \\ &\text{when } k = 4,\ l = 2, \end{aligned}$$

$\phi \equiv \phi(\alpha\,|\,x, \lambda) = \begin{pmatrix} u(\alpha\,|\,x, \lambda) \\ v(\alpha\,|\,x, \lambda) \end{pmatrix}$ is a vector satisfying (1.5) and (1.6) and [ ] (a) indicates the value of [ ] at $x = a$.

The vectors $\phi_j(a\,|\,x, \lambda)$ and $\phi_k(b\,|\,x, \lambda)$ are called the boundary condition vectors at $x = a$ and $x = b$ respectively.

For all x in $[a, b]$ and arbitrary λ we have

$$[\phi_1\phi_2] = 0 \qquad \text{(3.4)}$$

and

$$[\phi_3\phi_4] = 0. \qquad \text{(3.5)}$$

We note that (3.4) and (3.5) are respectively equivalent to (1.7) and (1.8) and $[\phi_1\phi_2]$ and $[\phi_3\phi_4]$ are independent of x and λ.

Since we stipulated in § 1 that the sets $\{a_{1k}\}$ and $\{a_{2k}\}$ are linearly independent, it follows that ϕ_1 and ϕ_2 are linearly independent over $[a, b]$ for all λ. Likewise ϕ_3 and ϕ_4 are linearly independent over $[a, b]$ for all λ.

We also note that ϕ_1 and ϕ_2 at $x = a$ and ϕ_3 and ϕ_4 at $x = b$ are independent of λ.

We now prove the following theorem:

Theorem 3.1. Let any two vectors $f(x) = \begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix}$ and $g(x) = \begin{pmatrix} g_1(x) \\ g_2(x) \end{pmatrix}$ satisfy the boundary conditions $M(a, \phi) = 0$ at $x = a$ and $N(b, \phi) = 0$ at $x = b$. Then $[f(x)g(x)] = 0$ both at $x = a$ and $x = b$.

First we prove that $[f(x)g(x)](a) = 0$.

Since $f(x)$ and $g(x)$ satisfy at $x = a$ the relations (1.5) where the relation (1.7) is satisfied, we have from (3.2) and (3.4)

$$\left.\begin{aligned} [f\phi_j](a) &= 0, \\ [g\phi_j](a) &= 0 \end{aligned}\right\} (j = 1, 2) \qquad \text{(3.6)}$$

and

$$[\phi_1\phi_2] = 0. \qquad \text{(3.7)}$$

From (3.6) and (3.7) we have $[fg](a) = 0$.

It follows similarly that $[fg](b) = 0$.

§ 4. Let $\phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$ and $\theta = \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix}$ be two vectors having continuous derivatives of the second order and A^T denote the transpose of the matrix A.

Then Green's formula for our boundary value problem is

$$\int_a^b (\theta^T L\phi - \phi^T L\theta)\, dx = [\phi\theta](b) - [\phi\theta](a). \qquad \text{(4.1)}$$

If ϕ and θ be the solutions of (1.3) and satisfy the boundary conditions (1.5) and (1.6) where conditions (1.7) and (1.8) are satisfied, then our boundary value problem (1.3), (1.5) and (1.6) is self-adjoint, i.e. $[\phi\theta](b) = [\phi\theta](a)$ and in particular by Thm. 3.1

$$[\phi\theta](b) = [\phi\theta](a) = 0.$$

Let the two eigenvectors

$y_1 \equiv \begin{pmatrix} u_1(x, \lambda_1) \\ v_1(x, \lambda_1) \end{pmatrix}$ and $y_2 \equiv \begin{pmatrix} u_2(x, \lambda_2) \\ v_2(x, \lambda_2) \end{pmatrix}$ be associated with two distinct eigenvalues λ_1 and λ_2 respectively. Then we obtain by Thm. 3.1 and Green's formula

$$\int_a y_1^T F y_2 \, dx = 0, \qquad (4.2)$$

the orthogonal relation for our eigenvector system.

