SOME PROBLEMS ON A PAIR OF SINGULAR SECOND-ORDER DIFFERENTIAL EQUATIONS

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(Communicated by N. S. Nagendra Nath, F.N.I.)

(*Received April 24, 1967*)

A pair of second-order homogeneous differential equations has been considered in the interval $[0, \infty)$. The nature of the singular surfaces has been discussed and it has been shown that there exist at least two solutions ψ_1 and ψ_2 of the system such that $\psi_r^T F \bar{\psi}_r$ belong to $L[0, \infty)$, F being a real valued positive definite symmetric matrix of order two. The existence of a Green's matrix in $[0, \infty)$ has been shown. The vector $\Phi(x, \lambda)$ has been defined which satisfies the non-homogeneous system associated with the homogeneous system considered and some of its properties have been proved.

§ 1. Let L denote the matrix operator

$$L \equiv \begin{pmatrix} \frac{d}{dx}\left[p_0(x)\frac{d}{dx}\right]+p_1(x) & r(x) \\ r(x) & \frac{d}{dx}\left[q_0(x)\frac{d}{dx}\right]+q_1(x) \end{pmatrix}, \qquad \ldots \quad (1.1)$$

F the symmetric matrix

$$F \equiv F(x) = \begin{pmatrix} F_{11}(x) & F_{12}(x) \\ F_{21}(x) & F_{22}(x) \end{pmatrix} \qquad \ldots \quad \ldots \quad \ldots \quad (1.2)$$

and $\phi \equiv \phi(x)$ a vector having two components $u \equiv u(x)$ and $v \equiv v(x)$ represented as a column matrix

$$\phi = \begin{pmatrix} u \\ v \end{pmatrix}.$$

Then consider the differential system

$$(L-\lambda F)\phi = 0 \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (1.3)$$

where λ is a parameter, real or complex.

The non-homogeneous system associated with (1.3) is

$$(L-\lambda F)\phi = -Ff \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (1.4)$$

where $f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$, f_1 and f_2 being real valued functions of x.

The case where the x-interval is finite and there are given boundary conditions at each end was considered by the author (1969). Here the finite x-interval is being extended to the interval $[0, \infty)$. The finite surfaces analogous to Weyl's circle are being discussed and the results analogous to Titchmarsh's book (1962, Chap. II) are being derived.

§ 2. The information about a pair of singular second-order differential equations and the corresponding eigenvector expansions is obtained by proceeding to the limit from a suitable finite case in the same way as in Titchmarsh (1961, 1962) and Everitt (1963, 1964) (*see also* Chakravarty 1963). We consider the differential system (1.3) for the interval $[0, b]$ for every $b > 0$ and then consider the limit as $b \to \infty$. The notations and results of Bhagat (1969) will be used here. We shall assume that

(i) $p_0(x)$, $q_0(x)$ are real valued and possess continuous derivatives of the first order in the closed interval $[0, b]$ for all $b > 0$;

(ii) $p_1(x)$, $q_1(x)$ and $r(x)$ are real valued and continuous in $[0, b]$ for all $b > 0$;

(iii) $p_0(x)$, $q_0(x) > 0$ for $0 \leqslant x \leqslant b$ and all $b > 0$;

(iv) the symmetric matrix F is real valued, continuous and positive definite for $0 \leqslant x \leqslant b$ and all $b > 0$.

The boundary conditions to be satisfied by any solution $\phi(x, \lambda)$ of (1.3) are taken in the form

$$[\phi(x, \lambda)\ \phi_j(0 \mid x, \lambda)] = 0\ ; \quad j = 1, 2 \qquad (2.1)$$

at $x = 0$, and

$$[\phi(x, \lambda)\ \phi_k(b \mid x, \lambda)] = 0\ ; \quad k = 3, 4 \qquad (2.2)$$

at $x = b$ for every $b > 0$.

The boundary condition vectors $\phi_j(0 \mid x, \lambda) = \begin{pmatrix} u_j(0 \mid x, \lambda) \\ v_j(0 \mid x, \lambda) \end{pmatrix}$ $(j = 1, 2)$ take real valued independent initial conditions at $x = 0$ so that they are linearly independent in x for all λ and the boundary condition vectors $\phi_k(b \mid x, \lambda) = \begin{pmatrix} u_k(b \mid x, \lambda) \\ v_k(b \mid x, \lambda) \end{pmatrix}$ take real valued independent initial conditions at $x = b$ for all $b > 0$ so that they are linearly independent in x for all λ [*see* Bhagat (1969), § 3].

The self-adjointness conditions for the boundary value problem (1.3), (2.1) and (2.2) are expressed in the form

$$[\phi_1(0 \mid x, \lambda)\ \phi_2(0 \mid x, \lambda)] = 0 \quad \text{and} \quad [\phi_3(b \mid x, \lambda)\ \phi_4(b \mid x, \lambda)] = 0, \qquad (2.3)$$

the latter to hold for all $b > 0$.

