TWO CIRCULAR DEFORMING INHOMOGENEITIES IN AN ISOTROPIC ELASTIC INFINITE MEDIUM

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Two circular inhomogeneities of an elastic material tend to undergo homogeneous deformations in a different otherwise homogeneous isotropic material. Owing to the constraints of the surrounding material, accommodation stresses develop in the system. This paper gives an analytical solution of the problem in terms of two complex functions.

1. Introduction

During the last two decades, a number of papers have appeared in literature dealing with the now well-known inclusion problems. It started with the papers of Nabarro (1940), Mott and Nabarro (1940) and Eshelby (1957). Three-dimensional problems turned out to be rather complicated. Jaswon and Bhargava (1961) gave a complex function formalism to two-dimensional inclusion problems. Then followed the work of Bhargava and Radhakrishna (1963*a*, 1963*b*, 1964), Bhargava and Sharma (1964), Bhargava and Kapoor (1964, 1966, 1963) and Willis (1965). But their work invariably concerned itself with a single inclusion. In this paper the authors have dealt with the problem of two inhomogeneities (not necessarily of equal size) of a material whose elastic constants could differ from those of the matrix.

Two circular regions in an otherwise homogeneous elastic medium have elastic constants differing from those of the remainder. They tend to undergo a change of form which, in the absence of the surrounding material, would be prescribed uniform strain. The effect is that the system gets self-stressed. The problem has been solved through an interesting process of superposition. Use has been made of some of the results obtained by the authors in earlier papers (Bhargava and Kapoor 1966 and Kapoor (unpublished paper)). In physical situations the problem can arise in metallurgical processes of formation of alloys, or in composite structures as strengthening reinforcement of as weakening impurities. Possibly there are applications in the design of rivets and joints.

2. Formulation of the Problem

Let two inhomogeneities be of radii a and unity, and the distance between their centres be l, $l > a+1$ so that they do not intersect. Choosing the coordinate system in accordance with Fig. 1, the inhomogeneities 1 and 2 have the equations $z\bar{z} \leqslant a^2$, $(z-l)\ (\bar{z}-l) \leqslant 1$ and the remaining portion of the complex plane represents the matrix. For convenience, the three regions will be called regions 1, 2 and 3 respectively. Let the dimensional

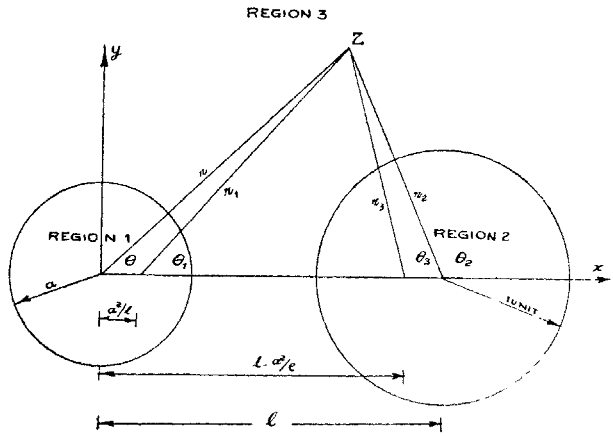

FIG. 1. Coordinate system and configuration.

changes, which the inhomogeneities 1 and 2 might have undergone in the absence of the matrix, be given in terms of cartesian components by the following displacement components

$$u = \delta_1 x + \delta_3 y, \quad v = \delta_3 x + \delta_2 y \qquad \ldots \quad \ldots \quad \ldots \quad (2.1)$$

for inhomogeneity 1,

$$u = \delta_4(x-l) + \delta_6 y, \quad v = \delta_6(x-l) + \delta_5 y \qquad \ldots \quad \ldots \quad (2.2)$$

for inhomogeneity 2.

Here and in subsequent analysis δ's lie within the elastic limits of linear elasticity theory. Due to elastic constraints of the surrounding material, locked-up accommodation stresses develop both in the inhomogeneities and matrix. The elastic fields so generated will be obtained in terms of the complex functions $\phi(z)$ and $\psi(z)$ which are related to the stresses and displacements by the following well-known equations

$$\left.\begin{aligned} \sigma_x + \sigma_y &= 4Rl[\phi'(z)] \\ \sigma_y - \sigma_x + 2i\tau_{xy} &= 2[\bar{z}\phi''(z) + \psi'(z)] \\ 2\mu(u+iv) &= \alpha\phi(z) - z\overline{\phi'(z)} - \overline{\psi(z)} \end{aligned}\right\} \quad \ldots \quad \ldots \quad \ldots \quad (2.3)$$

where $\alpha = (3-\sigma)/(1+\sigma)$ for plane stress case and $\alpha = 3-4\sigma$ for plane strain.

3. Reference to Some Solutions

In this paper it appears to be necessary to use the words 'inclusion' and 'inhomogeneity' in a definitive sense. Inclusion is a material region which has the same elastic properties as the matrix, while inhomogeneity is one which may have different elastic properties. The solutions of the following problems obtained in earlier papers by the authors form the starting point of the problem at hand:

(i) The problem of an oversize circular inclusion in an elastic infinite medium containing an inhomogeneity which does not tend to undergo any deformation (Bhargava and Kapoor 1966).

(ii) In an elastic infinite medium an inhomogeneity tends to undergo deformation (Kapoor, unpublished paper).

In Fig. 2, a flow chart is given indicating the process which leads to the solution. This helps in understanding and visualizing the procedure.

4. Solution to Problem of Circular Inclusion in an Elastic Medium with a Circular Inhomogeneity

An inhomogeneity occupies the region 1, and region 2 is occupied by the inclusion. The inclusion tends to undergo a homogeneous deformation characterized by (2.2). The complex functions which would give the elastic fields of this problem are given below (quoted with minor adjustments from Bhargava and Kapoor (1966)). The subscripts indicate the region which the subscripted quantities pertain to. The list of other symbols used is given in the Appendix.