§ 5. Let

$$y_j = \begin{pmatrix} u_j \\ v_j \end{pmatrix} \equiv \begin{pmatrix} u_j(x, \lambda) \\ v_j(x, \lambda) \end{pmatrix}, \ j = 1, 2, 3, 4,$$

be four solutions of (1.3). Then the determinant

$$W_x(y_1y_2y_3y_4)(\lambda) \equiv W(y_1(x, \lambda)y_2(x, \lambda)y_3(x, \lambda)y_4(x, \lambda)) \qquad (5.1)$$

defined by

$$W_x(y_1y_2y_3y_4)(\lambda) = \begin{vmatrix} u_1 & u_2 & u_3 & u_4 \\ v_1 & v_2 & v_3 & v_4 \\ u_1' & u_2' & u_3' & u_4' \\ v_1' & v_2' & v_3' & v_4' \end{vmatrix} \qquad (5.2)$$

is called the Wronskian for the system (1.3) as it plays the same role for the system (1.3) as does the Wronskian in the case of ordinary linear differential equations.

We have the identity

$$p_0(x)q_0(x)W_x(y_1y_2y_3y_4)(\lambda) \equiv -[y_1y_2][y_3y_4]+[y_1y_3][y_2y_4]-[y_1y_4][y_2y_3] \qquad (5.3)$$

It follows from (5.3) that $p_0(x)q_0(x)W_x(y_1y_2y_3y_4)(\lambda)$ is independent of x and depends only on λ, since each $[y_jy_k]$ is so. (*See* § 3(*d*).)

Let us now take the four solutions of (1.3) to be the four boundary condition vectors $\phi_i (i = 1, 2, 3, 4)$ introduced in § 3. Then by virtue of (5.3), (3.4) and (3.5)

$$p_0(x)q_0(x)W_x(\phi_1\phi_2\phi_3\phi_4)(\lambda) = [\phi_1\phi_3][\phi_2\phi_4]-[\phi_1\phi_4][\phi_2\phi_3]. \qquad (5.4)$$

It is independent of x and depends on λ alone. Let us denote it by $D(\lambda)$. It follows that $D(\lambda)$ is an integral function of λ, independent of x and takes real values when λ is real. $D(\lambda)$ is not identically zero in x over $[a, b]$ and so $W_x(\phi_1\phi_2\phi_3\phi_4)(\lambda)$ is also not identically zero in x over $[a, b]$. Hence the boundary condition vectors $\phi_i (i = 1, 2, 3, 4)$ are linearly independent over

$[a, b]$ and form a fundamental set for those values of λ for which $D(\lambda) \neq 0$. If $D(\lambda) = 0$ for some λ, then ϕ_3, ϕ_4 are linearly dependent on ϕ_1, ϕ_2. We have the following theorems:

Theorem 5.1. The necessary and sufficient condition that λ should be an eigenvalue is that it is a root of $D(\lambda) = 0$.

This can be proved by following Everitt (1957).

Theorem 5.2. The zeros of $D(\lambda)$, i.e. the eigenvalues are all real.

It follows on using the orthogonality relation (4.2) and the property of a positive definite Hermitian form.

The following lemmas hold:

LEMMA 5.3. Let $\phi_j(a\,|\,x, \lambda)(j = 1, 2)$ and $\phi_k(b\,|\,x, \lambda)$ $(k = 3, 4)$ be the boundary condition vectors. Then for all values of λ

$$\int_a^b \phi_j^T(a\,|\,x, \lambda)F\phi_k(b\,|\,x, \lambda)\,dx = \frac{d}{d\lambda}[\phi_j(a\,|\,x, \lambda)\phi_k(b\,|\,x, \lambda)], \quad (j = 1, 2;\ k = 3, 4). \qquad (5.5)$$

LEMMA 5.4. Let $U(x) = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ and $V(x) = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$ be any two vectors (real or complex) such that $U^T(x)F\overline{U(x)}$ and $V^T(x)F\overline{V(x)}$ belong to L.

Then

$$\text{(i)} \quad \left|\int U^T(x)FV(x)\,dx\right| \leqslant \left\{\int U^T(x)F\overline{U(x)}\,dx \int V^T(x)F\overline{V(x)}\,dx\right\}^{\frac{1}{2}}, \qquad (5.6)$$

$$\text{(ii)} \quad \left|\int U^T(x)F\overline{V(x)}\,dx\right| \leqslant \left\{\int U^T(x)F\overline{U(x)}\,dx \int V^T(x)F\overline{V(x)}\,dx\right\}^{\frac{1}{2}}, \qquad (5.7)$$

$$\text{(iii)} \quad \left|\int \overline{U^T}(x)F\overline{V(x)}\,dx\right| \leqslant \left\{\int U^T(x)F\overline{U(x)}\,dx \int V^T(x)F\overline{V(x)}\,dx\right\}^{\frac{1}{2}}, \qquad (5.8)$$

$$\text{(iv)} \quad \left\{\int (U(x)+V(x))^TF(\overline{U(x)}+\overline{V(x)})\,dx\right\}^{\frac{1}{2}}$$

$$\leqslant \left\{\int U^T(x)F\overline{U(x)}\,dx\right\}^{\frac{1}{2}} + \left\{\int V^T(x)F\overline{V(x)}\,dx\right\}^{\frac{1}{2}}, \qquad (5.9)$$

where the bar as usual denotes the complex conjugate. The interval of integration may be finite or infinite.