§ 3. The following notations will be used

$$[\phi_j \phi_k](b\,;\lambda) = [\phi_j(0 \mid x, \lambda)\ \phi_k(b \mid x, \lambda)], \quad j = 1, 2\,;\ k = 3, 4 \qquad (3.1)$$

and

$$D(b\,;\lambda) = [\phi_1\phi_3](b\,;\lambda)[\phi_2\phi_4](b\,;\lambda) - [\phi_1\phi_4](b, \lambda)[\phi_2\phi_3](b\,;\lambda) \qquad (3.2)$$

so that

$$D(b\,;\lambda) = p_0(x)\, q_0(x)\, W_x(\phi_1\phi_2\phi_3\phi_4)(b\,;\lambda).$$

The functions $[\phi_j\phi_k](b;\ \lambda)$ and $D(b;\ \lambda)$ are independent of x. They are integral functions of λ and are real for real λ. Thus

$$\overline{[\phi_j\phi_k](b\,;\lambda)} = [\phi_j\phi_k](b\,;\bar{\lambda}) \quad \text{and} \quad \overline{D(b\,;\lambda)} = D(b;\ \bar{\lambda}). \qquad .. \quad (3.3)$$

The Green's matrix $G(b;\, x,\, y,\, \lambda) = (G_{ij}(b;\, x,\, y,\, \lambda)) = \begin{pmatrix} G_{11} & G_{21} \\ G_{12} & G_{22} \end{pmatrix}$ for the finite boundary value problem given by (1.3), (2.1), (2.2) and (2.3) is defined by

$$\left.\begin{aligned} G(b\,;x,\,y,\,\lambda) &= \begin{pmatrix} \psi_{11}(b\,;x,\lambda) & \psi_{21}(b\,;x,\lambda) \\ \psi_{12}(b\,;x,\lambda) & \psi_{22}(b\,;x,\lambda) \end{pmatrix} \begin{pmatrix} u_1(0\,|\,y,\lambda) & v_1(0\,|\,y,\lambda) \\ u_2(0\,|\,y,\lambda) & v_2(0\,|\,y,\lambda) \end{pmatrix}, \quad 0 \leqslant y < x \\ &= \begin{pmatrix} u_1(0\,|\,x,\lambda) & u_2(0\,|\,x,\lambda) \\ v_1(0\,|\,x,\lambda) & v_2(0\,|\,x,\lambda) \end{pmatrix} \begin{pmatrix} \psi_{11}(b\,;y,\lambda) & \psi_{12}(b\,;y,\lambda) \\ \psi_{21}(b\,;y,\lambda) & \psi_{22}(b\,;y,\lambda) \end{pmatrix}, \quad x < y \leqslant b \end{aligned}\right\} \qquad .. \quad (3.4)$$

where

$$\psi_1(b\,;x,\lambda) = \begin{pmatrix} \psi_{11}(b;\ x,\ \lambda) \\ \psi_{12}(b;\ x,\ \lambda) \end{pmatrix}$$

$$= \frac{[\phi_2\phi_4](b\,;\lambda)\,\phi_3(b\,|\,x,\lambda) - [\phi_2\phi_3](b\,;\lambda)\,\phi_4(b\,|\,x,\lambda)}{D(b\,;\lambda)} \qquad .. \quad .. \quad (3.5)$$

and

$$\psi_2(b\,;x,\lambda) = \begin{pmatrix} \psi_{21}(b\,;x,\lambda) \\ \psi_{22}(b\,;x,\lambda) \end{pmatrix}$$

$$= \frac{[\phi_1\phi_3](b\,;\lambda)\,\phi_4(b\,|\,x,\lambda) - [\phi_1\phi_4](b\,;\lambda)\,\phi_3(b\,|\,x,\lambda)}{D(b\,;\lambda)}. \qquad .. \quad (3.6)$$

From (3.5), (3.6) and (3.2) we have

$$[\phi_j(0\,|\,x,\lambda)\,\psi_r(b\,;x,\lambda)] = \delta_{jr} \quad (1 \leqslant j,\ r \leqslant 2). \qquad .. \quad .. \quad (3.7)$$

Also we have from (3.5), (3.6) and (2.3)

$$[\psi_r(b\,;x,\lambda)\,\psi_s(b\,;x,\lambda)] = 0 \quad (1 \leqslant r,\ s \leqslant 2). \qquad .. \quad .. \quad (3.8)$$

We note that the eigenvalues $\lambda_{n,\,b}$ are the zeros of $D(b;\ \lambda)$ which are real. In general, $\lambda_{n,\,b}$ alters as b changes. Hence we discuss $\psi_r(b;\ x,\ \lambda)$ when λ is not real, i.e. $\nu \neq 0$ where $\lambda = \mu + i\nu$. Then $\psi_r(b;\ x,\ \lambda)$ are analytic in λ for fixed x, and regular in either of the half planes $\nu > 0$ or $\nu < 0$.

§ 4. Let $\theta_k(x,\lambda) \equiv \theta_k(0\,|\,x,\lambda) = \begin{pmatrix} x_k \\ y_k \end{pmatrix}$ $(k = 1,\ 2)$ be two solutions of (1.3) such that

$$[\phi_j\theta_k] = \delta_{jk} \quad (1 \leqslant j,\ k \leqslant 2), \qquad [\theta_1\theta_2] = 0. \qquad .. \quad .. \quad (4.1)$$

Like ϕ_j, θ_k take real values independent of λ at $x = 0$ and are integral functions of λ. Equations (4.1) do not ensure that θ_k can be selected uniquely; but this is not important for our purpose.

From Bhagat (1969), § 5, we have

$$p_0(x)\,q_0(x)\,W_x(\phi_1\phi_2\theta_1\theta_2)(\lambda) = 1.$$

Thus ϕ_1, ϕ_2, θ_1, θ_2 form a fundamental set of solutions for the system (1.3) for all $b > 0$ and all λ.