$$
\begin{aligned}
{}^1\phi_1'(z) &= \frac{\mu_m}{\alpha_m+1}(\delta_4-\delta_5+2i\delta_6)(\nu_2-1)\left(\frac{1}{z_2^2}-\frac{\beta-1}{2l^2(2\beta+\alpha_i-1)}\right)\\
&\quad -\frac{\mu_m}{\alpha_m+1}(\delta_4-\delta_5-2i\delta_6)(\nu_2-1)\frac{\beta-1}{2l^2(2\beta+\alpha_i-1)}\\
{}^1\psi_1'(z) &= \frac{\alpha_m-1}{\alpha_m+1}(1-\nu_1)(\lambda_m+\mu_m)(\delta_4+\delta_5)\frac{1}{z_2^2}\\
&\quad +(\nu_1-\nu_2)\frac{\mu_m}{\alpha_m+1}(\delta_4-\delta_5+2i\delta_6)\left(\frac{a^2}{l^2z_2^2}-\frac{2a^2}{lz_2^3}\right)\\
&\quad -(1-\nu_1)\frac{\mu_m}{\alpha_m+1}(\delta_4-\delta_5+2i\delta_6)\left(\frac{2l}{z_2^3}+\frac{3}{z_2^4}\right)\\
{}^1\phi_2'(z) &= \left(\frac{\lambda_m+\mu_m}{1}\right)\left(\frac{\delta_4+\delta_5}{2}\right)\left(\frac{1-\alpha_m}{1+\alpha_m}\right)\\
&\quad +\nu_1\left(\frac{\alpha_m-1}{\alpha_m+1}\right)(\delta_4+\delta_5)(\lambda_m+\mu_m)\frac{a^2}{l^2z_1^2}\\
&\quad -\nu_1\frac{\mu_m}{\alpha_m+1}(\delta_4-\delta_5-2i\delta_6)\left[(3a^2-2l^2+3)\frac{a^2}{l^4z_1^2}\right.\\
&\quad \left.+2(a^2-l^2+3)\frac{a^4}{l^5z_1^3}+\frac{3a^6}{l^6z_1^4}\right]
\end{aligned}
$$

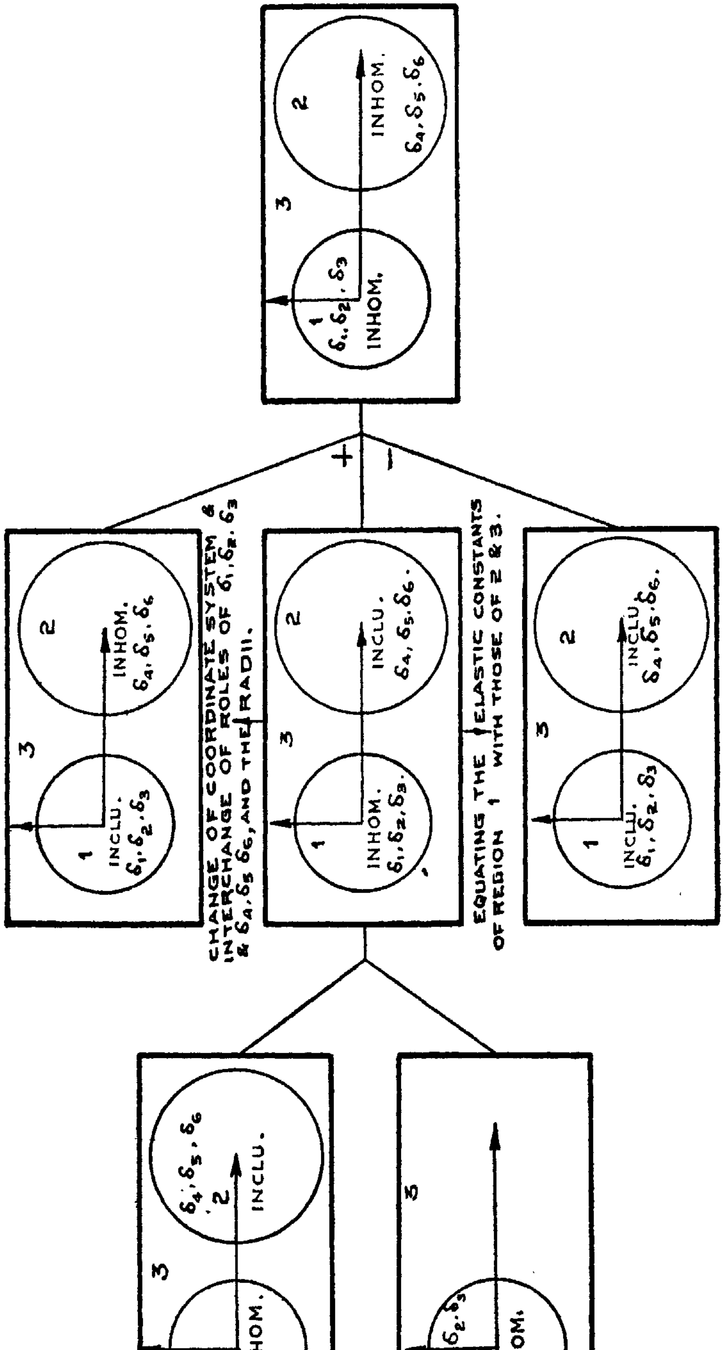

FIG. 2. Flow chart of the process of superposition

$$
\begin{aligned}
{}^{1}\psi_2'(z) = {} & \frac{\alpha_m-1}{\alpha_m+1}(\lambda_m+\mu_m)(\delta_4+\delta_5)\nu_1\left[\frac{2a^2}{lz_1^3}+\frac{1}{z^2}-\frac{1}{z_1^2}\right] \\
& -\frac{\mu_m}{\alpha_m+1}(\delta_4-\delta_5-2i\delta_6)\left[-1+(\nu_2-\nu_1)\frac{a^2}{l^2z_1^2}\right. \\
& +\left(\nu_1-\frac{(\nu_2-1)(\beta-1)}{2\beta+\alpha_i-1}\right)\frac{a^2}{l^2z^2}+\frac{2a^4\nu_1}{l^2z_1^3} \\
& \left.+\left(\frac{2a^2-2l^2+3}{1}\right)\frac{3a^4\nu_1}{l^4z_1^4}+\frac{12a^6\nu_1}{l^5z_1^5}\right] \\
& -\frac{\mu_m}{\alpha_m+1}(\delta_4-\delta_5+2i\delta_6)\left(\nu_1-\frac{(\nu_2-1)(\beta-1)}{2\beta+\alpha_i-1}\right)\frac{a^2}{l^2z^2}
\end{aligned}
$$

