STEADY MHD COUETTE FLOW BETWEEN TWO CONDUCTING WALLS OF AN ELECTRICALLY CONDUCTING, VISCOUS, INCOMPRESSIBLE RAREFIED GAS

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An investigation of the combined influence of magnetic field and electrically conducting walls on laminar, steady MHD Couette flow of an electrically conducting, incompressible, viscous, rarefied gas is carried out, in slip-flow regime under first order slip boundary conditions. It is observed that the flow field is completely affected by the rarefaction of the gaseous medium. Velocity profiles, magnetic field profiles, current density, rate of mass flow and viscous drag are presented graphically under different conditions and the numerical values of the magnetic drag coefficient, velocity-gradient, slip-velocity and centre line velocity are entered in tables. Some important conclusions are presented.

1. Introduction

The study of low-density or rarefied gases has been receiving attention during recent years. The flow of such slightly rarefied gases is studied under the slip-flow boundary conditions at the bounding walls with the momentum equations of continuum mechanics. The details of this kind of flow are described by Schaaf and Chambre (1961).

The study of electrically conducting, viscous incompressible, rarefied gas flowing between two stationary non-conducting walls, under the action of transversely applied magnetic field, was recently initiated by Inman (1965). Therein, he showed that the velocity profiles, slip-velocity, shear stress, centre-line velocity, the rate of mass flow are generally affected by the rarefaction of the gas, but the current distribution and, therefore, magnetic field, is not affected by the rarefaction of the gas. Also it was observed that velocity gradient at the upper wall was not affected by the magnetic field and the rarefaction of the gaseous medium. This study was extended by Soundalgekar (1967*a*, *b*) to the cases of (1) MHD Couette flow between non-conducting walls under the combined action of electric and magnetic fields and (2) channel flow between conducting walls under transversely applied magnetic field. In the case of Couette flow, it was observed by Soundalgekar that the velocity profiles, velocity-gradient at the upper wall, current distribution, slip-velocity and the shearing stress are affected by the electric field, magnetic field and the rarefaction of the gaseous medium. But in the case of channel flow of a rarefied gas between conducting walls, under a transverse magnetic field, the velocity profiles, slip-velocity, centre-line velocity and rate of mass flow are affected by both the magnetic field and the sum of the electrical conductance ratios of the walls, but the velocity-gradient at the upper wall, magnetic field and shearing stress are not affected by the rarefaction of the gaseous medium, but are affected only by the magnetic field and the sum of the wall electrical conductance ratios.

The object of this paper is to study the steady MHD Couette flow of a rarefied incompressible gas under a transversely applied magnetic field between two conducting walls in relative motion. The fluid properties are assumed constant. This problem was first studied between two non-conducting walls, under no-slip boundary conditions by Lehnert (1952) and between conducting walls by Yen and Chang (1964).

In § 2, closed form solutions of the problem are derived. Velocity profiles, magnetic field profiles, current distribution, viscous drag coefficient, rate of mass flow are shown graphically and the numerical values for magnetic drag coefficients are entered in Table I, whereas those for velocity-gradient and slip-velocity at the upper wall and centre-line velocity are entered in Table II. Some interesting conclusions are presented in § 3.

2. Governing Equations and Solutions

The flow of an incompressible, electrically conducting, viscous, rarefied gas is assumed between two conducting walls infinite in extent in both x_1- and z_1-directions. Hence all the physical variables are functions of y_1 only, except p, the pressure, which in plane Couette flow is such that $\partial p/\partial x_1 = 0$. Under these assumptions, the fully developed flow is governed by

$$\mu \frac{d^2 u_1}{dy_1^2} + \frac{\mu_c H_0}{4\pi} \frac{dH_x}{dy_1} = 0 \qquad (1)$$

$$H_0 \frac{du_1}{dy_1} + \eta \frac{d^2 H_x}{dy_1^2} = 0 \qquad (2)$$

where

$$\eta = 1/4\pi\mu_c\sigma$$

is the magnetic diffusivity and μ, μ_c, u_1, H_x, H_0 have their usual meaning given them earlier (Soundalgekar 1967*a*).

Introducing the following dimensionless quantities

$$y = y_1/L, \quad H = H_x/H_0, \quad u = u_1/U \qquad (3)$$

in eqns. (1) and (2), we get

$$R_m \frac{d^2 u}{dy^2} + M^2 \frac{dH}{dy} = 0 \qquad (4)$$

$$R_m \frac{du}{dy} + \frac{d^2 H}{dy^2} = 0 \qquad (5)$$

where U is the uniform velocity of the walls, L is the half-width of the channel, and M, R_m are defined as follows:

$$M = \mu_c H_0 L(\sigma/\mu)^{\frac{1}{2}} \text{ is Hartmann number}$$

$$R_m = 4\pi\mu_c \sigma LU \text{ is magnetic Reynolds number.}$$

Note that Gaussian system of units has been adopted here.