Hence

$$\psi_r(b\,;\,x,\,\lambda) = \sum_{s=1}^{2} (n_{rs}\theta_s + l_{rs}(b\,;\,\lambda)\phi_s) \quad (r = 1,\, 2).$$

From (2.3), (3.7) and (4.1) it follows that

$$n_{rs} = \delta_{rs} \qquad (1 \leqslant r,\, s \leqslant 2)$$

and

$$\psi_r(b\,;\,x,\,\lambda) = \theta_r + \sum_{s=1}^{2} l_{rs}(b\,;\,\lambda)\phi_s \quad (r = 1,\, 2) \qquad \ldots \quad \ldots \quad (4.2)$$

where

$$l_{rs}(b\,;\,\lambda) = [\psi_r(b\,;\,x,\,\lambda)\;\theta_s(x,\,\lambda)] \qquad (1 \leqslant r,\, s \leqslant 2). \qquad \ldots \quad \ldots \quad (4.3)$$

$l_{rs}(b;\,\lambda)$ can have infinite number of simple poles at the zeros of $D(b;\,\lambda)$ on the real axis whose only limit point is at infinity. Hence for any fixed b, if $\lambda = \mu + i\nu$

$$l_{rs}(b\,;\,\lambda) = O\left(\frac{1}{|\nu|}\right) \quad \text{as} \quad |\nu| \to 0 \;(\mu \text{ fixed}).$$

Also $l_{rs}(b;\,\lambda)$ are analytic functions of λ for every $b > 0$ regular in each of the half planes $\nu > 0$ and $\nu < 0$.

From (4.1), (4.2) and (3.8) it follows that

$$l_{12}(b\,;\,\lambda) = l_{21}(b\,;\,\lambda). \qquad \ldots \quad \ldots \quad \ldots \quad (4.4)$$

Since the vectors $\phi_k(b\,|\,x,\,\lambda)$, $k = 3,\, 4$ are independent of λ at $x = b$, we obtain

$$[\phi_j(b\,|x,\,\lambda_1)\;\phi_k(b\,|\,x,\,\lambda_2)](b) = 0 \qquad (3 \leqslant j,\, k \leqslant 4)$$

and so

$$[\psi_r(b\,;\,x,\,\lambda_1)\;\psi_s(b\,;\,x,\,\lambda_2)](b) = 0 \quad (1 \leqslant r,\, s \leqslant 2). \qquad \ldots \quad (4.5)$$

Hence by Green's formula we get

$$\int_0^b \psi_r^T(b\,;\,x,\,\lambda_1)\,F\psi_s(b\,;\,x,\,\lambda_2)\;dx = \frac{l_{sr}(b\,;\,\lambda_2) - l_{rs}(b\,;\,\lambda_1)}{\lambda_1 - \lambda_2},$$

whence putting $\lambda_1 = \lambda = \mu + i\nu$ and $\lambda_2 = \bar{\lambda} = \mu - i\nu$ we have

$$\int_0^b \psi_r^T(b\,;\,x,\,\lambda)\,F\bar{\psi}_r(b\,;\,x,\,\lambda)\,dx = -\frac{\text{im}\,[l_{rr}(b\,;\,\lambda)]}{\nu} \quad (r = 1,\, 2) \qquad \ldots \quad (4.6)$$

and

$$\int_0^b \psi_1^T(b\,;\,x,\,\lambda)\;F\bar{\psi}_2(b\,;\,x,\,\lambda)\,dx = -\frac{\text{im}\,[l_{12}(b\,;\,\lambda)]}{\nu} \qquad \ldots \quad \ldots \quad \ldots \quad (4.7)$$

for all $b > 0$.

§ 5. Let us write $\{\phi\theta\}(x) = -i[\phi\theta](x)$ for convenience for any two vectors $\phi(x)$ and $\theta(x)$ for which the right-hand side has a meaning.

From Green's formula it follows that

$$\left.\begin{aligned}\{\phi_r\bar{\phi}_s\}(b;\,\lambda) &= 2\nu\int_0^b \phi_r^T F\bar{\phi}_s\,dx\\ \{\theta_r\bar{\theta}_s\}(b;\,\lambda) &= 2\nu\int_0^b \theta_r^T F\bar{\theta}_s\,dx\end{aligned}\right\}\quad (1 \leqslant r,\, s \leqslant 2). \qquad (5.1)$$

From the representation (4.2) we have

$$\begin{aligned}-i[\psi_r(b;\,x,\,\lambda)\,\bar{\psi}_r(b;\,x,\,\lambda)](b) = {} & \{\phi_r\bar{\phi}_r\}(b;\,\lambda)[I_{rr}^2+R_{rr}^2]+\{\phi_s\bar{\phi}_s\}(b;\,\lambda)\\ & \times[R_{rs}^2+I_{rs}^2]+2\text{ im }[\phi_r\bar{\phi}_s](b;\,\lambda)[R_{rr}R_{rs}+I_{rr}I_{rs}]\\ & +2\text{ re }[\phi_r\bar{\phi}_s](b;\,\lambda)[R_{rs}I_{rr}-R_{rr}I_{rs}]\\ & +2\text{ im }[\phi_r\bar{\theta}_r](b;\,\lambda)R_{rr}+2\text{ re }[\phi_r\bar{\theta}_r](b;\,\lambda)I_{rr}\\ & +2\text{ im }[\phi_s\bar{\theta}_r](b;\,\lambda)R_{rs}+2\text{ re }[\phi_s\bar{\theta}_r](b;\,\lambda)I_{rs}\\ & +\{\theta_r\bar{\theta}_r\}(b;\,\lambda)\quad(\text{when } r=1,\, s=2 \text{ and}\\ & \qquad r=2,\, s=1) \qquad (5.2)\end{aligned}$$

where

$$l_{rs}(b;\,\lambda) = R_{rs}+iI_{rs}.$$

From (4.5)

$$-i[\psi_r(b;\,x,\,\lambda)\,\bar{\psi}_r(b;\,x,\,\lambda)](b) = 0 \quad (r = 1,\, 2). \qquad (5.3)$$

These are the equations of quadratic hypersurfaces in a four-dimensional real space. These are the finite surfaces of the boundary value problem on $[0,\, b]$ and are denoted here by $S_r(b;\,\lambda)$.