$$
\begin{aligned}
{}^{1}\phi_3'(z) = {} & \frac{\alpha_m-1}{\alpha_m+1}(\lambda_m+\mu_m)(\delta_4+\delta_5)\nu_1\frac{a^2}{l^2z_1^2} \\
& -\frac{\mu_m}{\alpha_m+1}(\delta_4-\delta_5+2i\delta_6)\frac{1}{z_2^2} \\
& -\frac{\mu_m\nu_1}{\alpha_m+1}(\delta_4-\delta_5-2i\delta_6)\left[(3a^2-2l^2+3)\frac{a^2}{l^4z_1^2}\right. \\
& \left.+(a^2-l^2+3)\frac{2a^4}{l^5z_1^3}+\frac{3a^6}{l^6z_1^4}\right]
\end{aligned}
$$

$$
\begin{aligned}
{}^{1}\psi_3'(z) = {} & \frac{\alpha_m-1}{\alpha_m+1}(\lambda_m+\mu_m)(\delta_4+\delta_5)\left[\frac{1}{z_2^2}+\frac{2a^2\nu_1}{lz_1^3}+\frac{\nu_1}{z^2}-\frac{\nu_1}{z_1^2}\right] \\
& -\frac{\mu_m}{\alpha_m+1}(\delta_4-\delta_5-2i\delta_6)\left[(\nu_2-\nu_1)\frac{a^2}{l^2z_1^2}+\left(\nu_1-\frac{(\nu_2-1)(\beta-1)}{2\beta+\alpha_i-1}\right)\frac{a^2}{l^2z^2}\right. \\
& \left.+\frac{2a^4\nu_1}{l^3z_1^3}+(2a^2-2l^2+3)\frac{3a^4\nu_1}{l^4z_1^4}+\frac{12a^6\nu_1}{l^5z_1^5}\right]-\frac{\mu_m(\delta_4-\delta_5+2i\delta_6)}{\alpha_m+1} \\
& \times\left[\left(\nu_2-\frac{(\nu_2-1)(\beta-1)}{2\beta+\alpha_i-1}\right)\frac{a^2}{l^2z^2}+\frac{2l}{z_2^3}+\frac{3}{z_2^4}\right].
\end{aligned}
$$

5. Solution to the Problem of a Circular Inhomogeneity

In this case the inhomogeneity of section 4 tends to undergo the deformation (2.1) in the absence of the matrix. The associated functions which give the elastic fields of this problem are given below (quoted from Kapoor, unpublished paper).

$$
{}^{2}\phi_1'(z) = \frac{1-\alpha_i}{2(2\beta+\alpha_i-1)}(\lambda_i+\mu_i)(\delta_1+\delta_2)
$$

$$
{}^{2}\psi_1'(z) = \frac{\alpha_m\nu_1+1}{\alpha_m+1}\mu_i(\delta_1-\delta_2-2i\delta_3)
$$

$$ {}^{2}\phi_2'(z) = {}^{2}\phi_3'(z) = -\frac{1+\nu_1\alpha_m}{\alpha_m+1}\mu_i\frac{a^2}{z^2}(\delta_1-\delta_2+2i\delta_3) $$

$$ {}^{2}\psi_2'(z) = {}^{2}\psi_3'(z) = \frac{\alpha_i-1}{2\beta+\alpha_i-1}(\lambda_i+\mu_i)(\delta_1+\delta_2)\frac{a^2}{z^2} $$

$$ -\frac{1+\nu_1\alpha_m}{\alpha_m+1}\mu_i(\delta_1-\delta_2+2i\delta_3)\frac{3a^4}{z^4}. $$

6. Solution to the Problem of Inhomogeneity and Inclusion

In this case both the inhomogeneity and the inclusion of section 4 tend to undergo the deformations (2.1) and (2.2) respectively. The complex function for this are obtained by adding the functions given in sections 4 and 5, i.e. they will be obtained as follows.

$$ {}^{3}\phi_1'(z) = {}^{1}\phi_1'(z)+{}^{2}\phi_1'(z) $$

$$ {}^{3}\psi_1'(z) = {}^{1}\psi_1'(z)+{}^{2}\psi_1'(z) $$

$$ {}^{3}\phi_2'(z) = {}^{1}\phi_2'(z)+{}^{2}\phi_2'(z) $$

$$ {}^{3}\psi_2'(z) = {}^{1}\psi_2'(z)+{}^{2}\psi_2'(z) $$

$$ {}^{3}\phi_3'(z) = {}^{1}\phi_3'(z)+{}^{2}\phi_3'(z) $$

$$ {}^{3}\psi_3'(z) = {}^{1}\psi_3'(z)+{}^{2}\psi_3'(z). $$

7. Solution to the Problem of Inclusion and Inhomogeneity

In this case the inhomogeneity occupies region 2, and the inclusion is at region 1. They tend to undergo the deformations (2.2) and (2.1) respectively. The complex functions for this case are obtained from the functions ${}^{3}\phi_1'(z)$, ${}^{3}\psi_1'(z)$, ... of section 6. This is done by a suitable transformation of the coordinate system, followed by the interchange of the roles of the constants δ_1, δ_2, δ_3 with δ_4, δ_5, δ_6 and making the radii of inhomogeneity and the inclusion unity and a respectively. This brings the configuration of section 6 to the configuration of the present section. Thus the functions obtained are given below:

$$ {}^{4}\phi_1'(z) = -\frac{1+\nu_1\alpha_m}{\alpha_m+1}\frac{1}{z_2^2}\mu_i(\delta_4-\delta_5+2i\delta_6)+\frac{(\lambda_m+\mu_m)(\delta_1+\delta_2)(1-\alpha_m)}{2(\alpha_m+1)} $$

$$ +\frac{\alpha_m-1}{\alpha_m+1}(\lambda_m+\mu_m)(\delta_1+\delta_2)\nu_1\frac{a^2}{l^2z_3^2}-\frac{\nu_1\mu_m}{\alpha_m+1}(\delta_1-\delta_2-2i\delta_3) $$

$$ \times\left[(3-2l^2+3a^2)\frac{a^2}{l^4z_3^2}+2(1-l^2+3a^2)\frac{a^2}{l^5z_3^3}+\frac{3a^4}{l^6z_3^4}\right] $$

