NON-STEADY FLOW IN A VISCOUS INCOMPRESSIBLE FLUID AROUND A POROUS ROTATING DISK

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(Communicated by R. S. Varma, F.N.I.)

(*Received March 31, 1967*)

The unsteady flow of a viscous incompressible fluid produced by a porous disk rotating with time-dependent angular velocity $\omega(t)$ is discussed. The resulting non-linear partial differential equations are solved by the Galerkin method and approximate solutions of the flow functions for different values of the parameters involved are obtained. The case of suddenly accelerated motion of the disk is deduced from the general case considered. It is found that at $\eta = 1$, the drag is independent of time in all cases of constant rotation of the disk irrespective of whether the suction (influx) velocity is time-dependent or constant. In the case of time-dependent angular velocity the drag at $\eta = 1$ attains that value at $\tau = 1$.

INTRODUCTION

The pioneering work on the flow due to a rotating disk has been done by von Kármán (1921). He studied the problem of steady state flow of a fluid due to a rotating disk, assuming that the disk is in motion for a sufficient length of time. Later the above problem was developed by Cochran (1934) by matching two solutions, one near and the other far away from the disk. Since, before attaining the steady state, at the start of the motion, the fluid rotation is essentially unsteady, much importance is to be attached to this transient phase of flow. Thiriot (1940) studied the problem of flow due to suddenly accelerated or suddenly stopped disk and obtained the solutions for the flow functions in a series of powers of the angle of rotation. The case of non-steady motion of a viscous liquid around a gradually accelerated rotating disk where the rotation velocity may be time-dependent was treated by Dolidze (1954) by setting up a system of integro-differential equations, which were solved by the method of successive approximations. Nigam (1951) also investigated the suddenly accelerated motion of a viscous incompressible fluid and obtained the solutions for the flow and pressure functions in the early stages of motion. Sparrow and Gregg (1960) investigated the flow about a disk rotating unsteadily with time-variant angular velocity $\omega(t)$

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by expanding the flow functions in a series involving $\beta_1 = \frac{d\omega}{dt}$, $\beta_2 = \frac{d^2\omega}{dt^2}$, etc. Batchelor (1951) made a qualitative study of the problem of a rotating disk.

More recently, Benton (1966) discussed the steady state problem and also investigated the flow due to a suddenly accelerated disk through the series used by Thiriot to a better approximation. He obtained the steady state solution by using the time-dependent initial value problem. The unsteady flow of a viscous incompressible fluid starting from rest between two infinite disks (at a given distance) both of which are rotating with different time-dependent angular velocities in the presence of uniform fluid injections at the disks has been studied by Tirskii (1958).

In the present paper the non-stationary flow due to a rotating disk starting from rest with a time-dependent velocity $\omega(t)$ has been discussed. Further it is assumed that the disk is porous [the general case when the suction (influx) velocity W is also time-dependent is considered] and different cases of suction and injection are investigated. To satisfy the initial conditions it is assumed that $\Omega(\tau) = L\tau^p$ and $W(\tau) = M\tau^q$, where τ is the non-dimensional time and L, M, p, q are arbitrary constants. It can be seen that the case of flow due to a suddenly accelerated disk is obtained when $p = q$, $M = 0$. The resulting non-linear partial differential equations have been solved by the Galerkin method. Numerical solutions for the flow functions are obtained for different values of the parameters involved. It is found that at $\eta = 1$, the drag is independent of time in all cases of constant rotation with or without suction or injection which may or may not be time-dependent. Further it is seen that in the case of time-dependent rotational velocity the drag at $\eta = 1$ attains the above value only at $\tau = 1$. It is observed that the presence of constant influx in the case of a suddenly accelerated disk has the effect of causing the tangential and radial flow grow in time, while the presence of a constant suction has the opposite effect of causing it decrease with time, always maintaining lower values than those in the corresponding injection case. In the case of time-dependent rotation of the disk the fluid mass at a certain distance away from the disk begins to rotate slowly in the direction opposite to the disk rotation. This distance increases slowly with the increase in time.

Equations of Motion

We assume that the infinite disk surrounded by a viscous incompressible fluid is rotating with a time-dependent angular velocity and also that a steady injection (or suction) of the same fluid with a time-dependent velocity $\omega(t)$ (+ve for influx and −ve for suction) is superimposed on the rotation. In a cylindrical polar coordinate system, the Navier-Stokes equations of motion in the absence of mass forces, together with the continuity equation, can be written as (azimuthal variation being neglected due to the axial symmetry)

$$\frac{\partial u}{\partial r}+\frac{u}{r}+\frac{\partial w}{\partial z}=0 \quad (1)$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial r}+w\frac{\partial u}{\partial z}-\frac{v^2}{r}=-\frac{1}{\rho}\frac{\partial p}{\partial r}+\nu\left[\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{\partial^2 u}{\partial z^2}-\frac{u}{r^2}\right] \quad (2)$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial r}+w\frac{\partial v}{\partial z}+\frac{uv}{r}=\nu\left[\frac{\partial^2 v}{\partial r^2}+\frac{1}{r}\frac{\partial v}{\partial r}+\frac{\partial^2 v}{\partial z^2}-\frac{v}{r^2}\right] \quad (3)$$

and

$$\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial r}+w\frac{\partial w}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial z}+\nu\left[\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r}+\frac{\partial^2 w}{\partial z^2}\right] \quad (4)$$

where u, v, w are the radial, tangential and axial components of the velocity. The no-slip condition at the disk, the presence of influx or suction, and the initial conditions determined from the absence of an initial motion velocity give us the boundary conditions

$$u(r, 0, t)=u(r, \infty, t)=0 \quad (5)$$

$$v(r, 0, t)=r\omega(t), \quad v(r, \infty, t)=0 \quad (6)$$

$$w(r, 0, t)=w_0(t) \quad (7)$$

and the initial condition

$$u(r, z, 0)=v(r, z, 0)=w(r, z, 0)=0. \quad (8)$$

Equations (1) to (8) form our basic problem to which an approximate solution is sought.

