VOLUME COMMON TO EQUAL, CIRCULAR CYLINDERS WITH CONCURRENT AXES

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Martin Gardner (1962) gives a method to find, without calculus, the volume common to two equal, circular cylinders whose axes intersect at right angles: if r is the radius of the cross-section of either (and also the sphere inscribed to both), the volume is $16r^3/3$.

Mahajani (1967) has pointed out that the method used is only an application of Cavalieri's theorem in pure solid geometry. He has further generalized the result as follows: if the axes of n equal, circular cylinders are co-planar and concur in a point symmetrically, the volume common to them is $\frac{8nr^3}{3}\tan\frac{\pi}{2n}$.

It is proposed here to generalize Mahajani's result in three directions, viz.:

(A) If the n equal, circular cylinders are not symmetrically disposed, though their axes, co-planar in the plane of the paper (ϕ), are concurrent in O, then the volume common to them is

$$\frac{8r^3}{3}(\tan\alpha_1+\tan\alpha_2+\ldots\tan\alpha_n) \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

where $\widehat{C_1, C_2}$ = angle between the axes of cylinders $\widehat{C_1, C_2} = 2\alpha_1$;

$$\widehat{C_2, C_3} = 2\alpha_2; \ \ldots \widehat{C_n, C_1} = 2\alpha_n$$

and

$$2\alpha_1+2\alpha_2+2\alpha_3+\ldots+2\alpha_n = \pi.$$

(B) If, in addition to the n equal, circular cylinders, symmetrically disposed in the plane ϕ, we introduce one $(n+1)$th equal, circular cylinder with its axis through O, and perpendicular to ϕ, then the volume common to all the $(n+1)$ cylinders is

$$\frac{8nr^3}{3}(2-\cos\alpha-\sec\alpha+\tan\alpha) = \frac{\pi}{2n}. \quad \ldots \quad \ldots \quad (2)$$

(C) If, in (A), where the n horizontal cylinders are not symmetrically disposed, we introduce C_{n+1} as in (B), then the volume common to all the $n+1$ cylinders is

$$\frac{8r^3}{3}\sum_{r=1}^{n}(2-\cos\alpha_r-\sec\alpha_r+\tan\alpha_r). \quad \ldots \quad \ldots \quad (3)$$

Proof of Result (A)

Imagine a sphere of radius r inside the volume common to the n cylinders whose axes are concurrent in the plane of the paper (ϕ) at O. The plane ϕ will slice the cylinders and the sphere in half (Fig. 1, drawn for $n = 3$). The cross-section in ϕ of the volume common to the cylinders will be a $2n$-sided polygon $PQRSTU$ in which opposite sides are equal and parallel. The cross-section of the sphere will be the circle (r) inscribed in the polygon.

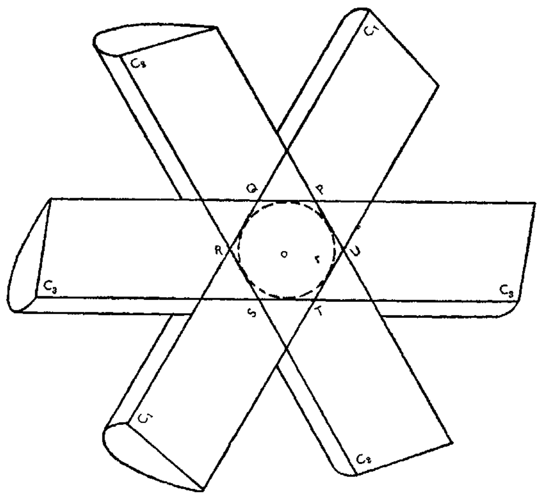

FIG. 1

Next suppose that the cylinders of the sphere are sliced by a plane parallel to ϕ; it will then shave off only a small portion of each cylinder (Fig. 2).

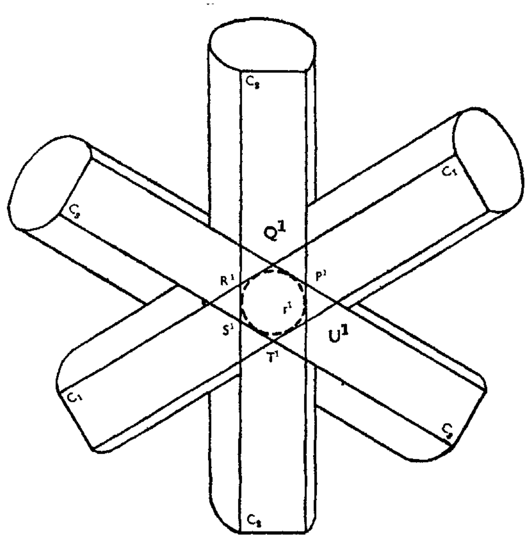

FIG. 2

The section will produce parallel tracks on each cylinder which will intersect to form, as before, a smaller $2n$-sided polygon $P^1Q^1R^1S^1T^1U^1$ as the cross-section common to all the n cylinders. Also, as before, the cross-section of the sphere will be the inscribed circle (r') of this polygon. It is obvious that this will be the case with all plane sections parallel to ϕ. The polygon and its inscribed circle will reduce to the point N where the plane distant r from ϕ will touch all the cylinders and the sphere (at its topmost point N).