By using the transformation

$$X_j = \sqrt{F_{11}}\left(x_j + \frac{F_{12}}{F_{11}}y_j\right), \quad Y_j = \sqrt{\frac{F_{11}F_{22}-F_{12}^2}{F_{11}}}\,y_j, \quad (j = 1, 2)$$

the inequality

$$|X_1X_2+Y_1Y_2|^2 \leqslant (|X_1|^2+|Y_1|^2)(|X_2|^2+|Y_2|^2) \qquad (5.10)$$

and Schwarz's inequality (i) follows.

By similar arguments (ii) and (iii) follow. The relation (iv) follows on using (ii).

§ 6. Let F be a positive definite Hermitian matrix of order two and $f_n(x) = \begin{pmatrix} f_{1n}(x) \\ f_{2n}(x) \end{pmatrix}$ be a sequence of vectors (real or complex) such that $f_n^T(x)F\overline{f_n(x)}$ belongs to L for each n. Let $f(x) = \begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix}$ be a given vector such that $f^T(x)F\,\overline{f(x)}$ also belongs to L. Then if the integral

$$\int (f_n(x)-f(x))^T F(\overline{f_n(x)}-\overline{f(x)})\,dx \quad \ldots \quad (6.1)$$

tends to zero as n tends to infinity, we say that $f_n(x)$ converges in the mean to $f(x)$.

The following fundamental result analogous to Riesz-Fischer theorem holds:

LEMMA 6.1. If a sequence $\{f_n(x)\}$ of vectors $f_n(x) = \begin{pmatrix} f_{1n}(x) \\ f_{2n}(x) \end{pmatrix}$ is given, then in order that there exists an element (vector) $f(x)$ towards which it converges in the mean, it is necessary and sufficient that

$$\int (f_m(x)-f_n(x))^T F(\overline{f_m(x)}-\overline{f_n(x)})\,dx$$

tends to zero as m and n tend independently to infinity. (*See* Riesz and Nagy 1956, pp. 58-59).

The following result also holds:

LEMMA 6.2. If $f_n(x)$ converges in the mean to $f(x)$ and $g_n(x)$ converges in the mean to $g(x)$, then

$$\underset{n\to\infty}{\mathrm{Lt}} \int f_n^T(x)F\overline{g_n(x)}\,dx = \int f^T(x)F\overline{g(x)}\,dx.$$

We shall frequently use the above results in our further discussion.

§ 7. We discuss the nature of the zeros of $D(\lambda)$ at an eigenvalue. There are two cases to be discussed here.

Firstly, suppose that $D(\lambda) = 0$ at $\lambda = \lambda_n$, but at least one of the $[\phi_j\phi_k]$, say $[\phi_2\phi_4]$, is not zero at $\lambda = \lambda_n$.

We have at $\lambda = \lambda_n$

$$D(\lambda) = [\phi_1\phi_3][\phi_2\phi_4]-[\phi_1\phi_4][\phi_2\phi_3] = 0, \quad \ldots \quad (7.1)$$

$$D'(\lambda) = [\phi_1\phi_3]'[\phi_2\phi_4]+[\phi_1\phi_3][\phi_2\phi_4]'-[\phi_1\phi_4]'[\phi_2\phi_3]-[\phi_1\phi_4][\phi_2\phi_3]', \quad (7.2)$$

where the accents denote differentiation with respect to λ.