$$
\begin{aligned}
{}^4\psi_1'(z) = {} & \frac{\alpha_i-1}{2\beta+\alpha_i-1}(\lambda_i+\mu_i)(\delta_4+\delta_5)\frac{1}{z_2^2}-\frac{1+\nu_1\alpha_m}{\alpha_m+1}\mu_i(\delta_4-\delta_5+2i\delta_6) \\
& \times\left(\frac{3}{z_2^4}+\frac{2l}{z_2^3}\right)+\frac{\alpha_m-1}{\alpha_m+1}(\lambda_m+\mu_m)(\delta_1+\delta_2)\nu_1\left(\frac{a^2}{z_2^2}-\frac{a^2}{z_3^2}\right) \\
& -\frac{\mu_m}{\alpha_m+1}(\delta_1-\delta_2-2i\delta_3)\Bigg[-1+(\nu_2-\nu_1)\frac{a^2}{l^2z_3^2} \\
& +\left(\nu_1-\frac{(\nu_2-1)(\beta-1)}{2\beta+\alpha_i-1}\right)\frac{a^2}{l^2z_2^2}+(2-2l^2+3a^2)\frac{2a^2\nu_1}{l^3z_3^3} \\
& \times(4-4l^2+9a^2)\frac{3a^2\nu_1}{l^4z_3^4}\Bigg]-\frac{\mu_m}{\alpha_m+1}(\delta_1-\delta_2+2i\delta_3) \\
& \times\left[\nu_2-\frac{(\nu_2-1)(\beta-1)}{2\beta+\alpha_i-1}\right]\frac{a^2}{l^2z_2^2}
\end{aligned}
$$

$$
\begin{aligned}
{}^4\phi_2'(z) = {} & \frac{1-\alpha_i}{2(2\beta+\alpha_i-1)}(\lambda_i+\mu_i)(\delta_4+\delta_5) \\
& +\frac{\mu_m}{\alpha_m+1}(\nu_2-1)(\delta_1-\delta_2+2i\delta_3)\left[\frac{a^2}{z^2}-\frac{(\beta-1)a^2}{2l^2(2\beta+\alpha_i-1)}\right] \\
& -\frac{\mu_m}{\alpha_m+1}(\delta_1-\delta_2-2i\delta_3)\frac{(\nu_2-1)(\beta-1)a^2}{2l^2(2\beta+\alpha_i-1)}
\end{aligned}
$$

$$
\begin{aligned}
{}^4\psi_2'(z) = {} & \frac{\nu_1\mu_m+1}{\alpha_m+1}\mu_i(\delta_4-\delta_5-2i\delta_6)+\frac{\alpha_m-1}{\alpha_m+1}(1-\nu_1)(\lambda_m+\mu_m)(\delta_1+\delta_2)\frac{a^2}{z^2} \\
& +(\nu_1-\nu_2)\frac{\mu_m}{\alpha_m+1}(\delta_1-\delta_2+2i\delta_3)\left(\frac{a^2}{l^2z^2}+\frac{2a^2}{lz^3}\right) \\
& -\frac{\mu_m}{\alpha_m+1}(\delta_1-\delta_2+2i\delta_3)\left(\frac{2l(\nu_1-\nu_2)a^2}{z^3}+\frac{3(1-\nu_1)a^4}{z^4}\right)
\end{aligned}
$$

$$
\begin{aligned}
{}^4\phi_3'(z) = {} & -\frac{1+\nu_1\alpha_m}{\alpha_m+1}\mu_i(\delta_4-\delta_5+2i\delta_6)\frac{1}{z_2^2}-\frac{\mu_m}{\alpha_m+1}(\delta_1-\delta_2+2i\delta_3)\frac{a^2}{z^2} \\
& +\nu_1\left(\frac{\alpha_m-1}{\alpha_m+1}\right)(\lambda_m+\mu_m)(\delta_1+\delta_2)\frac{a^2}{l^2z_3^2}-\frac{\nu_1\alpha_m}{\alpha_m+1}(\delta_1-\delta_2-2i\delta_3) \\
& \times\left[(3-2l^2+3a^2)\frac{a^2}{l^4z_3^2}-(1-l^2+3a^2)\frac{2a^2}{l^5z_3^3}+\frac{3a^4}{l^6z_3^4}\right]
\end{aligned}
$$

$$
\begin{aligned}
{}^4\psi_3'(z) = {} & \frac{\alpha_i-1}{2\beta+\alpha_i-1}(\lambda_i+\mu_i)(\delta_4+\delta_5)\frac{1}{z_2^2} - \frac{1+\nu_1\alpha_m}{\alpha_m+1}\mu_i(\delta_4-\delta_5+2i\delta_6) \\
& \times\left(\frac{3}{z_2^4}+\frac{2l}{z_2^3}\right)+(\lambda_m+\mu_m)\frac{\alpha_m-1}{\alpha_m+1}(\delta_1+\delta_2)\left[\frac{a^2}{z^2}+\frac{\nu_1 a^2}{z_2^2}-\frac{\nu_1 a^2}{z_3^2}\right] \\
& -\frac{\mu_m}{\alpha_m+1}(\delta_1-\delta_2-2i\delta_3)\left[(\nu_2-\nu_1)\frac{a^2}{l^2 z_3^2}\right. \\
& +\left(\nu_1-\frac{(\nu_2-1)(\beta-1)}{2\beta+\alpha_i-1}\right)\frac{a^2}{l^2 z_2^2}+(2-2l^2+3a^2)\frac{2a^2\nu_1}{l^3 z_3^3} \\
& \left.+3(4-4l^2+9a^2)\frac{a^2\nu_1}{l^4 z_3^4}\right]-\frac{\mu_m}{\alpha_m+1}(\delta_1-\delta_2+2i\delta_3) \\
& \times\left[\left(\nu_2-\frac{(\nu_2-1)(\beta-1)}{2\beta+\alpha_i-1}\right)\frac{a^2}{l^2 z_2^2}+\frac{3a^4}{z^4}\right].
\end{aligned}
$$

