ON THE DISTANCE AND RATIO SETS OF SOME NON-MEASURABLE SETS

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In this paper a new type of non-measurable set E has been constructed with some of its properties. Some mathematicians, Bose Majumder (1962), Jeffery (1951), Kestelman (1959) and Natanson (1961) have shown that a measurable set may or may not be an S-set (or an R-set). It has been demonstrated in this paper that a non-measurable set has also the same properties. Taking F to be the complementary set of E in [0, 1] which is also non-measurable, we have also discussed the nature of outer and inner measures of the sets E and F and also the measure of the sets E_1, E', F_1, F' and T where $E_1 \subset E$ is a set of maximum measure and $E' \subset E$ is the remaining non-measurable component (with similar meaning for F_1 and F') and T which is $E' \cup F'$ is found to be measurable with a positive measure.

INTRODUCTION

A well-known example of a non-measurable set A is dealt with by some authors, Jeffery (1951), Kestelman (1959) and Natanson (1961). Van Vleck (1908) has also constructed an example of a non-measurable set. We have constructed here another non-measurable set E of a different type where we used the ratios of points instead of the distances between pairs of points as used by the above authors.

NOTATIONS AND TERMS

A set E is defined to be an S-set, if its distance set fills an interval with 0 as its left-hand end point. A set E may also be defined to be an R-set, if its ratio set fills an interval with 1 as its left-hand end point.

(1) Lebesgue measure of any measurable set X will be denoted by $|X|$.

(2) Exterior and interior measures of the set E are denoted by $\overline{m}(E)$ and $\underline{m}(E)$.

(3) Union and intersection of any two sets are denoted by $A+B$ and $A \,.\, B$ respectively.

(4) r, r_1, r_2 . . . stand for rational numbers.

A NEW EXAMPLE OF A NON-MEASURABLE SET

Let I denote the interval (0, 1) and for $x \in I$, let $R(x)$ denote the set of numbers ξ, for which ξ/x is rational, where $0 < \xi \leqslant 1$.

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Properties of the class of sets $\{R(x)\}$.—(i) If x/y is irrational, then $R(x) . R(y) = 0$. For, suppose, there is ξ such that $\xi \in R(x) . R(y)$. Hence $\xi/x = r_1$, $\xi/y = r_2$ and $x/y = \frac{r_2}{r_1} =$ a rational. This contradicts the hypothesis that x/y is irrational and hence (i) follows.

(ii) If x/y is rational, then $R(x) = R(y)$. For, suppose, $\xi \in R(x)$ with $\xi/x = r_1$. Hence $\xi/y = \xi/x . x/y = r_1 \times r =$ a rational, where $r = x/y$. Therefore, $\xi \in R(y)$. Similarly, if $\xi \in R(y)$, we can show that $\xi \in R(x)$. Hence (ii) follows.

(iii) If $R(x)$ and $R(y)$ have one point common, then they are identical. This follows easily from (i) and (ii).

Thus we see that as x moves over I generating the sets of $\{R(x)\}$, the members of this class are distinct. That is to say, to each $x \in I$, there corresponds a single set $R(x)$. The correspondence is not, however, one to one; for example, by (ii) every rational number of I generates the same set.

Let E be the set formed by choosing one element ξ from each of the pairwise disjoint sets $\{R(x)\}$ according to the following procedure:

(iv) If x is rational and $x \in I$, choose $E . R(x)$ to consist of the single point 1. If x is irrational and $0 < x < \frac{1}{2}$, we choose $E . R(x)$ to consist of a single point ξ (obviously an irrational in (0, 1)), such that $\xi/x =$ a rational > 1. If x' is irrational and $\frac{1}{2} < x' < 1$, then $\frac{1}{4} < x'/2 < \frac{1}{2}$. We have $R(x') = R(x'/2)$ and hence $E . R(x') = E . R(x'/2)$.

(v) Let $R_n(E)[= r_n E]$ be the set of all points $\{r_n x\}$ where $x \in E$ and r_n is the nth rational in the enumeration of all rationals in [0, 1].

If $m \neq n$, we propose to show that $R_m(E) . R_n(E) = 0$. If possible, let $x \in R_m(E) . R_n(E)$.

Hence $x = r_m \xi$ and $x = r_n \eta$, where $\xi \in E$ and $\eta \in E$ which gives $\eta/\xi = \frac{r_m}{r_n} =$ a rational number.