§ 6. In order to facilitate the geometrical significance of the finite surfaces we consider the surface $S_1(b;\,\lambda)$ and reduce it to a canonical form. An entirely similar argument holds for the surface $S_2(b;\,\lambda)$.

Let us write (5.2) for $r = 1$ as

$$\sum_{r=1}^{4}\sum_{s=1}^{4} c_{rs}x_r x_s+2\sum_{r=1}^{4} b_r x_r+d = 0 \qquad (6.1)$$

where x_1, x_2, x_3, x_4 have been put for I_{11}, R_{11}, I_{12}, R_{12} respectively.

Let C denote the matrix (c_{rs}) and B the matrix

$$\begin{pmatrix} C & b\\ b^T & d\end{pmatrix} \quad \text{where } b = (b_1\, b_2\, b_3\, b_4)^T.$$

Define

$$\Delta(b;\,\lambda) \equiv \{\phi_1\bar{\phi}_1\}(b;\,\lambda)\,.\,\{\phi_2\bar{\phi}_2\}(b;\,\lambda)-[(\text{im }[\phi_1\bar{\phi}_2](b;\,\lambda))^2+(\text{re }[\phi_1\bar{\phi}_2](b;\,\lambda))^2].$$

Since ϕ_1 and ϕ_2 are linearly independent, by using inequality (5.6) of the author [Bhagat (1969)], it can be shown that

$$\Delta(b;\,\lambda) > 0.$$

Then by actual calculation it can be shown that $\det C = \Delta^2(b;\,\lambda)$ and the matrix C is positive definite if $\nu = \text{im }\lambda > 0$ and negative definite if $\nu < 0$. In order to evaluate $\det B$ we require the following result.

Lemma 6.1

Let $\theta_i = \begin{pmatrix} u_i \\ v_i \end{pmatrix}$, $(i = 1, 2, 3, 4, 5, 6)$ be any six vectors possessing differential coefficients of the first order in the interval $[0, b]$. Then

$$\begin{aligned}
&[\theta_1\theta_2]([\theta_3\theta_4][\theta_5\theta_6]-[\theta_3\theta_5][\theta_4\theta_6]+[\theta_3\theta_6][\theta_4\theta_5]) \\
&-[\theta_1\theta_3]([\theta_2\theta_4][\theta_5\theta_6]-[\theta_2\theta_5][\theta_4\theta_6]+[\theta_2\theta_6][\theta_4\theta_5]) \\
&+[\theta_1\theta_4]([\theta_2\theta_3][\theta_5\theta_6]-[\theta_2\theta_5][\theta_3\theta_6]+[\theta_2\theta_6][\theta_3\theta_5]) \\
&-[\theta_1\theta_5]([\theta_2\theta_3][\theta_4\theta_6]-[\theta_2\theta_4][\theta_3\theta_6]+[\theta_2\theta_6][\theta_3\theta_4]) \\
&+[\theta_1\theta_6]([\theta_2\theta_3][\theta_4\theta_5]-[\theta_2\theta_4][\theta_3\theta_5]+[\theta_2\theta_5][\theta_3\theta_4]) = 0.
\end{aligned}$$

The result follows by expanding the left-hand side of the following identity by Laplace's expansion theorem in terms of the first two rows and using (5.3) of Bhagat (1969), the determinants being represented by their leading diagonals,

$$p_0^2 q_0(u_1' u_2 u_3 v_4 u_5' v_6')+p_0 q_0^2 (v_1' v_2 u_3 v_4 u_5' v_6') = 0.$$

Expanding det B in terms of the first two rows by Laplace's expansion theorem we get after a bit of laborious calculations

$$\begin{aligned}
\det B = \Delta(b\,;\lambda)\,.\,i\{&[\phi_1\bar\phi_1](b\,;\lambda)[\phi_2\bar\phi_2](b\,;\lambda)[\theta_1\bar\theta_1](b\,;\lambda) \\
&+[\phi_1\bar\phi_1](b\,;\lambda)[\phi_2\bar\theta_1](b\,;\lambda)[\bar\phi_2\theta_1](b\,;\lambda)+[\phi_2\bar\phi_2](b\,;\lambda)[\phi_1\bar\theta_1](b\,;\lambda)[\bar\phi_1\theta_1](b\,;\lambda) \\
&+[\phi_1\bar\phi_2](b\,;\lambda)[\bar\phi_1\phi_2](b\,;\lambda)[\theta_1\bar\theta_1](b\,;\lambda)-[\phi_1\bar\phi_2](b\,;\lambda)[\phi_2\bar\theta_1](b\,;\lambda)[\bar\phi_1\theta_1](b\,;\lambda) \\
&+[\bar\phi_1\phi_2](b\,;\lambda)[\bar\phi_2\theta_1](b\,;\lambda)[\phi_1\bar\theta_1](b\,;\lambda)\} = \Delta(b\,;\lambda)\,.\,\Lambda, \quad \text{(say)}.
\end{aligned}$$

In the above lemma, putting the six functions $\phi_1(x, \lambda)$, $\bar\phi_1(x, \lambda)$, $\phi_2(x, \lambda)$, $\bar\phi_2(x, \lambda)$, $\theta_1(x, \lambda)$, $\bar\theta_1(x, \lambda)$ for $x = b$ and using (2.3) and (4.1), we get

$$\begin{aligned}
\Lambda = i[\phi_2\bar\phi_2](b\,;\lambda) &= -\{\phi_2\bar\phi_2\}(b\,;\lambda) \\
&= -2\nu \int_0^b \phi_2^T F \bar\phi_2 \, dx.
\end{aligned}$$