Since $D(\lambda) = 0$ at $\lambda = \lambda_n$, we have

$$A\phi_1+B\phi_2 = C\phi_3+D\phi_4,\ (x\epsilon[a,b]), \quad \ldots \quad (7.3)$$

where $A \neq 0$, $C \neq 0$, and

$$A([\phi_2\phi_4]\phi_1-[\phi_1\phi_4]\phi_2) = C([\phi_2\phi_4]\phi_3-[\phi_2\phi_3]\phi_4). \quad \ldots \quad (7.4)$$

Now

$$\int_a^b ([\phi_2\phi_4]\phi_1-[\phi_1\phi_4]\phi_2)^T F([\phi_2\phi_4]\phi_3-[\phi_2\phi_3]\phi_4)\,dx$$
$$= [\phi_2\phi_4]([\phi_1\phi_3]'[\phi_2\phi_4]+[\phi_1\phi_3][\phi_2\phi_4]'-[\phi_1\phi_4]'[\phi_2\phi_3]-[\phi_1\phi_4][\phi_2\phi_3]'),$$
by (5.5) and (7.1)
$$= [\phi_2\phi_4]D'(\lambda), \text{ by (7.2), at } \lambda = \lambda_n.$$

Using (7.4) we get

$$\frac{A}{C}\int_a^b ([\phi_2\phi_4]\phi_1-[\phi_1\phi_4]\phi_2)^T F([\phi_2\phi_4]\phi_1-[\phi_1\phi_4]\phi_2)\,dx = [\phi_2\phi_4]D'(\lambda) \qquad (7.5)$$

at $\lambda = \lambda_n$.

Since the integrand on the l.h.s. of (7.5) is a positive definite form, $D'(\lambda) \neq 0$ at $\lambda = \lambda_n$. Therefore $\lambda = \lambda_n$ is a simple zero of $D(\lambda)$.

It follows from (7.5) and (7.4) that an eigenvector corresponding to an eigenvalue λ_n is a constant multiple of $[\phi_2\phi_4]\phi_1-[\phi_1\phi_4]\phi_2$ or $[\phi_2\phi_4]\phi_3-[\phi_2\phi_3]\phi_4$. In particular,

$$\psi_n(x, \lambda_n) = \begin{pmatrix} u_n(x, \lambda_n) \\ v_n(x, \lambda_n) \end{pmatrix}$$
$$= \left\{\frac{k_n}{[\phi_2\phi_4]D'(\lambda_n)}\right\}^{\frac{1}{2}} \{[\phi_2\phi_4]\phi_1(x, \lambda_n)-[\phi_1\phi_4]\phi_2(x, \lambda_n)\} \quad .. \quad (7.6)$$

is a normalized eigenvector, corresponding to the eigenvalue λ_n, in the sense that

$$\int_a^b \psi_n^T(x, \lambda_n)F\psi_n(x, \lambda_n)\,dx = 1. \quad .. \quad .. \quad .. \quad (7.7)$$

Secondly, let all the $[\phi_j\phi_k]$ be zero at $\lambda = \lambda_n$.

Then we have at $\lambda = \lambda_n$

$$D(\lambda) = [\phi_1\phi_3][\phi_2\phi_4]-[\phi_1\phi_4][\phi_2\phi_3] = 0, \quad .. \quad .. \quad .. \quad .. \quad (7.8)$$
$$D'(\lambda) = [\phi_1\phi_3]'[\phi_2\phi_4]+[\phi_1\phi_3][\phi_2\phi_4]'-[\phi_1\phi_4]'[\phi_2\phi_3]-[\phi_1\phi_4][\phi_2\phi_3]' = 0, \quad (7.9)$$
$$D''(\lambda) = 2\{[\phi_1\phi_3]'[\phi_2\phi_4]'-[\phi_1\phi_4]'[\phi_2\phi_3]'\}. \quad .. \quad .. \quad .. \quad .. \quad (7.10)$$

At $\lambda = \lambda_n$ we have

$$\phi_3 = A\phi_1+B\phi_2,\ \phi_4 = C\phi_1+D\phi_2, \quad .. \quad .. \quad .. \quad (7.11)$$

where $AD-BC \neq 0$.