Let $\xi \in R(x')$ and $\eta \in R(y')$, $x' \in I$, $y' \in I$. Hence $\xi/x' =$ a rational and $\eta/y' =$ a rational. Therefore $y'/x' =$ a rational. Hence $R(x') = R(y')$ by (ii). Thus we have chosen two points ξ and η from the one and the same set $R(x')$ [or $R(y')$] which is not permissible. We conclude that $R_m(E) . R_n(E) = 0$ if $m \neq n$.

(vi) Let $x \in [0, \frac{1}{2}]$, then there must exist at least one set $R_n(E)$, for which $x \in R_n(E)$.

PROOF: For an irrational $x \in [0, \frac{1}{2}]$ there corresponds a set $R(x)$. Let the unique point contributed by $R(x)$ to E be ξ.

Then $\xi/x =$ a rational $r > 1$ by (iv) $= 1/r'$ (say), when r' is a positive rational < 1. Therefore, $x = r'\xi$.

Hence r' (< 1) is an r_n and thus $x = r_n \xi$. It follows that $x \in R_n(E)$.

If x is rational $0 < x \leqslant \frac{1}{2}$ then $R(x)$ contains 1 and $1 \epsilon E$. Obviously $1/x =$ a rational $(> 1) = 1/r'$, where r' is a rational < 1. Hence $x = r'.1 = r_m.1$ for a certain m where $0 < r_m < 1$ and r_m is the mth rational in the enumeration of rationals in $[0, 1]$ and $1 \epsilon E$. Therefore, $x \epsilon R_m(E)$. Hence the property.

Theorem 1. The set E constructed above is non-measurable.

Proof: If possible, let E be measurable with measure $|E|$. Hence $R_n(E)$ is measurable and $|R_n(E)| = r_n|E|$. Now $R_m(E) . R_n(E) = 0$, where $m \neq n$ and also $R_n(E) \subset [0, 1]$ for all n.

Therefore

$$\left|\sum_{n=1}^{\infty} R_n(E)\right| = \sum_{n=1}^{\infty} |R_n(E)| \leqslant 1,$$

or

$$|E| \sum_{n=1}^{\infty} r_n \leq 1. \quad \text{But} \sum_{n=1}^{\infty} r_n = \infty. \quad \text{Hence } |E| = 0$$

and thus $|R_n(E)| = 0$ for all n.

Therefore

$$\sum_{n=1}^{\infty} |R_n(E)| = 0. \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \qquad (1)$$

But

$$\sum_{n=1}^{\infty} R_n(E) \supset [0, \tfrac{1}{2}] \text{ by (vi)}.$$

Hence

$$\sum_{n=1}^{\infty} |R_n(E)| \geq \tfrac{1}{2}. \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \qquad (2)$$

But (2) contradicts (1).

Hence E is non-measurable.

Theorem 2. The set E is not an R-set.

Proof: If possible, let E be an R-set. Let r (> 0) be any rational and, if possible, set $\xi/\eta = r$, where $\xi \epsilon E$ and $\eta \epsilon E$. Hence ξ and η must have come from some sets $R(x)$ and $R(x')$ respectively where $\xi/x =$ a rational and $\eta/x' =$ a rational. Hence $R(x) = R(x')$ by (ii).

Thus $\xi = \eta$ and, therefore, $r = 1$ which is a contradiction as r is any rational. Hence no two points of E are in the ratio of a rational number. Therefore E cannot be an R-set.

Corollary 1. The above demonstration gives an alternative method to prove that the set E is non-measurable.

For, by Bose Majumder's theorem (1962), a set with positive measure must have infinite pairs of points in the ratio of rational numbers. Hence if

E is measurable, then its measure $|E|$ must be zero (as E has no pair of points in the ratio of a rational number).

That the set E is non-measurable can now be completed as is given in the concluding part of Theorem 1.

Corollary 2. A non-measurable set may or may not be an R-set.

We have seen above that the non-measurable set E is not an R-set.

Let us now take a non-measurable set $G = E+F$ where F is an arbitrary set with positive measure.

Obviously G is an R-set (as it contains a subset F with positive measure (Bose Majumder (1962)).

Hence G is a non-measurable R-set.