Hence

$$\det B = -\Delta(b, \lambda)\,.\,2\nu \int_0^b \phi_2^T F \bar\phi_2 \, dx.$$

Let $\mu_1, \mu_1, \mu_2, \mu_2$ be latent roots of the matrix C which are real and positive if $\nu > 0$, and real and negative if $\nu < 0$. Without loss of generality these may be ordered as

$$\mu_2 \geqslant \mu_1 > 0 \ \text{ if } \ \nu > 0 \ \text{ and } \ \mu_2 \leqslant \mu_1 < 0 \ \text{ if } \ \nu < 0.$$

Now transferring the origin to $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ where $\alpha_r = \dfrac{\det C_r}{\det C}$ $(r = 1, 2, 3, 4)$, C_r being the matrix obtained from C when the rth column in C is replaced by $(-b_1, -b_2, -b_3, -b_4)^T$, and taking the coordinate axes along principal axes of $S_1(b;\ \lambda)$ it follows that (6.1) is metrically equivalent to

$$\frac{z_1^2+z_3^2}{\rho_{11}^2(b;\ \lambda)}+\frac{z_2^2+z_4^2}{\rho_{12}^2(b;\ \lambda)}=1 \qquad (6.2)$$

where

$$\rho_{1s}^2(b;\ \lambda)=\frac{2\nu\int_0^b \phi_2^T F\bar{\phi}_2\, dx}{\mu_s\,\Delta(b;\ \lambda)}. \qquad (6.3)$$

By similar arguments the surface $S_2(b;\ \lambda)$ is metrically equivalent to

$$\frac{z_1^2+z_3^2}{\rho_{21}^2(b;\ \lambda)}+\frac{z_2^2+z_4^2}{\rho_{22}^2(b;\ \lambda)}=1 \qquad (6.4)$$

where

$$\rho_{2s}^2(b;\ \lambda)=\frac{2\nu\int_0^b \phi_1^T F\bar{\phi}_1\, dx}{\mu_s\,\Delta(b;\ \lambda)}. \qquad (6.5)$$

Clearly, for every $b>0$,

$$\rho_{r1}^2(b;\ \lambda)\geqslant\rho_{r2}^2(b;\ \lambda)>0. \qquad (6.6)$$

§ 7. Let us suppose that λ is fixed with $\text{im}\ \lambda=\nu>0$. Entirely similar results hold for $\nu<0$.

From (6.2) and (6.4) it follows that regions R_r of x-space occupied by the points inside and on the surface $S_r(b;\ \lambda)$ are bounded (*see* Mirsky 1955, example 26, p. 424). For fixed r ($r=1$ or 2) the point $(I_{r1},\ R_{r1},\ I_{r2},\ R_{r2})$ lies inside, on or outside the surface $S_r(b;\ \lambda)$ according as [from (4.5) and (4.6)]

$$\int_0^b \psi_r^T(b;\ x,\ \lambda)\,F\bar{\psi}_r(b;\ x,\ \lambda)\,dx+\frac{\text{im}\ [l_{rr}(b;\ \lambda)]}{\nu}$$

is negative, zero or positive.

Let the point $(I_{r1},\ R_{r1},\ I_{r2},\ R_{r2})$ lie on the surface $S_r(b;\ \lambda)$ and $0<b'<b$. Then since

$$\int_0^{b'}\left(\theta_r+\sum_{s=1}^{2} l_{rs}\phi_s\right)^T F\left(\bar{\theta}_r+\sum_{s=1}^{2}\bar{l}_{rs}\bar{\phi}_s\right)dx$$

$$<\int_0^{b}\left(\theta_r+\sum_{s=1}^{2} l_{rs}\phi_s\right)^T F\left(\bar{\theta}_r+\sum_{s=1}^{2}\bar{l}_{rs}\bar{\phi}_s\right)dx,$$

we have

$$\int_0^{b'}\left(\theta_r+\sum_{s=1}^{2} l_{rs}\phi_s\right)^T F\left(\bar{\theta}_r+\sum_{s=1}^{2}\bar{l}_{rs}\bar{\phi}_s\right)dx+\frac{\text{im}\ [l_{rr}]}{\nu}<0$$

so that the point $(I_{r1},\ R_{r1},\ I_{r2},\ R_{r2})$ which lies on the surface $S_r(b;\lambda)$ lies inside the surface $S_r(b';\ \lambda)$. Thus the surfaces $S_r(b;\ \lambda)$ are strictly contained inside each other for increasing b. It follows that, as $b\to\infty$, $S_r(b;\ \lambda)$ must tend to closed limit surfaces $S_r(\infty;\lambda)$. We call $S_r(\infty;\lambda)$ the 'singular surfaces'.

From monotonic nature of the surfaces $S_1(b;\,\lambda)$ as b increases it follows that ρ_{1s} $(s = 1, 2)$ also monotonically decrease as b increases and are positive (from 6.6). Hence $\underset{b\to\infty}{\text{Lt}}\, \rho_{1s}(b;\,\lambda)$ exist and are non-negative.