We non-dimensionalize the above equations (1) to (8) with the help of the following transformations:

$$\left.\begin{aligned} u&=\frac{r}{t_0}F(\eta, \tau), \quad v=\frac{r}{t_0}G(\eta, \tau), \quad w=\sqrt{\frac{\nu}{t_0}}H(\eta, \tau) \\ \frac{pt_0}{\rho\nu}&=\tfrac{1}{2}A(\tau)\frac{r^2}{\nu t_0}+B(\eta, \tau), \quad \eta=\frac{z}{\sqrt{\nu t_0}}, \quad \tau=\frac{t}{t_0} \end{aligned}\right\} \quad (9)$$

where t_0 is a characteristic time of the problem, ν the coefficient of kinematic viscosity, p the pressure and ρ the density. Here it may be pointed out that the expression assumed for pressure is different from the one assumed by Cochran (1934).

Substituting the transformations (9) in equations (1) to (8) we obtain a system of non-linear partial differential equations

$$\frac{\partial^2 G}{\partial \eta^2}=H\frac{\partial G}{\partial \eta}-\frac{\partial H}{\partial \eta}G+\frac{\partial G}{\partial \tau}, \quad 2F+\frac{\partial H}{\partial \eta}=0 \quad (10)$$

$$\frac{\partial^4 H}{\partial \eta^4}=H\frac{\partial^3 H}{\partial \eta^3}+4G\frac{\partial G}{\partial \eta}+\frac{\partial^3 H}{\partial \eta^2 \partial \tau} \quad (11)$$

together with the boundary conditions

$$H(0, \tau) = \sqrt{\frac{t_0}{\nu}}\, w_0(t) = W(\tau) \qquad \ldots \quad \ldots \quad \ldots \quad (12)$$

$$\frac{\partial H(0, \tau)}{\partial \eta} = \frac{\partial H(\infty, \tau)}{\partial \eta} = 0 \qquad \ldots \quad \ldots \quad \ldots \quad (13)$$

$$G(0, \tau) = \omega(t) \,.\, t_0 = \Omega(\tau), \;\; G(\infty, \tau) = 0 \qquad \ldots \quad \ldots \quad (14)$$

and the initial condition

$$G(\eta, 0) = H(\eta, 0) = 0. \qquad \ldots \quad \ldots \quad \ldots \quad (15)$$

The functions $A(\tau)$ and $B(\tau)$ can be determined after solving equations (10) to (15) from the equations

$$A(\tau) = \frac{\partial^2 F}{\partial \eta^2} + G^2 - F^2 - H\frac{\partial F}{\partial \eta} - \frac{\partial F}{\partial \tau} \qquad \ldots \quad \ldots \quad \ldots \quad (16)$$

$$\frac{\partial B(\eta, \tau)}{\partial \eta} = \frac{\partial^2 H}{\partial \eta^2} - H\frac{\partial H}{\partial \eta} - \frac{\partial H}{\partial \tau}. \qquad \ldots \quad \ldots \quad \ldots \quad (17)$$

Solution of the Equations

We assume for G and H approximate solutions satisfying the boundary conditions (12) to (15) as

$$G \approx \Omega e^{-\eta} + \sum_{n=1}^{\infty} a_n \eta^n \tau^n e^{-n\eta} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (18)$$

$$H \approx W + \sum_{m=1}^{\infty} b_m \tau^m (1 - e^{-m\eta^2}) \qquad \ldots \quad \ldots \quad \ldots \quad (19)$$

where $\Omega(\tau)$ and $W(\tau)$ are to be chosen to satisfy the prescribed initial conditions $\Omega(0) = W(0) = 0$.

The solution of the above equations (10) to (15) are obtained, taking into account two arbitrary constants in each of the equations (18) and (19). Better approximations can be obtained by considering more number of arbitrary constants in the functions G and H.

Substituting the values of the functions G and H in the equations (10) and (11) we obtain the defect functions D_G and D_H as

$$\begin{aligned} D_G \equiv \frac{\partial^2 G}{\partial \eta^2} - H\frac{\partial G}{\partial \eta} + \frac{\partial H}{\partial \eta} G - \frac{\partial G}{\partial \tau} &\approx \Omega e^{-\eta} - 2a_1\tau e^{-\eta} + 2a_2\tau^2 e^{-2\eta} \\ &+ a_1\eta\tau e^{-\eta} - 8a_2\eta\tau^2 e^{-2\eta} + 4a_2\eta^2\tau^2 e^{-2\eta} + [\{(W + b_1\tau + b_2\tau^2) \\ &- b_1\tau e^{-\eta^2} - b_2\tau^2 e^{-2\eta^2}\}\{(\Omega - a_1\tau)e^{-\eta} + a_1\tau\eta e^{-\eta} - 2a_2\tau^2\eta e^{-2\eta} \\ &+ 2a_2\tau^2\eta^2 e^{-2\eta}\}] + (\Omega e^{-\eta} + a_1\eta\tau e^{-\eta} + a_2\eta^2\tau^2 e^{-2\eta})(2b_1\tau\eta e^{-\eta^2} \\ &+ 4b_2\tau^2\eta e^{-2\eta^2}) - \frac{\partial \Omega}{\partial \tau} e^{-\eta} - a_1\eta e^{-\eta} - 2a_2\tau\eta^2 e^{-2\eta} \end{aligned}$$