Let

$$I_{jk} \equiv I_{jk}(\lambda_n) = \int_a^b \phi_j^T F\phi_k dx (1 < j, k < 2). \quad .. \quad .. \quad (7.12)$$

Then

$$I_{jj} > 0\ (j = 1, 2) \text{ and } I_{jk} = I_{kj}.$$

From (7.11), (7.12) and (5.5) we have at $\lambda = \lambda_n$

$$[\phi_1\phi_3]' = AI_{11}+BI_{12},\ [\phi_2\phi_3]' = AI_{21}+BI_{22},$$
$$[\phi_1\phi_4]' = CI_{11}+DI_{12},\ [\phi_2\phi_4]' = CI_{21}+DI_{22}. \quad .. \quad .. \quad (7.13)$$

Putting in (7.10) we get

$$\tfrac{1}{2}D''(\lambda_n) = (AD-BC)(I_{11}I_{22}-I_{12}^2).$$

Now

$$I_{12}^2 = \left\{\int_a^b \phi_1^T F\phi_2\, dx\right\}^2 < \int_a^b \phi_1^T F\phi_1\, dx \int_a^b \phi_2^T F\phi_2\, dx, \text{ by (5.6)}.$$

The equality sign is omitted since ϕ_1 and ϕ_2 are linearly independent over $[a, b]$.

Thus $I_{12}^2 < I_{11}I_{22}$ and consequently $D''(\lambda) \neq 0$ at $\lambda = \lambda_n$, i.e. zero of $D(\lambda)$ at $\lambda = \lambda_n$ for this type of eigenvalue is of order two. It follows from (7.11) that corresponding to the eigenvalue λ_n there are two eigenvectors, one $\psi_n^{(1)}(x, \lambda_n)$, a constant multiple of ϕ_3 and the other $\psi_n^{(2)}(x, \lambda_n)$, a constant multiple of ϕ_4, which are linearly independent and normalized in the sense of (7.7).

Also

$$\int_a^b \psi_n^{(1)^T}(x, \lambda_n)F\psi_n^{(2)}(x, \lambda_n)\, dx = 0. \qquad \text{(7.14)}$$

Define $\psi_n(x, \lambda_n)$ by

$$\psi_n(x, \lambda_n) = \frac{A_n}{\sqrt{A_n^2+B_n^2}}\psi_n^{(1)}(x, \lambda_n) + \frac{B_n}{\sqrt{A_n^2+B_n^2}}\psi_n^{(2)}(x, \lambda_n), \qquad \text{(7.15)}$$

where

$$A_n = \int_a^b \psi_n^{(1)^T}(x, \lambda_n)Ff(x)\, dx,\ B_n = \int_a^b \psi_n^{(2)^T}(x, \lambda_n)Ff(x)\, dx, \qquad \text{(7.16)}$$

$f(x)$ being a fixed vector such that $f^T(x)\, Ff(x) \in L[a, b]$.

By using a relation contained in Mirsky (1955, p. 362, 12.1.14) it follows that the vector $\psi_n(x, \lambda_n)$ is normalized in the sense of (7.7). $\psi_n(x, \lambda_n)$, thus defined, is taken as the eigenvector corresponding to an eigenvalue λ_n when λ_n is a double zero of $D(\lambda)$.

§ 8. We now construct Green's matrix for our boundary value problem (1.3), (1.5) and (1.6) satisfying the usual properties. (*See* Hilbert (1912, p. 206), Courant & Hilbert (1953, p. 393), Neumark (1960, p. 105) and Chakravarty (1965)).

Let $\phi_j \equiv \phi_j(a\,|\,x, \lambda)$ $(j = 1, 2)$, $\phi_k \equiv \phi_k(b\,|\,x, \lambda)$ $(k = 3, 4)$ be the boundary condition vectors and $D(\lambda)$ given by (5.4) be not zero. Then if we assume that

$$\psi_1 \equiv \psi_1(x, \lambda) = \begin{pmatrix}\psi_{11}(x, \lambda)\\ \psi_{12}(x, \lambda)\end{pmatrix} = \frac{[\phi_2\phi_4]\phi_3(b\,|\,x, \lambda)-[\phi_2\phi_3]\phi_4(b\,|\,x, \lambda)}{D(\lambda)} \qquad \text{(8.1)}$$

and

$$\psi_2 \equiv \psi_2(x, \lambda) = \begin{pmatrix}\psi_{21}(x, \lambda)\\ \psi_{22}(x, \lambda)\end{pmatrix} = \frac{[\phi_1\phi_3]\phi_4(b\,|\,x, \lambda)-[\phi_1\phi_4]\phi_3(b\,|\,x, \lambda)}{D(\lambda)}, \qquad \text{(8.2)}$$

then the expression for Green's matrix $G \equiv G(x, y; \lambda) = \begin{pmatrix} G_{11} G_{21} \\ G_{12} G_{22} \end{pmatrix}$ $\equiv (G_{ij}(x, y; \lambda))$ can be written as a product of two matrices given by