Let

$$\underset{b\to\infty}{\text{Lt}}\ \rho_{1s}(b;\,\lambda) = \sigma_{1s}(\lambda) \equiv \sigma_{1s} \quad (s = 1,\, 2).$$

Then

$$\sigma_{11} \geqslant \sigma_{12} \geqslant 0.$$

The following are now the possibilities: (i) $\sigma_{11} \geqslant \sigma_{12} > 0$, (ii) $\sigma_{11} > 0$, $\sigma_{12} = 0$, and (iii) $\sigma_{11} = 0$, $\sigma_{12} = 0$.

In case (i) $S_1(\infty,\,\lambda)$ is a four-dimensional ellipsoid, in (ii) it is a circle and in (iii) it is a point. Similar arguments hold for the surface $S_2(\infty,\,\lambda)$.

From monotonic and bounded nature of the surfaces $S_r(b;\,\lambda)$ it follows that $l_{rs}(b;\,\lambda)$ are also uniformly bounded. Hence by selection principle there exist a sequence

$$\{b_n\,;\, n > 1\} \text{ such that } b_n \to \infty \text{ as } n \to \infty \text{ and}$$

$$\underset{b_n\to\infty}{\text{Lt}}\ l_{rs}(b_n\,;\,\lambda) = m_{rs}(\lambda) \ \ (\text{say}) \ \ (1 \leqslant r,\, s \leqslant 2).$$

From (4.4) it follows that

$$m_{12}(\lambda) = m_{21}(\lambda). \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (7.1)$$

Also following Everitt (1963, § A8) it follows that

$$m_{11}m_{22} - m_{12}^2 \neq 0, \quad \text{im}\,\lambda \neq 0.$$

$m_{rs}(\lambda)$, in general, will have singularities on the real axis, $m_{rs}(\lambda)$ are analytic functions of λ regular in either of the half planes $\nu > 0$ and $\nu < 0$.

We now prove the following theorems.

Theorem 7.1

In the singular case $[0,\,\infty)$, for values of λ other than the real values, there exist at least two linearly independent solutions $\psi_1(x,\,\lambda)$ and $\psi_2(x,\,\lambda)$ of the differential system (1.3) such that $\psi_r^T(x,\,\lambda)\ F\bar{\psi}_r(x,\,\lambda)$ belong to $L[0,\,\infty)$ $(r = 1,\,2)$.

In particular, if det $F \geqslant 1$, then $\psi_r(x,\,\lambda)$ belong to $L^2[0,\,\infty)$.

Let $\nu = \text{im}\,\lambda \neq 0$. Define the vectors $\psi_r(x,\,\lambda)$ $(r = 1,\,2)$ by

$$\psi_r(x,\,\lambda) = \begin{pmatrix}\psi_{r1}(x,\,\lambda)\\ \psi_{r2}(x,\,\lambda)\end{pmatrix}$$

$$= \theta_r(x,\,\lambda) + \sum_{s=1}^{2} m_{rs}\phi_s(x,\,\lambda) \quad (r = 1,\,2) \qquad \ldots \quad \ldots \quad (7.2)$$

for $0 \leqslant x < \infty$.

Then the vectors $\psi_r(x,\,\lambda)$ $(r = 1,\,2)$ are the linearly independent solutions of (1.3) and, for fixed x, they are analytic functions of λ regular in either of the half planes $\nu > 0$ and $\nu < 0$.

Since ϕ_1 and ϕ_2 are linearly independent, we also have

$$\operatorname*{Lt}_{n\to\infty} \psi_r(b_n\,;\lambda) = \psi_r(x, \lambda) \quad (r = 1, 2) \qquad \ldots \quad \ldots \quad (7.3)$$

for all λ (im $\lambda \neq 0$) and $0 \leqslant x < \infty$.

We also note that

$$[\phi_j(x, \lambda)\, \psi_r(x, \lambda)] = \delta_{jr} \quad (1 \leqslant j, r \leqslant 2). \qquad \ldots \quad \ldots \quad (7.4)$$

Since the points (im $[m_{r1}(\lambda)]$, re $[m_{r1}(\lambda)]$, im $[m_{r2}(\lambda)]$, re $[m_{r2}(\lambda)]$) lie inside the surface $S_r(b;\lambda)$, for every $b > 0$, we have just as above

$$\int_0^b \psi_r^T(x, \lambda)\, F\bar{\psi}_r(x, \lambda)\, dx < -\frac{\text{im}\,[m_{rr}(\lambda)]}{\nu}.$$

Hence, as $b \to \infty$, we get

$$\int_0^\infty \psi_r^T(x, \lambda)\, F\bar{\psi}_r(x, \lambda)\, dx \leqslant -\frac{\text{im}\,[m_{rr}(\lambda)]}{\nu} \qquad \ldots \quad \ldots \quad (7.5)$$

and the result follows.

In particular, if det $F \geqslant 1$, then

$$|\psi_r(x, \lambda)|^2 = \psi_r^T(x, \lambda)\, \bar{\psi}_r(x, \lambda) \leqslant \psi_r^T(x, \lambda)\, F\bar{\psi}_r(x, \lambda)$$

(*see* Mirsky 1955, Prob. 37, p. 426) and it follows that $\psi_r(x, \lambda)$ belong to $L^2[0, \infty)$.

Theorem 7.2

If for one complex value of λ, say λ_0, every solution $\phi \equiv \phi(x, \lambda_0)$ of the differential system (1.3) is such that $\phi^T F\bar{\phi} \in L[0, \infty)$, then for arbitrary complex λ every solution $\phi \equiv \phi(x, \lambda) = \begin{pmatrix} u \\ v \end{pmatrix}$ of (1.3) is such that

$$\phi^T(x, \lambda)\, F\bar{\phi}(x, \lambda) \in L[0, \infty).$$

In particular, if det $F \geqslant 1$, then $\phi(x, \lambda_0) \in L^2[0, \infty)$ and consequently $\phi(x, \lambda) \in L^2[0, \infty)$.