8. Solution to the Problem of Two Inclusions

In this case inclusions occupy the regions 1 and 2, and tend to undergo the deformations (2.1) and (2.2) respectively. The complex functions of this case can be easily obtained from those given in section 6, by equating the elastic constants of regions 1 and 2 with those of region 3. Thus

$$
\begin{aligned}
{}^5\phi_1'(z) = {} & \frac{1-\alpha_m}{2(\alpha_m+1)}(\lambda_m+\mu_m)(\delta_1+\delta_2) \\
& -\frac{\mu_m}{\alpha_m+1}(\delta_4-\delta_5+2i\delta_6)\frac{1}{z_2^2}
\end{aligned}
$$

$$
\begin{aligned}
{}^5\psi_1'(z) = {} & \frac{\mu_m}{\alpha_m+1}(\delta_1-\delta_2-2i\delta_3)+\frac{\alpha_m-1}{\alpha_m+1}(\lambda_m+\mu_m)(\delta_4+\delta_5)\frac{1}{z_2^2} \\
& -\frac{\mu_m}{\alpha_m+1}(\delta_4-\delta_5+2i\delta_6)\left(\frac{2l}{z_2^3}+\frac{3}{z_2^4}\right)
\end{aligned}
$$

$$
{}^5\phi_2'(z) = \frac{1-\alpha_m}{2(\alpha_m+1)}(\lambda_m+\mu_m)(\delta_4+\delta_5)-\frac{\mu_m}{\alpha_m+1}(\delta_1-\delta_2+2i\delta_3)\frac{a^2}{z^2}
$$

$$
\begin{aligned}
{}^5\psi_2'(z) = {} & \frac{\alpha_m-1}{\alpha_m+1}(\lambda_m+\mu_m)(\delta_1+\delta_2)\frac{a^2}{z^2}-\frac{\mu_m}{\alpha_m+1}(\delta_1-\delta_2+2i\delta_3)\frac{3a^4}{z^4} \\
& +\frac{\mu_m}{\alpha_m+1}(\delta_4-\delta_5-2i\delta_6)
\end{aligned}
$$

$$
{}^5\phi_3'(z) = -\frac{\mu_m}{\alpha_m+1}(\delta_1-\delta_2+2i\delta_3)\frac{a^2}{z^2}-\frac{\mu_m}{\alpha_m+1}(\delta_4-\delta_5+2i\delta_6)\frac{1}{z_2^2}
$$

$$
{}^{5}\psi_3'(z) = \frac{\alpha_m-1}{\alpha_m+1}(\lambda_m+\mu_m)(\delta_1+\delta_2)\frac{a^2}{z^2} - \frac{\mu_m}{\alpha_m+1}(\delta_1-\delta_2+2i\delta_3)\frac{3a^4}{z^4}
$$
$$
+\frac{\alpha_m-1}{\alpha_m+1}(\lambda_m+\mu_m)(\delta_4+\delta_5)\frac{1}{z_2^2}
$$
$$
-\frac{\mu_m}{\alpha_m+1}(\delta_4-\delta_5+2i\delta_6)\left(\frac{2l}{z_2^3}+\frac{3}{z_2^4}\right).
$$

9. Solution to the Problem in Hand

The complex function for this case is obtained by adding the corresponding function of sections 6 and 7, and then subtracting from the resultant functions the functions of section 8. Thus the final functions are given below:

$$
{}^{6}\phi_1'(z) = \frac{1-\alpha_i}{2(2\beta+\alpha_i-1)}(\lambda_i+\mu_i)(\delta_1+\delta_2) + \frac{\mu_m}{\alpha_m+1}(\delta_4-\delta_5+2i\delta_6)\left(\frac{\nu_2}{z_2^2} - \frac{(\nu_2-1)(\beta-1)}{2l^2(2\beta+\alpha_i-1)}\right)
$$
$$
-\frac{1+\nu_1\alpha_m}{\alpha_m+1}\mu_i(\delta_4-\delta_5+2i\delta_6)\frac{a^2}{z_2^2} - \frac{\mu_m}{\alpha_m+1}(\delta_4-\delta_5-2i\delta_6)\frac{(\nu_2-1)(\beta-1)}{2l^2(2\beta+\alpha_i-1)}
$$
$$
+\frac{\nu_1(\alpha_m-1)}{\alpha_m+1}(\lambda_m+\mu_m)(\delta_1+\delta_2)\frac{a^2}{l^2z_3^2} - \frac{\mu_m\nu_1}{\alpha_m+1}(\delta_1-\delta_2-2i\delta_3)
$$
$$
\times\left[(3-2l^2+3a^2)\frac{a^2}{l^4z_3^2} - (1-l^2+3a^2)\frac{2a^2}{l^5z_3^3} + \frac{3a^4}{l^6z_3^4}\right] \quad \ldots \quad (9.1)
$$

$$
{}^{6}\psi_1'(z) = \frac{1+\nu_1\alpha_m}{\alpha_m+1}\mu_i(\delta_1-\delta_2-2i\delta_3) - \frac{\alpha_m-1}{\alpha_m+1}(\lambda_m+\mu_m)(\delta_4+\delta_5)\frac{\nu_1}{z_2^2}
$$
$$
+(\nu_1-\nu_2)\frac{\mu_m}{\alpha_m+1}(\delta_4-\delta_5+2i\delta_6)\left(\frac{a^2}{l^2z_2^2} - \frac{2a^2}{lz_2^3}\right) + \frac{\mu_m\nu_1}{\alpha_m+1}(\delta_4-\delta_5+2i\delta_6)
$$
$$
\times\left(\frac{2l}{z_2^3}+\frac{3}{z_2^4}\right) + \frac{\alpha_i-1}{(2\beta+\alpha_i-1)}(\lambda_i+\mu_i)(\delta_4+\delta_5)\frac{1}{z_2^2} - \frac{1+\nu_1\alpha_m}{\alpha_m+1}\mu_i(\delta_4-\delta_5+2i\delta_6)
$$
$$
\times\left(\frac{3}{z_2^4}+\frac{2l}{z_2^3}\right) + \frac{\alpha_m-1}{\alpha_m+1}(\lambda_m+\mu_m)(\delta_1+\delta_2)\nu_1\left(\frac{a^2}{z_2^2} - \frac{a^2}{z_3^2}\right)
$$
$$
-\frac{\mu_m}{\alpha_m+1}(\delta_1-\delta_2-2i\delta_3)\left[(\nu_2-\nu_1)\frac{a^2}{l^2z_3^2} + \left(\nu_1 - \frac{(\nu_2-1)(\beta-1)}{2\beta+\alpha_i-1}\right)\frac{a^2}{l^2z_2^2}\right.
$$
$$
\left.+2(2-2l^2+3a^2)\frac{a^2\nu_1}{l^3z_3^3} - \frac{9a^4\nu_1}{l^4z_3^4}\right] - \frac{\mu_m}{\alpha_m+1}(\delta_1-\delta_2+2i\delta_3)
$$
$$
\times\left(\nu_2 - \frac{(\nu_2-1)(\beta-1)}{2\beta+\alpha_i-1}\right)\frac{a^2}{l^2z_2^2}. \quad \ldots \quad (9.2)
$$