$$\left.\begin{aligned} G(x, y; \lambda) = G^T(y, x; \lambda) &= \begin{pmatrix} \psi_{11}(x, \lambda) & \psi_{21}(x, \lambda) \\ \psi_{12}(x, \lambda) & \psi_{22}(x, \lambda) \end{pmatrix} \begin{pmatrix} u_1(y, \lambda) & v_1(y, \lambda) \\ u_2(y, \lambda) & v_2(y, \lambda) \end{pmatrix}, y \epsilon [a, x] \\ &= \begin{pmatrix} u_1(x, \lambda) & u_2(x, \lambda) \\ v_1(x, \lambda) & v_2(x, \lambda) \end{pmatrix} \begin{pmatrix} \psi_{11}(y, \lambda) & \psi_{12}(y, \lambda) \\ \psi_{21}(y, \lambda) & \psi_{22}(y, \lambda) \end{pmatrix}, y \epsilon [x, b]. \end{aligned}\right\} \quad \ldots \quad (8.3)$$

For the boundary condition vectors ϕ_j $(j = 1, 2)$ and the vectors $\psi_i \equiv \psi_i(x, \lambda)$ $(i = 1, 2)$ given by (8.1) and (8.2) we have the following lemma:

LEMMA 8.1

(i) $(\psi_{11} v_1 + \psi_{21} v_2 - \psi_{12} u_1 - \psi_{22} u_2) = 0,$ .. .. (8.4)

(ii) $p_0(\psi'_{11} u_1 + \psi'_{21} u_2 - \psi_{11} u'_1 - \psi_{21} u'_2) = -1,$ .. .. (8.5)

(iii) $(\psi'_{11} v_1 + \psi'_{21} v_2 - \psi_{12} u'_1 - \psi_{22} u'_2) = 0,$ .. .. .. (8.6)

(iv) $(\psi'_{12} u_1 + \psi'_{22} u_2 - \psi_{11} v'_1 - \psi_{21} v'_2) = 0,$ .. .. .. (8.7)

(v) $q_0(\psi'_{12} v_1 + \psi'_{22} v_2 - \psi_{12} v'_1 - \psi_{22} v'_2) = -1.$ .. .. (8.8)

These results are proved directly by using (3.4) and (3.5).

§ 9. Let $f(x) = \begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix}$ be a real-valued continuous vector in $a \leqslant x \leqslant b$ and let us suppose that λ is not an eigenvalue. Then define a vector

$$\Phi \equiv \Phi(x, \lambda) = \begin{pmatrix} \Phi_1(x, \lambda) \\ \Phi_2(x, \lambda) \end{pmatrix}$$

by

$$\Phi(x, \lambda) = \int_a^b (G_{ij}(x, y; \lambda)) Ff(y) \, dy, \quad \ldots \quad \ldots \quad (9.1)$$

where the matrix (G_{ij}) is given by (8.3).

It can be easily shown on using lemma (8.1) that $\Phi(x, \lambda)$ satisfies eqn. (1.4) and the boundary conditions (1.5) and (1.6) at $x = a$ and $x = b$ respectively.

§ 10. We obtain some results in connection with $\Phi(x, \lambda)$ defined by (9.1).

LEMMA 10.1. Let $f(x) = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$ be a vector having continuous derivatives up to second order in $[a, b]$. Suppose that $f(x)$ satisfies the boundary conditions (1.5) and (1.6) at $x = a$ and $x = b$ respectively and λ is not an eigenvalue.

Then

$$\Phi(x, \lambda) = \frac{1}{\lambda} \left\{ f(x) + \Phi^*(x, \lambda) \right\}, \quad \ldots \quad \ldots \quad \ldots \quad (10.1)$$

where Φ^* depends on Lf in the same way as Φ depends on Ff.

Writing explicitly the formula for $\Phi(x, \lambda)$ in terms of ϕ_j and ψ_j ($j = 1, 2$) and applying Green's formula we have

$$\Phi(x, \lambda) = \frac{1}{\lambda}[\Phi^*(x, \lambda)+\psi_1[\phi_1 f](x)+\psi_2[\phi_2 f](x)-\phi_1[\psi_1 f](x)-\phi_2[\psi_2 f](x)].$$

Using (8.4)–(8.8) it follows that

$$\psi_1[\phi_1 f](x)+\psi_2[\phi_2 f](x)-\phi_1[\psi_1 f](x)-\phi_2[\psi_2 f](x) = f(x),$$

and the result follows.