Let

$$\phi_r \equiv \phi_r(c\,|\,x, \lambda_0) = \begin{pmatrix} u_r(c\,|\,x, \lambda_0) \\ v_r(c\,|\,x, \lambda_0) \end{pmatrix} \quad (r = 1, 2, 3, 4)$$

be a set of four linearly independent solutions of (1.3) for $\lambda = \lambda_0$ which are such that

$$\phi_r^T F\bar{\phi}_r \in L[0, \infty) \quad (r = 1, 2, 3, 4),$$

$$[\phi_1\phi_2] = [\phi_3\phi_4] = [\phi_1\phi_4] = [\phi_2\phi_3] = 0,$$

$$[\phi_1\phi_3] = 1 = [\phi_2\phi_4]$$

and they take real values at $x = c$ ($c \geqslant 0$).

Then

$$D(\lambda_0) = p_0(x)\, q_0(x)\, W_x(\phi_1\phi_2\phi_3\phi_4)(\lambda_0)$$
$$= [\phi_1\phi_3][\phi_2\phi_4]-[\phi_1\phi_2][\phi_3\phi_4]-[\phi_1\phi_4][\phi_2\phi_3] = 1.$$

Hence ϕ_1, ϕ_2, ϕ_3 and ϕ_4 form a fundamental set of solutions for $\lambda = \lambda_0$. Let $\phi(x) \equiv \phi(x, \lambda) = \begin{pmatrix} u(x, \lambda) \\ v(x, \lambda) \end{pmatrix}$ be any solution of (1.3) for arbitrary λ.

Now the equation (1.3) may be written as

$$(L-\lambda_0 F)\,\phi = -(\lambda_0-\lambda)\, F\phi.$$

Then the variation of constant formula yields

$$\phi(x) = \sum_{r=1}^{4} A_r\phi_r(x)-(\lambda_0-\lambda)P_3\phi_1(x)-(\lambda_0-\lambda)P_4\phi_2(x)$$
$$+(\lambda_0-\lambda)P_1\phi_3(x)+(\lambda_0-\lambda)P_2\phi_4(x) \qquad \ldots \quad (7.6)$$

where A_r $(r = 1, 2, 3, 4)$ is a constant and

$$P_r = \int_c^x \phi_r^T(t)\, F\phi(t)\, dt.$$

Let

$$A = \max\,(|A_1|,\ |A_2|,\ |A_3|,\ |A_4|), \quad S = \left[\int_c^x |\phi^T(x)\, F\bar{\phi}(x)|\, dx\right]^{\frac{1}{2}}$$

$$M = \max\left[\int_c^\infty |\phi_r^T(x)\, F\bar{\phi}_r(x)|\, dx\right]^{\frac{1}{2}} \quad (r = 1, 2, 3, 4).$$

Then

$$|P_r| = \left|\int_c^x \phi_r^T(t)\, F\phi(t)\, dt\right|$$
$$\leqslant \left\{\int_c^x \phi_r^T(t)\, F\bar{\phi}_r(t)\, dt \int_c^x \phi^T(t)\, F\bar{\phi}(t)\, dt\right\}^{\frac{1}{2}}$$

by the inequality (5.6) of Bhagat (1969).

$$\therefore\quad |P_r| \leqslant SM. \qquad \ldots \quad (7.7)$$

Similarly, by the inequality (5.8) of Bhagat (1969), we have

$$|\bar{P}_r| \leqslant SM. \qquad \ldots \quad (7.8)$$

Using (7.7) and (7.8) we have

$$|\phi^T(x)\, F\bar{\phi}(x)| \leqslant (A+|\lambda_0-\lambda|\, MS)^2 \sum_{r=1}^{4}\sum_{s=1}^{4} |\phi_r^T(x)\, F\bar{\phi}_s(x)|. \qquad \ldots \quad (7.9)$$

Integrating (7.9) from c to x we obtain by the inequality (5.7) of Bhagat (1969)

$$S \leqslant \frac{4AM}{1-|\lambda_0-\lambda|\, 4M^2}.$$

Let us choose c so large that

$$|\lambda_0-\lambda|\, M^2 < \tfrac{1}{8},$$

then

$$S \leqslant 8AM$$

and the result follows.

§ 8. We define the Green's matrix for our problem for the singular case $[0, \infty)$. Let us define the matrix

$$G(x, y; \lambda) = \begin{pmatrix} G_{11} & G_{21} \\ G_{12} & G_{22} \end{pmatrix}$$

for all λ with im $\lambda \neq 0$, by

$$\left.\begin{aligned} G(x, y; \lambda) &= \begin{pmatrix} \psi_{11}(x, \lambda) & \psi_{21}(x, \lambda) \\ \psi_{12}(x, \lambda) & \psi_{22}(x, \lambda) \end{pmatrix} \cdot \begin{pmatrix} u_1(y, \lambda) & v_1(y, \lambda) \\ u_2(y, \lambda) & v_2(y, \lambda) \end{pmatrix}; \quad y \in [0, x) \\ &= \begin{pmatrix} u_1(x, \lambda) & u_2(x, \lambda) \\ v_1(x, \lambda) & v_2(x, \lambda) \end{pmatrix} \cdot \begin{pmatrix} \psi_{11}(y, \lambda) & \psi_{12}(y, \lambda) \\ \psi_{21}(y, \lambda) & \psi_{22}(y, \lambda) \end{pmatrix}; \quad y \in (x, \infty) \end{aligned}\right\}. \tag{8.1}$$

Then since $\psi_r^T F \bar{\psi}_r \in L[0, \infty)$ it can be shown that the vectors $G_k \equiv G_k(x, y; \lambda) = \begin{pmatrix} G_{k1} \\ G_{k2} \end{pmatrix}$, $(k = 1, 2)$ are such that $G_k^T F \bar{G}_k$ belong to $L[0, \infty)$.