$$
{}^6\phi_2'(z) = \frac{1-\alpha_i}{2(2\beta+\alpha_i-1)}(\lambda_i+\mu_i)(\delta_4+\delta_5)+\frac{\mu_m}{\alpha_m+1}(\delta_1-\delta_2+2i\delta_3)\left(\frac{\nu_2 a^2}{z^2}-\frac{(\beta-1)(\nu_2-1)a^2}{2l^2(2\beta+\alpha_i-1)}\right)
$$

$$
-\frac{\mu_m}{\alpha_m+1}(\delta_1-\delta_2-2i\delta_3)\frac{a^2(\nu_2-1)(\beta-1)}{2l^2(2\beta+\alpha_i-1)}-\frac{1+\nu_1\alpha_m}{\alpha_m+1}\mu_i(\delta_1-\delta_2+2i\delta_3)\frac{a^2}{z^2}
$$

$$
+\nu_1\frac{\alpha_m-1}{\alpha_m+1}(\lambda_m+\mu_m)(\delta_4+\delta_5)\frac{a^2}{l^2z_1^2}-\frac{\mu_m\nu_1}{\alpha_m+1}(\delta_4-\delta_5-2i\delta_6)
$$

$$
\times\left[(3a^2-2l^2+3)\frac{a^2}{l^4z_1^2}+(a^2-l^2+3)\frac{2a^4}{l^5z_1^3}+\frac{3a^4}{l^6z_1^4}\right]. \quad \ldots \quad (9.3)
$$

$$
{}^6\psi_2'(z) = \frac{\alpha_i-1}{2\beta+\alpha_i-1}(\lambda_i+\mu_i)(\delta_1+\delta_2)\frac{a^2}{z^2}-\frac{1+\nu_1\alpha_m}{\alpha_m+1}\mu_i(\delta_1-\delta_2+2i\delta_3)\frac{3a^4}{z^4}
$$

$$
+\frac{\alpha_m-1}{\alpha_m+1}(\lambda_m+\mu_m)(\delta_4+\delta_5)\nu_1\left(\frac{2a^2}{lz_1^3}+\frac{1}{z^2}-\frac{1}{z_1^2}\right)-\frac{\mu_m}{\alpha_m+1}(\delta_4-\delta_5-2i\delta_6)
$$

$$
\times\left[(\nu_2-\nu_1)\frac{a^2}{l^2z_1^2}+\left(\nu_1-\frac{(\nu_2-1)(\beta-1)}{2\beta+\alpha_i-1}\right)\frac{a^2}{l^2z^2}+\frac{2a^4\nu_1}{l^3z_1^3}+(2a^2-2l^2+3)\frac{3a^4\nu_1}{l^4z_1^4}\right.
$$

$$
\left.+\frac{12a^6\nu_1}{l^5z_1^5}\right]-\frac{\mu_m}{\alpha_m+1}(\delta_4-\delta_5+2i\delta_6)\left(\nu_2-\frac{(\nu_2-1)(\beta-1)}{2\beta+\alpha_i-1}\right)\frac{a^2}{l^2z^2}+\frac{1+\alpha_m\nu_1}{\alpha_m+1}\mu_i
$$

$$
\times(\delta_4-\delta_5-2i\delta_6)-\frac{\alpha_m-1}{\alpha_m+1}\nu_1(\lambda_m+\mu_m)(\delta_1+\delta_2)\frac{a^2}{z^2}
$$

$$
+\frac{\mu_m(\nu_1-\nu_2)}{\alpha_m+1}(\delta_1-\delta_2+2i\delta_3)\left(\frac{a^2}{l^2z^2}+\frac{2a^2}{lz^3}\right)-\frac{\mu_m}{\alpha_m+1}(\delta_1-\delta_2+2i\delta_3)
$$

$$
\times\left[2l(\nu_1-\nu_2)\frac{a^2}{z^3}-\frac{3\nu_1a^4}{z^4}\right]. \quad \ldots \quad (9.4)
$$

$$
{}^6\phi_3'(z) = -\frac{(1+\nu_1\alpha_m)}{\alpha_m+1}\mu_i(\delta_1-\delta_2+2i\delta_3)\frac{a^2}{z^2}+\frac{\alpha_m-1}{\alpha_m+1}(\lambda_m+\mu_m)(\delta_4+\delta_5)\nu_1\frac{a^2}{l^2z_1^2}
$$

$$
-\frac{\nu_1\mu_m}{\alpha_m+1}(\delta_4-\delta_5-2i\delta_6)\left[(3a^2-2l^2+3)\frac{a^2}{l^4z_1^2}+2(a^2-l^2+3)\frac{a^4}{l^5z_1^3}+\frac{3a^6}{l^6z_1^4}\right]
$$

$$
-\frac{1+\nu_1\alpha_m}{\alpha_m+1}\mu_i(\delta_4-\delta_5+2i\delta_6)\frac{1}{z_2^2}+\nu_1\frac{\alpha_m-1}{\alpha_m+1}(\lambda_m+\mu_m)(\delta_1+\delta_2)\frac{a^2}{l^2z_3^2}
$$

$$
-\frac{\nu_1\mu_m}{\alpha_m+1}(\delta_1-\delta_2-2i\delta_3)\left[(3-2l^2+3a^2)\frac{a^2}{l^4z_3^2}-2(1-l^2+3a^2)\frac{a^2}{l^5z_3^3}+\frac{3a^4}{l^6z_3^4}\right].
$$