$$D_H \equiv \frac{\partial^4 H}{\partial \eta^4} - H\frac{\partial^3 H}{\partial \eta^3} - 4G\frac{\partial G}{\partial \eta} - \frac{\partial^3 H}{\partial \eta^2 \partial \tau} \approx -12b_1\tau e^{-\eta^2} - 48b_2\tau^2 e^{-2\eta^2}$$
$$+48b_1\eta^2\tau e^{-\eta^2} - 16b_1\tau\eta^4 e^{-\eta^2} + 384b_2\tau^2\eta^2 e^{-2\eta^2} - 256b_2\tau^2\eta^4 e^{-2\eta^2}$$
$$+\{(W+b_1\tau+b_2\tau^2) - b_1\tau e^{-\eta^2} - b_2\tau^2 e^{-2\eta^2}\}\{12b_1\tau\eta e^{-\eta^2} + 48b_2\tau^2\eta e^{-2\eta^2}$$
$$-8b_1\tau\eta^3 e^{-\eta^2} - 64b_2\tau^2\eta^3 e^{-2\eta^2}\} + 4(\Omega e^{-\eta} + a_1\eta\tau e^{-\eta} + a_2\eta^2\tau^2 e^{-2\eta})$$
$$\times(\Omega e^{-\eta} - a_1\tau e^{-\eta} - 2a_2\eta\tau^2 e^{-2\eta} + a_1\eta\tau e^{-\eta} + 2a_2\eta^2\tau^2 e^{-2\eta})$$
$$-2b_1 e^{-\eta^2} - 8b_2\tau e^{-2\eta^2} + 4b_1\eta^2 e^{-\eta^2} + 32b_2\tau\eta^2 e^{-2\eta^2}.$$

Now we choose the time-dependent functions Ω and W satisfying the initial conditions, as $\Omega = L\tau^p$ and $W = M\tau^q$, where L, p, M, q are constants and can have prescribed values. Applying the Galerkin technique of orthogonalizing the defect functions D_G and D_H with the constituent functions of G and H respectively, we arrive at (after performing the necessary integrations) four non-linear simultaneous algebraic equations for the arbitrary constants a_1, a_2, b_1, b_2 for different combinations of L, M, p and q. The equations are given as

$$\frac{5a_1}{24} + \frac{11a_2}{162} + \frac{L}{p+3}\left(\frac{5}{4} - 2\sqrt{\pi}\, e \operatorname{erfc} 1\right)b_1 + \frac{L}{4(p+4)}\left(2 - 5\sqrt{\frac{\pi e}{2}} \operatorname{erfc}\frac{1}{\sqrt{2}}\right)b_2$$
$$= \frac{1}{4}\left(\frac{LM}{p+q+2} - \frac{L(p^2+p-1)}{(p+1)(p+2)}\right) + \frac{3}{4}\left(1 - \frac{5}{4}\sqrt{\pi}\, e \operatorname{erfc} 1\right)a_1 b_1$$
$$-\frac{1}{16}\left(38 - 67\sqrt{\pi}\, e^{\frac{9}{4}} \operatorname{erfc}\frac{3}{2}\right)a_2 b_1 + \frac{3}{40}\left(3 - 4\sqrt{\frac{\pi e}{2}} \operatorname{erfc}\frac{1}{\sqrt{2}}\right)a_1 b_2$$
$$-\frac{5}{384}\left(29 - \frac{115}{2}\sqrt{\frac{\pi}{2}}\, e^{\frac{9}{8}} \operatorname{erfc}\frac{3}{2\sqrt{2}}\right)a_2 b_2 \qquad \ldots \qquad (20)$$

$$\frac{7a_1}{162} + \frac{23a_2}{1280} + \left(7\sqrt{\pi}\, e^{\frac{9}{4}} \operatorname{erfc}\frac{3}{2} - \frac{110}{27}\right)\frac{L}{p+4}b_1 + \left(\frac{19}{8}\sqrt{\frac{\pi}{2}}\, e^{\frac{9}{8}} \operatorname{erfc}\frac{3}{2\sqrt{2}} - \frac{143}{108}\right)\frac{L}{p+5}b_2$$
$$= \frac{2L}{27}\left(\frac{M}{p+q+3} - \frac{p^2+2p-2}{(p+2)(p+3)}\right) + \frac{1}{20}\left(67\sqrt{\pi}\, e^{\frac{9}{4}} \operatorname{erfc}\frac{3}{2} - 38\right)a_1 b_1$$
$$+\left(9 - \frac{477}{24}e^4\sqrt{\pi} \operatorname{erfc} 2\right)a_2 b_1 + \frac{1}{192}\left(115\sqrt{\frac{\pi}{2}}\, e^{\frac{9}{8}} \operatorname{erfc}\frac{3}{2\sqrt{2}} - 58\right)a_1 b_2$$
$$+\frac{1}{112}\left(90 - 213e^2\sqrt{\frac{\pi}{2}} \operatorname{erfc}\sqrt{2}\right)a_2 b_2 \qquad \ldots \qquad (21)$$

$$\frac{2L}{p+3}(3\sqrt{\pi}\, e \operatorname{erfc} 1 - 2)a_1 + \frac{L}{p+4}\left(13 - \frac{45}{2}\sqrt{\pi}\, e^{\frac{9}{4}} \operatorname{erfc}\frac{3}{2}\right)a_2$$
$$+\frac{3}{4}\sqrt{\frac{\pi}{2}}\, b_1 + \left(\frac{10\sqrt{3}}{27}\sqrt{\pi} - \frac{4}{9}\frac{M}{(q+4)}\right)b_2$$

$$= \frac{L^2}{p+1}(\sqrt{\pi}\, e \operatorname{erfc} 1 - 1) - \left(1 - \frac{5}{4}\sqrt{\pi}\, e \operatorname{erfc} 1\right) a_1^2$$

$$- \left(12 - \frac{53}{2} e^4 \sqrt{\pi} \operatorname{erfc} 2\right) a_2^2 - \left(\frac{201}{20}\sqrt{\pi}\, e^{\frac{9}{4}} \operatorname{erfc} \frac{3}{2} - \frac{57}{10}\right) a_1 a_2$$

$$+ \frac{1}{9} b_1^2 + \frac{104}{675} b_2^2 + \frac{43}{180} b_1 b_2 \qquad \ldots \qquad (22)$$

$$\frac{2L}{p+4}\left(2\sqrt{\frac{\pi e}{2}} \operatorname{erfc} \frac{1}{\sqrt{2}} - 1\right) a_1 + \frac{L}{8(p+5)}\left(34 - 63\sqrt{\frac{\pi}{2}}\, e^{\frac{9}{8}} \operatorname{erfc} \frac{3}{2\sqrt{2}}\right) a_2$$

$$+ \left(\frac{8\sqrt{\pi}}{27} + \frac{4M}{9(q+4)}\right) b_1 + \frac{17}{20}\sqrt{\pi}\, b_2$$