LEMMA 10.2. Let $\Phi(x, \lambda)$ be defined by (9.1) where $f(x) = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$ is a real-valued vector such that $f^T(x)Ff(x)$ belongs to $L[a, b]$ and $\lambda = \mu + i\nu$ ($\nu \neq 0$). Then

$$\int_a^b \Phi^T F\bar{\Phi}\, dx \leqslant \frac{1}{\nu^2}\int_a^b f^T Ff\, dx. \quad \text{.. .. .. (10.2)}$$

Let us define a sequence $\{Q_n(x)\}$ of real-valued vectors $Q_n(x) = \begin{pmatrix} Q_{1n}(x) \\ Q_{2n}(x) \end{pmatrix}$ which are continuous in $a \leqslant x \leqslant b$. Then the vector $\Phi(x, \lambda\,;\, Q_n)$ satisfies the differential system

$$(L-\lambda F)\Phi(x, \lambda\,;\, Q_n) = -FQ_n$$

and the boundary conditions (1.5) and (1.6) and is analytic in λ. By principle of reflection

$$\overline{\Phi(x, \lambda\,;\, Q_n)} = \Phi(x, \bar{\lambda}\,;\, Q_n).$$

$\Phi(x, \bar{\lambda};\, Q_n)$ also satisfies the boundary conditions (1.5) and (1.6). By Green's formula and inequality (5.6) we have

$$\int_a^b \Phi^T(x, \lambda\,;\, Q_n)F\Phi(x, \bar{\lambda}\,;\, Q_n)\, dx \leqslant \frac{1}{\nu^2}\int_a^b Q_n^T FQ_n\, dx. \quad \text{.. (10.3)}$$

If we approximate in the mean to the vector $f(x)$ by the vectors $Q_n(x)$, we have

$$Lt \int_a^b Q_n^T FQ_n\, dx = \int_a^b f^T Ff\, dx \text{ (see § 6).} \quad \text{.. .. (10.4)}$$

By (9.1) and inequality (5.6)

$$\Phi(x, \lambda\,;\, Q_n) \to \Phi(x, \lambda\,;\, f) \quad \text{.. .. .. (10.5)}$$

uniformly over $[a, b]$.

The result for any vector f such that $f^T Ff$ belongs to $L[a, b]$ follows from (10.3), (10.4) and (10.5).

LEMMA 10.3. Let the vector $f(x) = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$ satisfy the same conditions as in lemma (10.1) and $\psi_n \equiv \psi_n(x, \lambda_n)$ be an eigenvector corresponding to the eigenvalue λ_n.

Let

$$c_n = \int_a^b \psi_n^T F f\, dx \text{ and } c_n^* = \int_a^b \psi_n^T L f\, dx.$$

Then $c_n^* = \lambda_n c_n$.

This follows from Green's formula applied to ψ_n and f and using $[\psi_n f]_a^b = 0$.

LEMMA 10.4. The poles of $\Phi(x, \lambda)$ defined by (9.1) are all simple.

If possible, let Φ have a pole of order $m > 1$ at $\lambda = \lambda_n$. Then near $\lambda = \lambda_n$

$$\Phi(x, \lambda) = \frac{h_m(x)}{(\lambda-\lambda_n)^m} + \frac{h_{m-1}(x)}{(\lambda-\lambda_n)^{m-1}} + \dots \quad .. \quad .. \ (10.6)$$

Since

$(L-\lambda_n F)\Phi = (\lambda-\lambda_n)F\Phi - Ff$, $[\Phi\phi_j](a) = 0$ $(j = 1, 2)$ and $[\Phi\phi_k](b) = 0$ $(k = 3, 4)$,

we have

$$(L-\lambda_n F)h_m(x) = 0,\ (L-\lambda_n F)h_{m-1}(x) = Fh_m(x), \dots$$

and

$$[h_i\phi_j](a) = 0,\ j = 1, 2 \text{ and } i = m, m-1, \dots$$

$$[h_i\phi_k](b) = 0,\ k = 3, 4 \text{ and } i = m, m-1, \dots$$

Hence by Thm. (3.1)

$$[h_r h_s](a) = 0 \text{ and } [h_r h_s](b) = 0,\ r, s = m, m-1, \dots$$

Thus it follows from Green's formula that

$$\int_a^b h_m^T F h_m\, dx = 0.$$

Hence $h_m = 0$.