By Cavalieri's theorem

$$\frac{\text{required common volume, } V}{\text{volume of the sphere}}$$

$$= \frac{\text{area of the polygon}}{\text{area of the circle}}$$

$$= \frac{2r^2\,(\tan\alpha_1+\tan\alpha_2+\ldots+\tan\alpha_n)}{\pi r^2}$$

$$\therefore\quad V = \frac{8r^3}{3}\,(\tan\alpha_1+\tan\alpha_2+\ldots+\tan\alpha_n). \qquad \ldots \qquad \ldots \qquad (4)$$

We may note here that the prism which circumscribes the sphere with the $2n$-sided polygonal base has its volume equal to

$4nr^3\,(\tan\alpha_1+\tan\alpha_2+\ldots+\tan\alpha_3)$, and hence

$V =$ the volume of the solid portion common to the n cylinders
$= \frac{2}{3}$ times the volume of the circumscribing right prism on the $2n$-sided polygon

(This was pointed out by Archimedes for $n = 2$).

Proof of Result (B)

In this case

$$\alpha_1 = \alpha_2 = \alpha_3 = \ldots = \alpha_n = \frac{\pi}{2n}.$$

The C_{n+1} will trace on ϕ exactly the same trace of the circle, as done by the inscribed sphere.

We have now to take the volume as in (A) above, but only that portion of it as is contained in the vertical C_{n+1}.

Consider Fig. 3. The polygon $P_1Q_1R_1S_1T_1U_1$ lies entirely within all the cylinders. A prism on this as base, of whatever height, will be entirely within C_{n+1}; but we must take only that part of it which lies in all the $C_1, C_2 \ldots C_n$.

In Fig. 3, $P_1M = MQ_1 = r\sin\alpha$. If we rotate the circular section about the diameter so that it becomes vertical then it will be the right section of C_3. In other words P_1Q_1, if lifted above the plane of the paper (ϕ) through $r\sin\alpha$, becomes a generator of C_3. Similarly Q_1R_1, if lifted through $r\sin\alpha$, becomes a generator of C_1, etc. The same is true below ϕ.

Thus the prism on the polygonal base $P_1Q_1R_1S_1T_1U_1$ with total height $= 2r \sin \alpha$ lies entirely within all the $n+1$ cylinders. Let this volume be denoted by X.

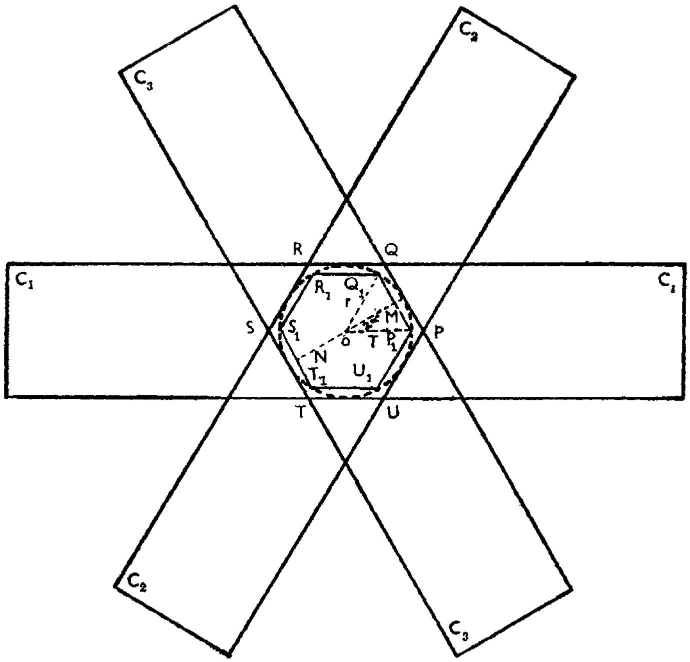

FIG. 3

Obviously (Fig. 4)

$$X = 2r \sin \alpha \,.\, 2nr^2 \sin \alpha \cos \alpha$$

$$= 4nr^3 \sin^2 \alpha \cos \alpha. \qquad \qquad (5)$$

Next, on each of the two bases of the pyramid (one above and one below ϕ), we have to find the portion common to the n co-planar cylinders. Call this Y.