Note: We thus see (since an R-set may be of measure zero, may be of positive measure or a non-measurable set) that nothing definite may be said about an R-set, so far as its measure is concerned. Put in another way, we may say that when in future, a necessary and sufficient condition for a given set to be an R-set is found (this problem still remains open), it will have nothing to do with the measure of the set.

Under 'Introduction' we have mentioned about the well-known non-measurable set A, which is formed by taking one point from each of the mutually exclusive sets $\{B(x)\}$, where $B(x)$ is the set of points $\{\xi\}$ and $\xi - x$ = a rational number, x and ξ being points of the interval $(0, 1)$.

Theorem 3. The set A is not an S-set.

PROOF: There exists no rational number which is the distance between a pair of points of A. If possible, let there exist a rational number d, $(0 < d < 1)$, which is the distance between a pair of points ξ and η of A.

That is, let $\xi - \eta = d$.

Therefore, there exist points x and x' of $(0, 1)$ such that $B(x)$ contributes ξ to A and $B(x')$ contributes η to A.

Hence $\xi - x = r_1$ and $\eta - x' = r_2$. Therefore, the rational number $d = \xi - \eta = (\xi - x) - (\eta - x') + x - x' = r_1 - r_2 + (x - x')$.
It follows that $x - x'$ = a rational number.

Therefore, $B(x) = B(x')$ and thus ξ and η come from the same set which is a contradiction.

Hence no two points of A are at a rational distance.

Therefore, A cannot be an S-set.

Corollary 1. The above demonstration gives us an alternative method to prove that the set A is non-measurable.

For, by Steinhaus' theorem (1920), a set with positive measure must have infinite pairs of points at rational distances. Hence, if A is measurable, then its measure $|A|$ must be zero (since we have shown above that the set A has no pair of points at a rational distance).

That the set A is non-measurable can now be shown as given in the concluding part given by Jeffery (1951).

Corollary 2. A non-measurable set may or may not be an S-set.

Proof is the same as that of the Corollary 2 of Theorem 2.

Note: The same observations may be made as in the note after Theorem 2.

Having shown that E is non-measurable, it would be interesting to investigate about the nature of its exterior and interior measures, $\overline{m}(E)$ and $\underline{m}(E)$ respectively and also $\overline{m}(F)$ and $\underline{m}(F)$ where F is the complementary set of E, i.e. $F = [0, 1]-E$, and obviously F is also non-measurable.

Let $E = E_1+E'$ where E_1 is a measurable subset with maximum measure and E' is the remaining non-measurable subset of E (Van Vleck 1908).

Also let $F = F_1+F'$ with similar meanings as above.

Obviously $T = E'+F'$ is a measurable set.

Theorem 4. To show that

(i) $\underline{m}(E) = 0$, (ii) $\overline{m}(F) = 1$, (iii) $m(E_1) = 0$, (iv) $\overline{m}(E) = \overline{m}(E') = \overline{m}(F')$ $= m(T) = 1-m(F_1)$, (v) $0 < \overline{m}(E) \leqslant \frac{1}{2}$; $0 < m(T) \leqslant \frac{1}{2}$; $\underline{m}(F) \geqq \frac{1}{2}$; $m(F_1) \geqq \frac{1}{2}$.

Since no two points of E are in rational ratio, hence by Bose Majumder's theorem (1962), it follows that $\underline{m}(E) = 0$ which is (i).

Hence, since, $\underline{m}(E) = 1-\overline{m}(F)$, $\overline{m}(F) = 1$ which is (ii).

The other results may be easily proved by using some of Van Vleck's (1908) results.

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References

Bose Majumder, N. C. (1962). On some properties of sets with positive measures. *Annali Univ. Ferrara*, Vol. X, No. 1, pp. 1–12.

Jeffery, R. L. (1951). The Theory of Functions of a Real Variable. University of Toronto Press, Toronto, p. 62.

Kestelman, H. (1959). Modern Theories of Integration, Dover Publication Inc., p. 90.

Natanson, I. P. (1961). Theory of Functions of a Real Variable. Frederick Ungar Publishing Co., New York, Vol. I, p. 76.

Steinhaus, H. (1920). Surles distances des points des ensembles de mesure positive. *Fundamenta Math.*, Vol. I, pp. 93–104.

Van Vleck, E. B. (1908). On non-measurable sets of points with an example. *Trans. Am. Math. Soc.*, 9, pp. 237–244.