$$= \frac{2L^2}{2p+3}\left(\sqrt{\frac{\pi e}{2}} \operatorname{erfc} \frac{1}{\sqrt{2}} - 1\right) - \frac{1}{10}\left(3 - 4\sqrt{\frac{\pi e}{2}} \operatorname{erfc} \frac{1}{\sqrt{2}}\right) a_1^2$$

$$- \frac{1}{28}\left(30 - 71 e^2 \sqrt{\frac{\pi}{2}} \operatorname{erfc} \sqrt{2}\right) a_2^2 - \frac{1}{64}\left(115\sqrt{\frac{\pi}{2}}\, e^{\frac{9}{8}} \operatorname{erfc} \frac{3}{2\sqrt{2}} - 58\right) a_1 a_2$$

$$+ \frac{11}{180} b_1^2 + \frac{8}{63} b_2^2 + \frac{112}{675} b_1 b_2. \qquad \ldots \qquad (23)$$

It may be observed here that any values can be prescribed for the constants L, M, p and q. Without any loss of generality we assume here that $L = 1$ and prescribe the values 1, -1, 0 for M to correspond to the three important cases of fluid influx, fluid suction and the absence of either influx or suction respectively. The case of the suddenly accelerated disk can be studied by putting $p = 0$ and the case of constant fluid influx (or suction) can be studied by assigning to q the value zero (*see* Appendix) in equations (20) to (23). The other cases of interest examined are when $p = \frac{1}{2}$, 1 and $q = \frac{1}{2}$, 1.

A simple iteration technique to solve the above equations (20) to (23) for the arbitrary constants a_1, a_2, b_1, b_2 is used. In the first instance, all the non-linear terms involving a_1, a_2, b_1, b_2 are neglected and the resulting linear equations are solved. These rough values of a_1, a_2, b_1, b_2 are substituted in the non-linear terms on the right-hand side of the equations (20) to (23) and the equations were again solved for a_1, a_2, b_1, b_2. This procedure is repeated till the values of the constants a_1, a_2, b_1, b_2 obtained by the above iteration are stabilized up to six decimal places. (I.B.M. 1620 is used for computation). Using these values of the constants obtained for different values of p, q and M, the flow functions and the drag are calculated and the results are discussed in the next section.

For a disk of finite but large radius R, the torque $M(\tau)$ is given as

$$\begin{aligned} M(\tau) &= -2\pi \int_0^R r^2\, T_{z\phi}\, dr \\ &= -2\pi\mu \int_0^R \left(\frac{\partial v}{\partial z}\right)_{z=0} r^2\, dr \\ &= -\frac{2\pi\mu}{t_0\sqrt{\nu t_0}} \int_0^R G'(0,\tau)\, r^3\, dr \\ &= -\frac{\pi\rho R^4}{2}\left(\frac{\nu}{t_0^3}\right)^{\frac{1}{2}} G'(0,\tau). \end{aligned}$$

The tangential shear stress at the disk is given by

$$\mu\left(\frac{\partial v}{\partial z}\right)_{z=0} = \frac{\mu r}{t_0\sqrt{\nu t_0}}\, G'(0,\tau) = \rho\left(\frac{\nu}{t_0^3}\right)^{\frac{1}{2}} r\, G'(0,\tau).$$

The values of $G'(\eta, \tau)$ are obtained for different values of η and τ for fixed p, q and M and are often referred to as drag, during the discussion of the results in the next section.

Discussion of the Results

As mentioned earlier, the numerical values of the flow functions and drag have been calculated for different values of η and τ for fixed p, q and M. [Only a few of the results are exhibited in graphs and a few are tabulated. The authors will be glad to furnish any result on request.] The following observations have been made from the numerical results obtained:

(i) $p = 0$, $M = 0$, i.e. the case of suddenly accelerated disk in the absence of suction or influx (Fig. 1).

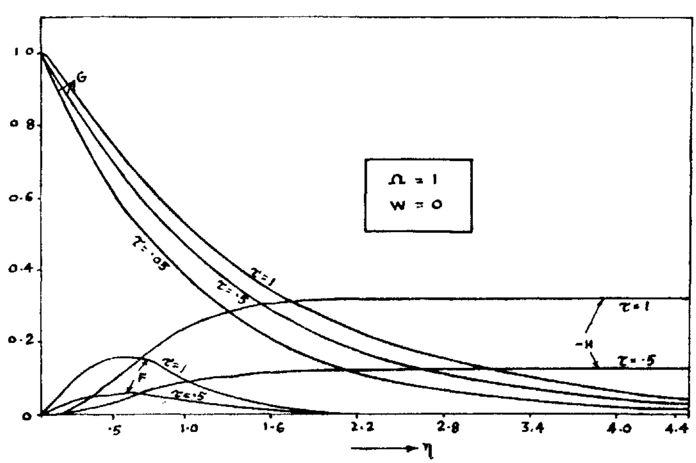

FIG. 1. Flow functions F, G, $-H$.

This case corresponds to the problem discussed by Thiriot (1940) and Benton (1966). Limiting our analysis for $\tau \leqslant 1$, we observe that at $\tau = 1$ the flow functions attain values close to the steady state solution obtained by Cochran (1934), near the disk. We further notice that near the disk drag decreases with time while far away it increases with time. At $\eta = 1$, drag is independent of time and maintains a constant value 0·36787944. The values of drag at $\tau = 1$ are lower than the corresponding values at $\tau = 0{\cdot}5$ in the ranges $0 \leqslant \eta \leqslant 0{\cdot}5$ and $1 \leqslant \eta \leqslant 1{\cdot}6$, and higher than the values in the ranges $0{\cdot}5 < \eta < 1$ and $1{\cdot}6 < \eta < \infty$.