Y is the cylindro-pyramid on the polygonal section $P^1Q^1R^1S^1T^1U^1$ distant $r \sin \alpha$ from O, contained in the spherical cap (Fig. 5).

Note: The lateral faces are parts of cylinders, opposite ones belonging to the same cylinder.

The area of the polygonal base is $2n \tan \alpha/\pi$ times the area of the inscribed circular trace of the sphere.

Therefore again by Cavalieri's theorem

$$\frac{Y}{\text{spherical cap}} = \frac{2n \tan \alpha}{\pi}$$

$$\therefore \quad Y = \frac{2n \tan \alpha}{\pi} \cdot \frac{\pi r^3}{3} (2 - 3 \sin \alpha + \sin^3 \alpha).$$

$\therefore$ The total volume of the two cylindro-pyramids is

$$2Y = \frac{4n \tan \alpha}{3} r^3 (2-3 \sin \alpha + \sin^3 \alpha). \quad .. \quad .. \quad .. \quad (6)$$

Finally, we have the $2n$ lateral, square faces of the prism X (Figs. 4 and 6).

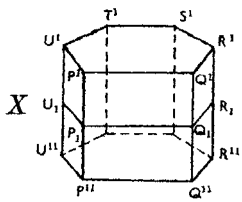

FIG. 4

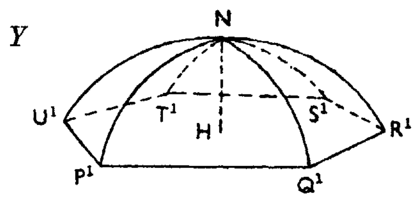

FIG. 5

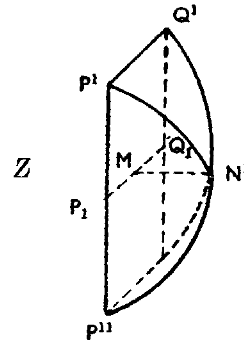

FIG. 6

Consider the lateral face through P_1Q_1. On this square face, we have to calculate the portion common only to the two cylinders C_3 and the vertical C_{n+1}.

Here also, if we call this volume Z, $Z = 4/\pi$ the volume of the spherical cap.

Here, the distance of the base of the cap from $O = OM = r \cos \alpha$

$$\therefore \quad Z = \frac{4r^3}{3} (2-3 \cos \alpha + \cos^3 \alpha). \quad .. \quad .. \quad .. \quad (7)$$

There will be $2n$ such Z.

Therefore, the total volume common to all the $(n+1)$ cylinders is $X+2Y+2nZ$, that is

$$4nr^3 \sin^2 \alpha \cos \alpha$$
$$+\frac{4nr^3}{3} \tan \alpha \, (2-3 \sin \alpha + \sin^3 \alpha)$$
$$+\frac{8nr^3}{3} (2-3 \cos \alpha + \cos^3 \alpha).$$

Surprisingly enough this reduces to the simple result

$$\frac{8nr^3}{3} [2-\cos \alpha - \sec \alpha + \tan \alpha]. \quad .. \quad .. \quad .. \quad (8)$$

Note: (1) As $n \to \infty$, and $\alpha \to 0$, the expression reduces to $4\pi r^3/3$, which is the volume of the inscribed sphere. (2) If we put $n = 2$, we get the expression reduced to $8r^3(2-\sqrt{2})$, which is the volume common to three equal circular cylinders whose axes concur orthogonally.

The Case (C)

In this case $\alpha_1, \alpha_2, \alpha_3 \ldots \alpha_n$ are not equal and we will not get the prism X, as when they were equal.

The best way is to consider the parts of the common volume which correspond to ΔOP_1Q_1, ΔOQ_1R_1, ... separately.

The result (8) indicates that the volume corresponding to each of the $2n$ triangles must be

$$\frac{4r^3}{3}(2-\cos\alpha-\sec\alpha+\tan\alpha).$$

$\therefore$ When $\alpha_1, \alpha_2, \alpha_3 \ldots$ are distinct from each other, the volume corresponding to the triangular base of $2\alpha_r$ is

$$\frac{4r^3}{3}(2-\cos\alpha_r-\sec\alpha_r+\tan\alpha_r).$$

$\therefore$ The total volume is given by

$$\frac{8r^3}{3}\sum_{r=1}^{r=n}[2-\cos\alpha_r-\sec\alpha_r+\tan\alpha_r]. \qquad \ldots \quad \ldots \quad \ldots \quad (9)$$

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