$$
\ldots \quad (9.5)
$$

$$
\begin{aligned}
{}^{6}\psi_3'(z) &= \frac{\alpha_i-1}{2\beta+\alpha_i-1}(\lambda_i+\mu_i)\left((\delta_1+\delta_2)\frac{a^2}{z^2}+(\delta_4+\delta_5)\frac{1}{z_2^2}\right) \\
&\quad -\frac{1+\nu_1\alpha_m}{\alpha_m+1}\mu_i(\delta_1-\delta_2+2i\delta_3)\frac{3a^4}{z^4}+\frac{\alpha_m-1}{\alpha_m+1}(\lambda_m+\mu_m)(\delta_4+\delta_5) \\
&\quad \times\left[\frac{2a^2\nu_1}{lz_1^3}+\frac{\nu_1}{z^2}-\frac{\nu_1}{z_1^2}\right]-\frac{\mu_m}{\alpha_m+1}(\delta_4-\delta_5-2i\delta_6)\left[(\nu_2-\nu_1)\frac{a^2}{l^2z_1^2}\right. \\
&\quad +\left(\nu_1-\frac{(\nu_2-1)(\beta-1)}{2\beta+\alpha_i-1}\right)\frac{a^2}{l^2z^2}+\frac{2a^4\nu_1}{l^3z_1^3}+3(2a^2-2l^2+3)\frac{a^4\nu_1}{l^4z_1^4} \\
&\quad \left.+\frac{12a^6\nu_1}{l^5z_1^5}\right]-\frac{\mu_m}{\alpha_m+1}(\delta_4-\delta_5+2i\delta_6)\left(\nu_2-\frac{(\nu_2-1)(\beta-1)}{2\beta+\alpha_i-1}\right)\frac{a^2}{l^2z^2} \\
&\quad -\frac{1+\nu_1\alpha_m}{\alpha_m+1}\mu_i(\delta_4-\delta_5+2i\delta_\delta)\left(\frac{3}{z_2^4}+\frac{2l}{z_2^3}\right)+\frac{\alpha_m-1}{\alpha_m+1}(\lambda_m+\mu_m) \\
&\quad \times(\delta_1+\delta_2)\left(\frac{a^2\nu_1}{z_2^3}-\frac{a^2\nu_1}{z_3^2}\right)-\frac{\mu_m}{\alpha_m+1}(\delta_1-\delta_2-2i\delta_3)\left[(\nu_2-\nu_1)\frac{a^2}{l^2z_3^2}\right. \\
&\quad +\left(\nu_1-\frac{(\nu_2-1)(\beta-1)}{2\beta+\alpha_i-1}\right)\frac{a^2}{l^2z_2^2}+(2-2l^2+3a^2)\frac{2a^2\nu_1}{l^3z_3^3} \\
&\quad \left.+(4-4l^2+9a^2)\frac{3a^4\nu_1}{l^4z_3^4}\right]-\frac{\mu_m}{\alpha_m+1}(\delta_1-\delta_2+2i\delta_3) \\
&\quad \times\left(\nu_2-\frac{(\nu_2-1)(\beta-1)}{2\beta+\alpha_i-1}\right)\frac{a^2}{l^2z_2^2} \qquad \qquad (9.6)
\end{aligned}
$$

It may be worth while to point out that the solution to the problem of a single inhomogeneity alone fails to yield the solution to the problem of two inhomogeneities through the process of superposition. The reason is as follows. Let us suppose that a deforming inhomogeneity is present at region 1, and regions 2 and 3 are of the same elastic material. The solution is known (Kapoor, unpublished paper). Next, suppose a deforming inhomogeneity is now present at region 2, and regions 1 and 3 of the same elastic material. The solution is known from Kapoor (unpublished paper) by merely shifting the origin. Therefore, for superposition to work, one has to make the assumption that regions 1, 2 and 3 are of the same material. In other words simple superposition can yield result for two inclusions and in fact for any number of inclusions.

Elastic Fields.—The stress distribution in the various regions can now be obtained from their respective functions using the equation (2.3). A check on the analysis can be made at each step by verifying that the normal and tangential stresses across the boundaries $z\bar{z} = a^2$ and $(z-l)(\bar{z}-l) = 1$ are continuous. The hoop stress is discontinuous as it should be. Some numerical work was done on the computer. A few figures (Figs. 3–10) are

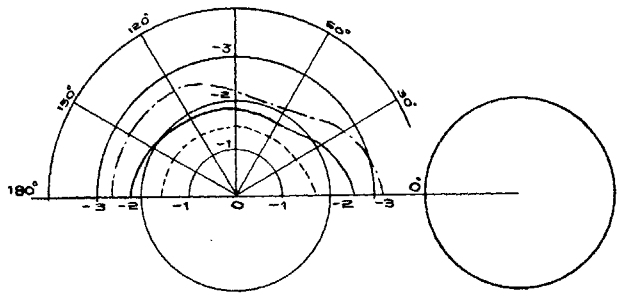

FIG. 3. Normal stress σ_n/μ_m . $\delta_1 = \delta_2$ - - - - - , $\beta = 0{\cdot}5$; ——, $\beta = 1$; – · – · –, $\beta = 2$.

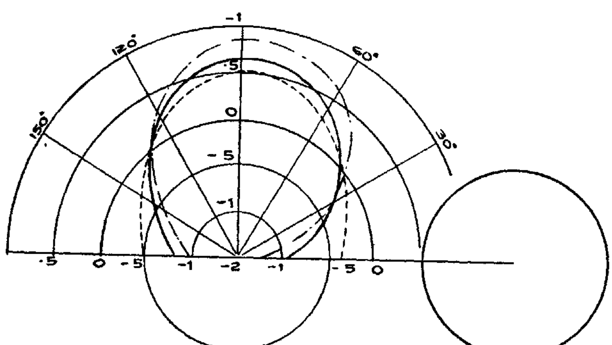

FIG. 4. Normal stress σ_n/μ_m . $\delta_1 = -\delta_2$. - - - - , $\beta = 0{\cdot}5$; —— , $\beta = 1$; – · – · –, $\beta = 2$.

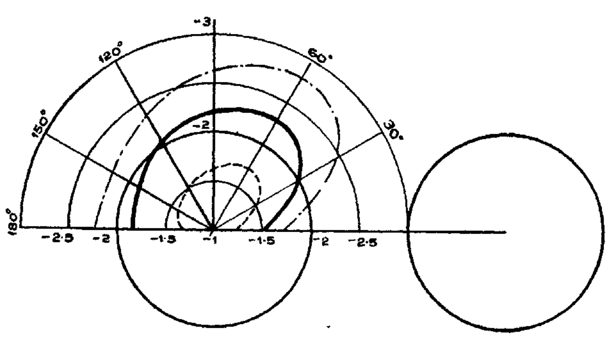

FIG. 5. Hoop stress σ_0/μ_m inside. $\delta_1 = \delta_2$. - - - - , $\beta = 0{\cdot}5$; —— $\beta = 1$; – · – · –, $\beta = 2$.

included here showing the variation of normal, shearing, hoop stress inside and the hoop stress outside the boundary of the region 1 for plane stress case with the following data

$$\sigma_l = \sigma_m = \tfrac{1}{3};\ \beta = \tfrac{1}{2}, 1, 2;\ a = 1, l = 3;\ \delta_3 = 0.$$