Initially ($\tau = 0{\cdot}05$) the drag function $-G'$ registers a steady decrease with distance from the disk, but with the elapse of time it shows an upward trend for a very small distance from the disk, before it begins to decrease with increasing distance. At the disk, the drag is as low as $-G'(0) = 0{\cdot}31526$ for $\tau = 1$. It attains its maximum value at $\eta = 0{\cdot}3$ and then starts decreasing to zero at infinity. The axial flow is everywhere directed towards the disk and increases with time. At $\eta = 2{\cdot}8$ it attains its maximum which is maintained till infinity. The tangential flow decreases very rapidly near the disk till $\eta = 1$, after which the decrease is rather slow. As is expected, there is an outward radial flow everywhere which grows stronger with time. At $\tau = 1$, this flow rate is highest near the plane $\eta = 0{\cdot}6$ beyond which it gradually dies down.

(ii) $p = 1$, $M = 0$, i.e. the case of time-dependent rotation $\Omega = \tau$ of the disk without suction or injection (Figs. 2, 8(*b*)).

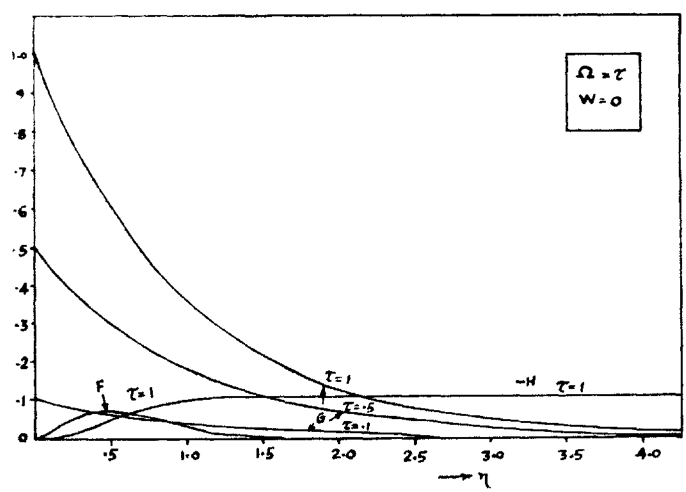

FIG. 2. Flow functions F, G, $-H$.

In this case we observe that for $\eta < 0{\cdot}2$, the axial flow is towards the disk and the radial flow is outward, whereas for $\eta > 0{\cdot}2$ there is a small axial flow away from the disk and the radial flow is inward. At $\eta = 0{\cdot}2$, the radial flow is inward and the axial flow towards the disk. The inward radial flow exists in the region $\eta \geqslant 0{\cdot}9$ for $\tau = 0{\cdot}1$, in $\eta \geqslant 1{\cdot}6$ for $\tau = 0{\cdot}5$ and in $\eta \geqslant 1{\cdot}8$ for $\tau = 1$. In the rest of fluid space, the radial flow is outwards. For $\tau = 1$, the strongest inward and outward radial flows occur at $\eta = 2$ and $\eta = 0{\cdot}5$ respectively and near $\eta = 1{\cdot}7$ there exists a region with no radial flow at all. In this case the fluid rotates much slower than the corresponding layers in case (i) and the drag gradually increases almost in proportion with time. The drag at the disk at $\tau = 1$ is more than three times the value in case (i) though at $\eta = 1$ drags in both the cases are the same. The axial flow in this case is feebler.

(iii) $p = 0$, $M = 1$, $q = 0$, i.e. the case of a constant fluid influx, in the case of a suddenly accelerated disk (Figs. 3, 8(d)).

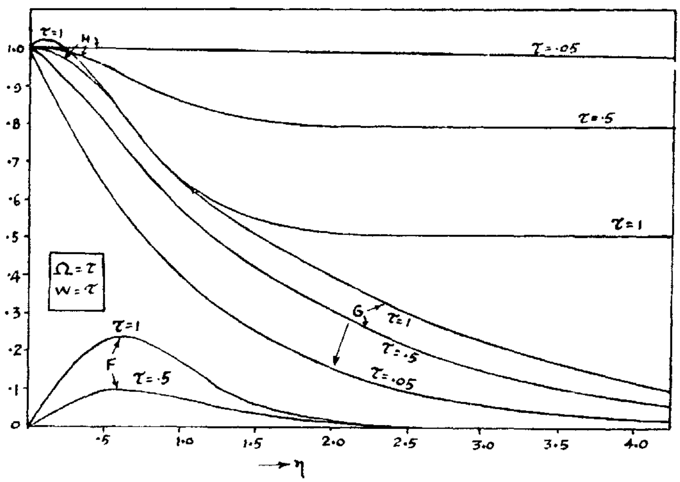

FIG. 3. Flow functions F, G, H (with influx).

The influx lowers down the drag at the disk and in all fluid planes till $\eta = 1$, after which the drag values are higher as compared to case (i) without influx or suction. Though initially ($\tau = 0{\cdot}05$) drag decreases with distance from the disk, at $\tau = 0{\cdot}5$ we see that it first increases till $\eta = 0{\cdot}4$, after which it begins to decrease and at $\tau = 1$ G' is positive at the plate and in a region very near the plate on account of the fact that the fluid rotation speed is higher than the disk rotation in that region due to the presence of influx. The axial velocity, which is away from the disk, decreases with time throughout the fluid, the decrease being more pronounced with increasing distance from the disk. At $\eta = 3$, it falls to nearly half the value of influx velocity, and stabilizes at that value. The radial flow is everywhere and always outwards attaining its maximum at $\eta = 0{\cdot}6$ (same as case (i), without injection). In this case this maximum is one and a half times the maximum of case (i). The tangential velocity is maximum at $\eta = 0{\cdot}1$, with a value slightly higher than the disk velocity.

(iv) $p = 0$, $M = 1$, $q = 1$, i.e. the case of a suddenly accelerated disk in presence of a time-dependent fluid influx velocity $W = \tau$ (Table I).