The following results analogous to lemma 8.1 of Bhagat (1969) involving $\psi_{ij}(x, \lambda)$ and ϕ_1, ϕ_2 can be proved by using (7.1), (7.2) and (4.1).

Lemma 8.1

(i) $(\psi_{11}v_1+\psi_{21}v_2-\psi_{12}u_1-\psi_{22}u_2) = 0$

(ii) $p_0(\psi'_{11}u_1+\psi'_{21}u_2-\psi_{11}u'_1-\psi_{21}u'_2) = -1$

(iii) $(\psi'_{11}v_1+\psi'_{21}v_2-\psi_{12}u'_1-\psi_{22}u'_2) = 0$

(iv) $(\psi'_{12}u_1+\psi'_{22}u_2-\psi_{11}v'_1-\psi_{21}v'_2) = 0$

(v) $q_0(\psi'_{12}v_1+\psi'_{22}v_2-\psi_{12}v'_1-\psi_{22}v'_2) = -1.$

By utilizing these results it can be proved that the matrix $G(x, y; \lambda)$ defined by (8.1) satisfies all the properties of a Green's matrix.

§ 9. Let the vector $f(x) = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$ be such that $f^T F f$ belongs to $L[0, \infty)$ and im $\lambda \neq 0$.

We define the vector

$$\Phi(x, \lambda) = \begin{pmatrix} \Phi_1(x, \lambda) \\ \Phi_2(x, \lambda) \end{pmatrix}$$

by

$$\Phi(x, \lambda) = \int_0^\infty G(x, y; \lambda) F f(y)\, dy \quad (0 \leqslant x < \infty) \quad \ldots \quad \ldots \tag{9.1}$$

where $G(x, y; \lambda)$ is the Green's matrix for the singular case $[0, \infty)$. Then the integral on the right exists by the inequality (5.6) of Bhagat (1969).

It can be shown that $\Phi(x, \lambda)$ defined by (9.1) satisfies the differential system (1.4) if $f(x)$ is continuous in x.

We now prove a property of this vector.

Lemma 9.1

Let $f(x) = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$ be a vector such that $f^T F f$ belongs to $L[0, \infty)$ and let the vector $\Phi(x, \lambda)$ be defined by (9.1) and im $\lambda = \nu \neq 0$. Then

$$\int_0^\infty \Phi^T(x, \lambda) F \overline{\Phi(x, \lambda)}\, dx \leqslant \frac{1}{\nu^2} \int_0^\infty f^T F f\, dx. \qquad \text{.. .. (9.2)}$$

By § 7 $l_{rs}(b; \lambda)$ are uniformly bounded for $b > b_0$. Using (4.6) and (4.7) it follows from the explicit expressions of $G(b; x, y, \lambda)$ that, for every fixed λ,

$$\int_0^b G_k^T(b; x, y, \lambda) F \bar{G}_k(b; x, y, \lambda)\, dx < A \quad (0 \leqslant x,\ y \leqslant b \text{ and } b > b_0). \qquad (9.3)$$

Now $G(b; x, y, \lambda) \to G(x, y; \lambda)$ pointwise in x and y as $b \to \infty$ and $G_k^T(x, y; \lambda) F \bar{G}_k(x, y; \lambda)$ belongs to $L[0, \infty)$. Let $G_k^b \equiv G_k(b; x, y, \lambda)$ and $G_k \equiv G_k(x, y; \lambda)$, then we have

$$\begin{aligned} &\left| \int_0^b (G_k^b - G_k)^T F f(y)\, dy \right| \\ &\leqslant \left| \int_0^X (G_k^b - G_k)^T F f(y)\, dy \right| + \left| \int_X^b (G_k^b - G_k)^T F f(y)\, dy \right| \\ &\leqslant \left\{ \int_0^X (G_k^b - G_k)^T F (\bar{G}_k^b - \bar{G}_k)\, dy \int_0^\infty f^T(y) F f(y)\, dy \right\}^{\frac{1}{2}} \\ &+ \left\{ \int_0^b (G_k^b - G_k)^T F (\bar{G}_k^b - \bar{G}_k)\, dy \int_X^b f^T(y) F f(y)\, dy \right\}^{\frac{1}{2}}. \end{aligned}$$

The second term can be made less than ϵ, for all b, by properly choosing X and having fixed X, the first term tends to zero as b tends to infinity. Hence

$$\Phi(x, \lambda) = \operatorname*{Lt}_{b \to \infty} \Phi(b; x, \lambda)$$

For $c < b$,

$$\begin{aligned} \int_0^c \Phi^T(b; x, \lambda) F \overline{\Phi(b; x, \lambda)}\, dx &< \int_0^b \Phi^T(b; x, \lambda) F \overline{\Phi(b; x, \lambda)}\, dx \\ &\leqslant \frac{1}{\nu^2} \int_0^b f^T(x) F f(x)\, dx \end{aligned}$$

by (10.2) of Bhagat (1969). By making first b tend to infinity and then c tend to infinity the result follows.

In a paper sequel to this it will be proved that, under certain conditions to be satisfied, the Green's matrix is unique.

ACKNOWLEDGEMENT

The author is thankful to Dr. N. K. Chakravarty for encouragement and suggestions in preparation of this paper.

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