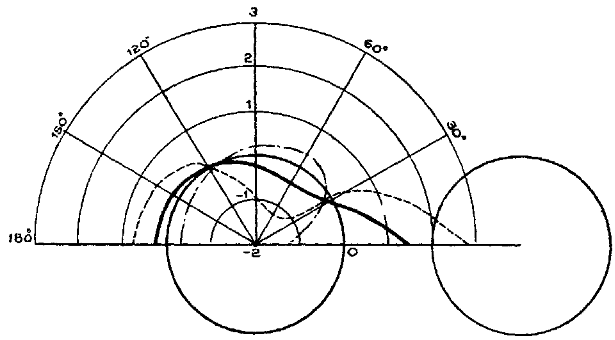

FIG. 6. Hoop stress σ_0/μ_m outside. $\delta_1 = \delta_2$. - - - - - , $\beta = 0{\cdot}5$, —— , $\beta = 1$; –·–·– , $\beta = 2$.

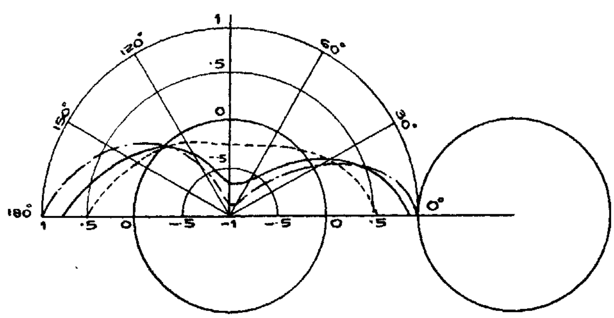

FIG. 7. Hoop stress σ_0/μ_m inside. $\delta_1 = -\delta_2$. - - - - , $\beta = 0{\cdot}5$; —— , $\beta = 1$; –·–·– , $\beta = 2$.

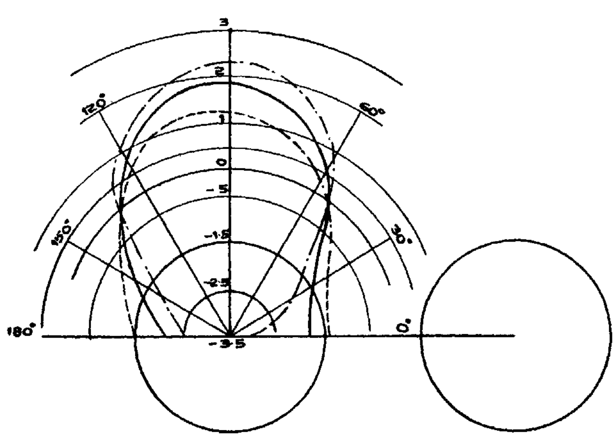

FIG. 8. Hoop stress $\sigma\theta/\mu$ outside. $\delta_1 = -\delta_2$. - - - - - , $\beta = 0{\cdot}5$; —— , $\beta = 1$; –·–·–, $\beta = 2$.

For the displacement fields, one can find out the functions ${}^6\phi_1(z)$, ${}^6\psi_1(z)$, ${}^6\phi_2(z)$, ${}^6\psi_2(z)$, ${}^6\phi_3(z)$ and ${}^6\psi_3(z)$ by integrating the corresponding functions in equations (9.1) to (9.6) with respect to z. One can choose the constants of integration so as to satisfy the continuity conditions on displacements across the two circular boundaries.

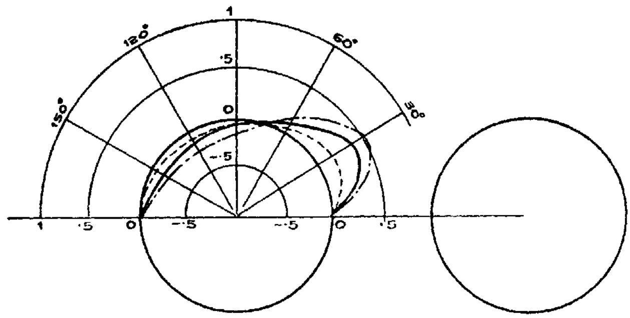

FIG. 9. Tangential stress $\tau_{r\theta}/\mu$. $\delta_1 = \delta_2$. ---- , $\beta = 0\cdot5$; —— , $\beta = 1$; -·-·- , $\beta = 2$.

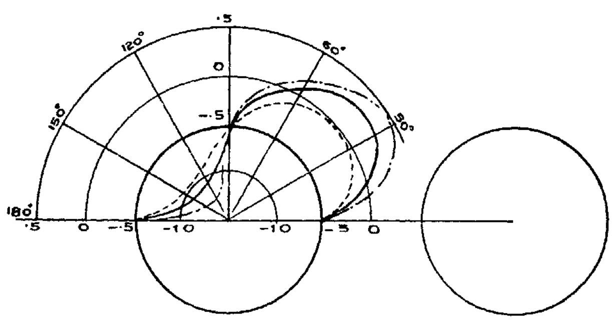

FIG. 10. Tangential stress $\tau_{r\theta}/\mu$. $\delta_1 = -\delta_2$. ---- , $\beta = 0\cdot5$; —— , $\beta = 1$; -·-·- , $\beta = 2$.

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Appendix

List of Symbols

λ_i, μ_i Lame' constants for the inhomogeneity.
λ_m, μ_m Lame' constants for the matrix.
σ_i Poisson's ratio for the inhomogeneity.
σ_m Poisson's ratio for the inclusion.

$$\alpha_i = \frac{3-\sigma_i}{1+\sigma_i} \text{ for plane stress}$$

$$\alpha_i = 3-4\sigma_i \text{ for plane strain}$$

$$\alpha_m = \frac{3-\sigma_m}{1+\sigma_m} \text{ for plane stress}$$

$$\alpha_m = 3-4\sigma_m \text{ for plane strain}$$

$$\beta = \frac{\mu_i}{\mu_m}$$

$$\nu_1 = (1-\beta)/(\beta\alpha_m+1)$$

$$\nu_2 = (\alpha_i-\beta\alpha_m)/\beta+\alpha_i$$

$$z_1 = z-a^2/l$$

$$z_2 = z-l$$

$$z_3 = z-(l-a^2/l)$$