TABLE I

Disk suddenly accelerated to $\Omega = 1$ ($p = 0$, $M = 1$, $q = 1$)

τ	η	F	G	H	$-G'$
0·05	0	0·000000	1·000000	0·050000	0·997391
	0·05	0·000094	0·951353	0·049995	0·948869
	0·1	0·000186	0·905073	0·049926	0·817016
	0·2	0·000359	0·819158	0·049839	0·739459
	0·5	0·000699	0·607324	0·049593	0·605734
	0·7	0·000742	0·497495	0·049299	0·496193
	0·8	0·000715	0·450270	0·049153	0·449092
	1·0	0·000601	0·368843	0·048888	0·367879
	2·0	0·000054	0·136043	0·048341	0·135690
	3·0	0·000000	0·050177	0·048314	0·050047
	3·6	0·000000	0·027580	0·048314	0·027509
0·1	0	0·000000	1·000000	0·100000	0·994782
	0·1	0·000450	0·905310	0·099954	0·900571
	0·2	0·000865	0·819588	0·099822	0·815289
	0·5	0·001640	0·608123	0·099029	0·604927
	0·6	0·001709	0·550542	0·098693	0·547649
	0·7	0·001693	0·498412	0·098351	0·495796
	1·2	0·000952	0·303095	0·096980	0·301513
	2·0	0·000110	0·136756	0·096285	0·136049
	3·0	0·000001	0·050569	0·096231	0·050309
	3·6	0·000000	0·027838	0·096231	0·027696
0·5	0	0·000000	1·000000	0·500000	0·973910
	0·1	0·005367	0·907221	0·499458	0·883175
	0·2	0·010194	0·823078	0·497889	0·801038
	0·5	0·017842	0·614702	0·488892	0·598100
	0·6	0·017831	0·557708	0·485306	0·542677
	1·0	0·010849	0·377858	0·473418	0·367879
	2·0	0·000571	0·142603	0·465537	0·139072
	3·0	0·000005	0·053746	0·465260	0·052468
	3·6	0·000000	0·029917	0·465258	0·029216
1·0	0	0·000000	1·000000	1·000000	0·947820
	0·1	0·018526	0·909651	0·998130	0·860683
	0·2	0·035063	0·827577	0·992723	0·782135
	0·5	0·059788	0·623391	0·962148	0·588633
	0·6	0·058875	0·567216	0·950215	0·535727
	1·0	0·032454	0·388601	0·912475	0·367879
	2·0	0·001196	0·150284	0·891349	0·143223
	3·0	0·000011	0·057832	0·890781	0·055318
	3·6	0·000000	0·032565	0·890777	0·031188

In this case the circumferential velocity profile is almost stable. As expected, the axial velocity increases with time. Initially ($\tau = 0{\cdot}05$) it stabilizes beyond $\eta = 2{\cdot}6$ at 97 per cent of its value at the disk. This percentage decreases with time and is 89 per cent of the value at the disk at $\tau = 1$. At $\eta = 1$, drag is independent of time. Drag increases with time for $\eta > 1$ and decreases in the fluid region $\eta < 1$. The radial flow which is outwards increases with time. The plane of maximum radial flow shifts from $\eta = 0{\cdot}7$ at $\tau = 0{\cdot}05$ towards the disk and occurs for $\tau = 1$ at $\eta = 0{\cdot}5$. The maximum value of the radial flow in this case is much lower than in the case without influx (case i) and the case with constant influx (case iii).

(v) $M = 1$, $p = 1$, $q = 0$, i.e. the case of time-dependent rotation $\Omega = \tau$ with constant influx (Fig. 4).

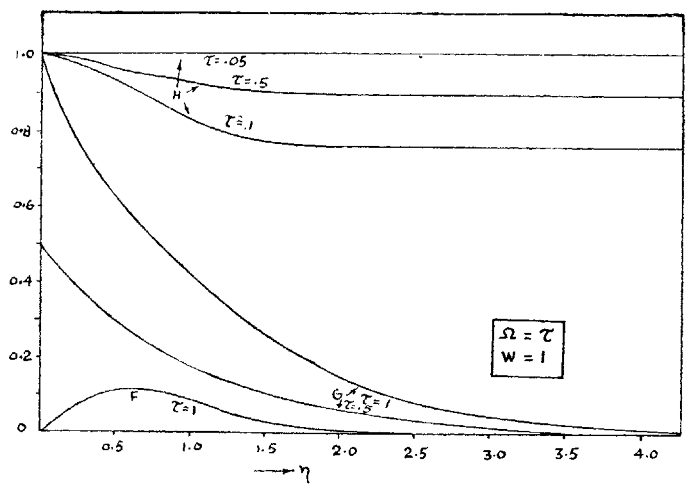

FIG. 4. Flow functions F, G, H (with influx).

Drag in this case is higher at the disk and very near it, compared to case (ii) without influx, but decreases very rapidly for $0{\cdot}2 \leqslant \eta < 1$ and maintains lower values compared to case (ii). For $\eta = 1$ at $\tau = 1$ the drags are equal after which the drags at different layers are higher than at the corresponding layers of case (ii). Axial velocity at infinity which is 99 per cent initially ($\tau = 0{\cdot}05$) and 89·5 per cent at $\tau = 0{\cdot}5$ stabilizes at 75·6 per cent of the influx velocity at $\tau = 1$. The radial flow is everywhere outwards, unlike case (ii) where an inward radial flow exists beyond $\eta = 1{\cdot}8$ at $\tau = 1$. The maximum value of the outward radial flow is much higher in this case.

(vi) $p = \frac{1}{2}$, $q = \frac{1}{2}$, $M = 1$, i.e. the case when disk rotation and influx velocities are time-dependent and each is equal to $\sqrt{\tau}$ (Fig. 5).

In this case we see that for $\tau = 1$, a feeble outward radial flow exists attaining its maximum at $\eta = 0{\cdot}6$. Drag at $\eta = 1$ is the same as in other cases and the axial flow away from the disk at infinity is 79 per cent of its value at the disk.

(vii) $M = 1$, $p = 0$, $q = 0$, i.e. the case of a sudden acceleration with a constant suction at the disk.