This holds for all f, hence $\Phi(x, \lambda)$ has at most a simple pole.

§ 11. Let the vector $\Phi(x, \lambda) = \begin{pmatrix} \Phi_1(x, \lambda) \\ \Phi_2(x, \lambda) \end{pmatrix}$ be defined by (9.1) and λ be a complex parameter. Then by lemma (10.4), the poles of $\Phi(x, \lambda)$ are all simple. From the definition it is obvious that the singularities of $\Phi(x, \lambda)$ are at the zeros of $D(\lambda)$. Let us now calculate the residues of $\Phi(x, \lambda)$ at the zeros of $D(\lambda)$.

If λ_n be a simple zero of $D(\lambda)$, then the residue at $\lambda = \lambda_n$ is given by

$$\operatorname*{Lt}_{\lambda \to \lambda_n} (\lambda-\lambda_n)\Phi(x, \lambda).$$

After a little manipulation, the residue of $\Phi(x, \lambda)$ at $\lambda = \lambda_n$ comes to be

$$\psi_n(x, \lambda_n) \int_a^b \psi_n^T(y, \lambda_n) F f(y)\, dy, \quad .. \quad .. \quad .. \ (11.1)$$

where

$$\psi_n(x, \lambda_n) = \begin{pmatrix} \psi_{1n}(x, \lambda_n) \\ \psi_{2n}(x, \lambda_n) \end{pmatrix}$$

are the normalized eigenvectors (7.6).

If λ_n be a double zero of $D(\lambda)$, by considering the $\underset{\lambda\to\lambda_n}{\text{Lt}} (\lambda-\lambda_n)\Phi(x, \lambda)$ the residue comes to be

$$\psi_n^{(1)}(x, \lambda_n)\int_a^b \psi_n^{(1)^T}(y, \lambda_n)Ff(y)\,dy+\psi_n^{(2)}(x, \lambda_n)\int_a^b \psi_n^{(2)^T}(y, \lambda_n)Ff(y)\,dy, \tag{11.2}$$

where

$$\psi_n^{(1)} = \begin{pmatrix}\psi_{1n}^{(1)}\\ \psi_{2n}^{(1)}\end{pmatrix} = \frac{\phi_1(x, \lambda_n)}{\{I_{11}(\lambda_n)\}^{\frac{1}{2}}}$$

and

$$\psi_n^{(2)} = \begin{pmatrix}\psi_{1n}^{(2)}\\ \psi_n^{(2)}\end{pmatrix} = \frac{I_{11}\phi_2(x, \lambda_n)-I_{12}\phi_1(x, \lambda_n)}{\{I_{11}(I_{11}I_{22}-I_{12}^2)\}^{\frac{1}{2}}},$$

I_{ij} having the same definition as given in (7.12), are each normalized eigenvectors in the sense of (7.7).
Also

$$\int_a^b \psi_n^{(1)^T}F\psi_n^{(2)}\,dx = 0. \quad \dots \quad \dots \quad \dots \quad \dots \tag{11.3}$$

§ 12. Bessel's inequality in our case takes the form

$$\sum_{-\infty}^{\infty} c_n^2 \leqslant \int_a^b f^T Ff\,dx, \quad \dots \quad \dots \quad \dots \tag{12.1}$$

where $c_n = \int_a^b \psi_n^T Ff\,dx$ is the *Fourier coefficient* of $f = \begin{pmatrix}f_1\\ f_2\end{pmatrix}$, ψ_n being the eigenvector corresponding to the eigenvalue λ_n.
Following exactly Titchmarsh (1962) we now have the following expansion theorem and Parseval's formula.

Theorem 12.1. Let the vector $f(x) = \begin{pmatrix}f_1(x)\\ f_2(x)\end{pmatrix}$ be twice differentiable, $f''(x)$ be $L^2[a, b]$ and let $f(x)$ satisfy the boundary conditions (1.5) and (1.6) at $x = a$ and $x = b$ respectively.
Then

$$f(x) = \sum_{n=-\infty}^{\infty} c_n\psi_n(x),$$

the series being absolutely and uniformly convergent in $[a, b]$.

Theorem 12.2. Let the vector $f(x)$ be such that $f^T(x)Ff(x)$ belongs to $L[a, b]$. Then

$$\sum_{-\infty}^{\infty} c_n^2 = \int_a^b f^T(x)Ff(x)\,dx.$$

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