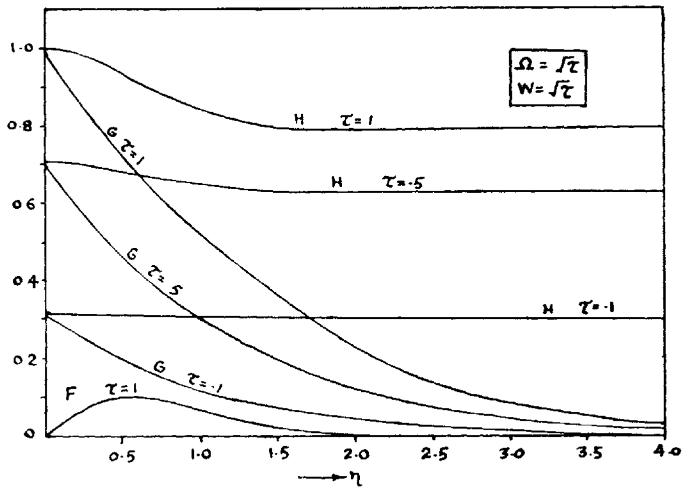

FIG. 5. Flow functions F, G, H (with influx).

Here we observe that the axial velocity towards the disk is everywhere higher than its value at the disk and increases with distance η as also with time τ. At infinity, for $\tau = 1$ it is about 20 per cent higher than the suction velocity. The tangential velocity profile does not suffer much change in time. The outward radial flow is less and fluid rotation slower, compared with the non-suction case (i). Drag increases with time between the disk and $\eta = 1$, while it decreases beyond $\eta = 1$. Also suction increases the drag in the range $0 \leqslant \eta < 1$ while decreases it for $\eta > 1$, compared to the non-suction case (i). At $\eta = 1$, the drag is independent of suction and time. [Compare this with the constant influx case (iii) where influx decreases the drag for $0 \leqslant \eta < 1$ and increases the values beyond $\eta = 1$, in comparison to case (i)].

(viii) $M = -1$, $p = 0$, $q = 1$, i.e. the case of a sudden acceleration with time-dependent suction velocity $-\tau$ (Table II).

Initially ($\tau = 0{\cdot}05$) an inward radial flow is introduced beyond $\eta = 0{\cdot}3$, the outward radial flow being confined to a very small region near the disk, viz. $0 < \eta < 0{\cdot}3$. With elapsing time the starting plane of inward radial flow recedes farther from the disk. For instance, at $\tau = 1$, it starts at as far away as $\eta = 1{\cdot}8$ from the disk. But the inward radial flow is much weaker than the outward radial flow, which, at $\tau = 1$, is maximum at $\eta = 0{\cdot}5$. The tangential velocity profile does not change with time. The axial velocity, which is directed towards the disk, attains its highest at $\eta = 1{\cdot}8$, where it is

20 per cent higher than the suction velocity and distribution is roughly the same as in the case of constant suction (case vii).

TABLE II

Disk suddenly accelerated to $\Omega = 1$ ($p = 0$, $M = -1$, $q = 1$)

τ	η	F	G	$-H$	$-G'$
0·05	0	0·000000	1·000000	0·050000	1·000000
	0·1	0·000003	0·904831	0·050000	0·904892
	0·2	0·000000	0·818719	0·050001	0·818775
	0·3	−0·000013	0·740803	0·049999	0·740853
	0·9	−0·000227	0·406545	0·049855	0·406572
	1·0	−0·000239	0·367854	0·049808	0·367879
	1·1	−0·000239	0·332846	0·049760	0·332868
	3·6	0·000000	0·027317	0·049494	0·027319
	4·4	0·000000	0·012273	0·049494	0·012274
0·1	0	0·000000	1·000000	0·100000	1·000000
	0·1	0·000223	0·904825	0·100022	0·904947
	0·3	0·000526	0·740788	0·100181	0·740889
	0·4	0·000561	0·670283	0·100291	0·670375
	0·5	0·000515	0·606489	0·100400	0·606572
	0·8	0·000089	0·449280	0·100593	0·449341
	0·9	−0·000061	0·406519	0·100596	0·406575
	1·3	−0·000312	0·272483	0·100406	0·272520
	1·4	−0·000294	0·246550	0·100345	0·246583
	3·6	0·000000	0·027310	0·100091	0·027314
0·5	0	0·000000	1·000000	0·500000	1·000673
	0·1	0·009769	0·904775	0·500987	0·905402
	0·4	0·028447	0·670131	0·513546	0·670614
	0·5	0·029346	0·606316	0·519370	0·606755
	0·6	0·027800	0·548578	0·525120	0·548975
	1·4	0·000979	0·246351	0·544993	0·246523
	1·6	−0·000253	0·201667	0·545085	0·201806
	1·8	−0·000441	0·165088	0·544927	0·165201
	2·0	−0·000313	0·135145	0·544773	0·135236
	3·6	0·000000	0·027256	0·544597	0·027274
1·0	0	0·000000	1·000000	1·000000	1·001347
	0·1	0·041171	0·904712	1·004159	0·905999
	0·4	0·120998	0·669927	1·057311	0·670957
	0·5	0·125623	0·606081	1·082159	0·607020
	0·6	0·120056	0·548320	1·106878	0·549171
	1·6	0·001604	0·201415	1·199861	0·201698
	1·8	−0·000272	0·164859	1·200035	0·165086
	2·0	−0·000478	0·134938	1·199858	0·135120
	2·2	−0·000307	0·110448	1·199698	0·110595
	3·6	0·000000	0·027187	1·199545	0·027222

(ix) $M = 1$, $p = 1$, $q = 0$, i.e. the case of constant suction imposed on a time-dependent rotation velocity τ of the disk (Figs. 6 and 7).

The axial velocity initially ($\tau = 0{\cdot}05$) is slightly less than suction velocity, but it increases with time and soon exceeds the suction velocity, though by a very small margin. There is a steep fall in the tangential velocity profile

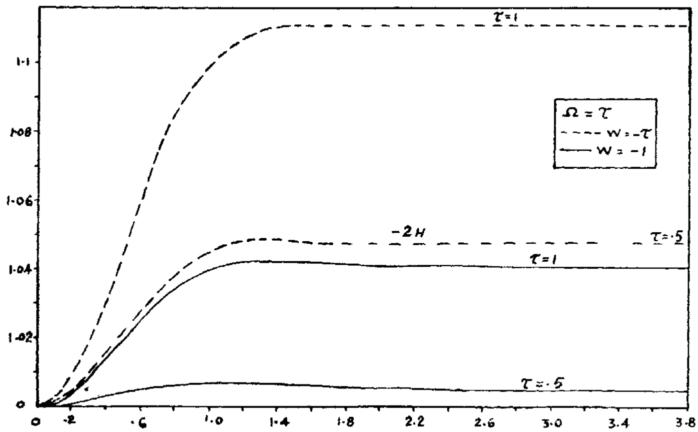

FIG. 6. Axial flow function $-H$ (with suction).

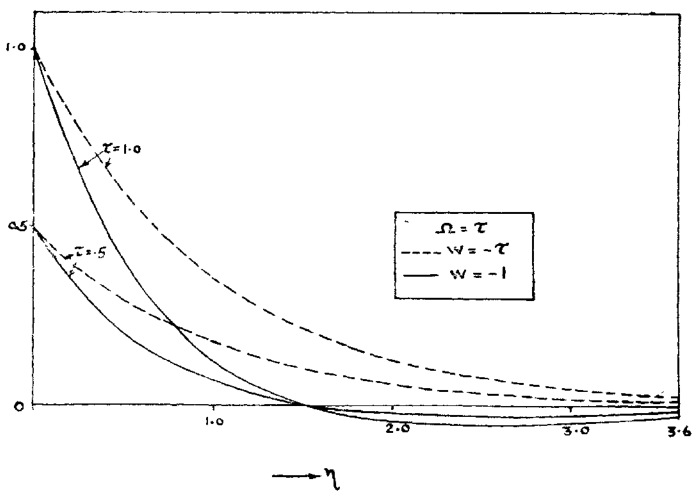

FIG. 7. Tangential velocity vs η (with suction).

with distance. Soon after the start of the motion the radial flow is inward throughout the fluid space, but at $\tau = 0{\cdot}5$ it starts beyond $\eta = 1$ and $\tau = 1$ beyond $\eta = 1{\cdot}3$. In the rest of the fluid mass there is outward radial flow. Both the inward and outward radial flows are very weak. An interesting behaviour of the fluid motion in this case is that while just after the start of the motion the fluid mass between the disk and the plane $\eta = 1{\cdot}6$ rotates with the disk in the same sense, the speed decreasing with the distance from the disk, the fluid beyond the plane $\eta = 1{\cdot}6$ rotates very slowly in the opposite direction. After $\tau = 0{\cdot}5$ this dividing plane of oppositely moving fluid mass is $\eta = 1{\cdot}8$.

(x) $M = -1$, $p = 1$, $q = 1$, i.e. the case of a time-dependent suction imposed on time-dependent rotation of the disk—each velocity being of magnitude τ (Figs. 6 and 7).

In this case the axial velocity which is directed towards the disk increases with distance from the disk and with time. At $\tau = 1$, the axial velocity attains its maximum at $\eta = 1{\cdot}6$ with a value which is nearly 11 per cent higher than the suction velocity. The inward radial flow which exists everywhere immediately after the start of the motion shifts away from the disk with time and occurs beyond $\eta = 1{\cdot}8$ at $\tau = 1$.

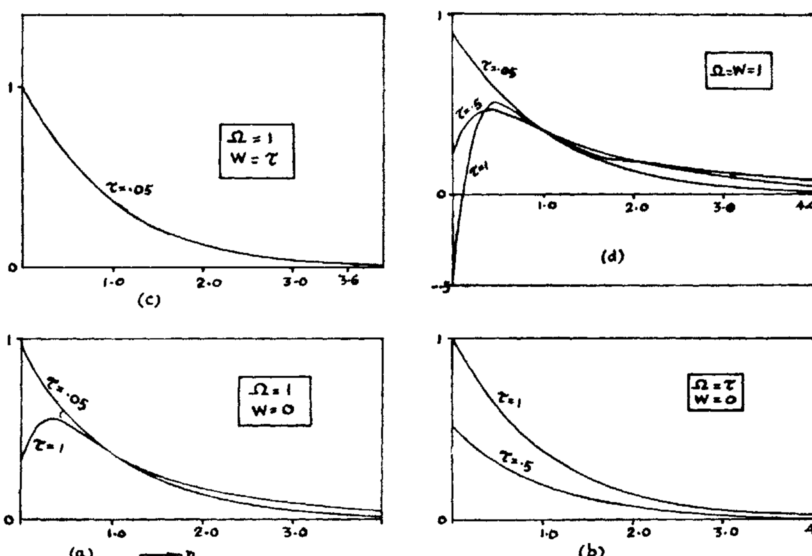

FIG. 8. $-G'$ vs η.

ACKNOWLEDGEMENT

The authors wish to express their gratitude to Prof. M. K. Jain, D.Sc., Head of the Department of Mathematics, Indian Institute of Technology, Hauz Khas, New Delhi, for his constant encouragement throughout the preparation of this paper.

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Appendix

The case of sudden acceleration can be included with the following explanation:

We define in the case of sudden acceleration of the disk and constant suction or iuflux that

$$G(\eta, \tau) = H(\eta, \tau) = 0 \quad \text{when } \tau = 0$$

$$\left.\begin{aligned} G(\eta, \tau) &= \Omega e^{-\eta} + a_1\eta\tau e^{-\eta} + a_2\eta^2\tau^2 e^{-2\eta} + \ldots \\ H(\eta, \tau) &= W + b_1\tau(1-e^{-\eta^2}) + b_2\tau^2(1-e^{-2\eta^2}) + \ldots \end{aligned}\right\} \quad \tau > 0$$

Let $\epsilon \ll 1$ be an infinitesimally small positive number. In the orthogonalization of the defect functions, the integrations are performed in the range $(\epsilon, 1)$ for τ and in the limit when $\epsilon \to 0$ we can see that the values of the integrals turn out to be the same as have been obtained in other cases where Ω and W are functions of τ. Hence the results are valid for $p = 0$, $q = 0$, when Ω and W turn out to be constants for any positive value of τ excluding zero.