The dimensionless form of slip-flow boundary condition that permits a slip-velocity $\bar{u}_s$ at the walls ($y = \pm 1$) is written as (Schaaf and Chambre 1961; Inman 1965)

$$u(\pm 1) = \bar{u}_s = \mp 1 \mp \lambda \left(\frac{du}{dy}\right)_{y=\pm 1} \quad \ldots \quad \ldots \quad \ldots \quad (6)$$

where $\bar{u}_s = u_s/U$, u_s being the dimensional slip-velocity at the walls. The rarefaction parameter λ is defined as

$$\lambda = \xi_u/L$$

where the slip coefficient ξ_u is given by the expression (1),

$$\xi_u = \left[\frac{2-\beta}{\beta}\right]\bar{l}$$

where $\bar{l}$ is the mean free path given by

$$\bar{l} = (\sqrt{\pi/8}/0{\cdot}499)\mu(\sqrt{RT}/p) \text{ (Schaaf and Chambre 1961).}$$

Here β is termed Maxwell's reflection coefficient and R is the gas constant.

The dimensionless MHD boundary conditions as given by Shercliff (1956) and Yen and Chang (1964) are written as

$$\left.\begin{aligned} \frac{dH}{dy} + \frac{1}{\phi_u} H = 0 \text{ at } y = +1 \\ \frac{dH}{dy} - \frac{1}{\phi_l} H = 0 \text{ at } y = -1 \end{aligned}\right\} \quad \ldots \quad \ldots \quad \ldots \quad (7)$$

where the subscripts u and l stand respectively for the upper and lower walls.

The parameters ϕ_u, ϕ_l are called the 'electrical conductance ratios' and are defined as

$$\phi_u = \frac{\sigma_{w,u} l_u}{\sigma_f L} \quad \phi_l = \frac{\sigma_{w,l} l_l}{\sigma_f L}$$

where $\sigma_{w,u}$, $\sigma_{w,l}$ and σ_f are respectively the electrical conductivities of the upper wall, lower wall and the fluid, and l_u, l_l are the thicknesses of the upper and the lower walls respectively. It may be noted here that the variation in ϕ does not imply a change in the conductivity ratio $\sigma_{w,u}/\sigma_f$ (or $\sigma_{w,l}/\sigma_f$) but implies change in thickness ratio l_u/L (or l_l/L).

The solutions of eqns. (4) and (5), subject to the boundary conditions (6) and (7), are

$$u = \frac{\Phi \cosh My - \sinh My}{\sinh M(1+\lambda M \coth M)} - \frac{\Phi(\coth M + \lambda M)}{(1+\lambda M \coth M)} \quad \ldots \quad \ldots \quad \ldots \quad (8)$$

$$H = -\frac{R_m}{M \sinh M(1+\lambda M \coth M)}[\Phi M(\sinh M - \phi_u M \cosh M - \sinh M) - \cosh My + \cosh M + \phi_u M \sinh M] \quad \ldots \quad (9)$$

in terms of ϕ_u, and

$$H = -\frac{R_m}{M \sinh M(1+\lambda M \coth M)}[\Phi M(\sinh My+\phi_l M \cosh M+\sinh M) \\ -\cosh My+\cosh M+\phi_l M \sinh M] \quad \ldots \quad (10)$$

where

$$\Phi = \frac{\phi_u-\phi_l}{[(\phi_u+\phi_l)M \coth M+2]}.$$

Hence the induced current is given

$$j = \frac{j_z}{4\pi\mu_c H_0 U\sigma} = \frac{1}{R_m}\frac{dH}{dy} \\ = \frac{\sinh My-\Phi M \cosh My}{\sinh M(1+\lambda M \coth M)}. \quad \ldots \quad (11)$$

The velocity profiles, magnetic field profiles and distribution of induced current are plotted in Figs. 1–7.

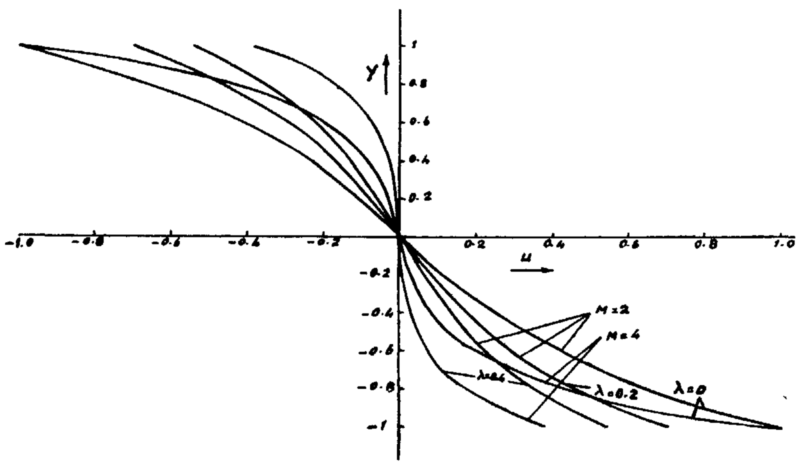

FIG. 1. Velocity profiles for $\phi_u = \phi_l = 0$; $\lambda = 0, 0{\cdot}2, 0{\cdot}4$; $M = 2, 4$.

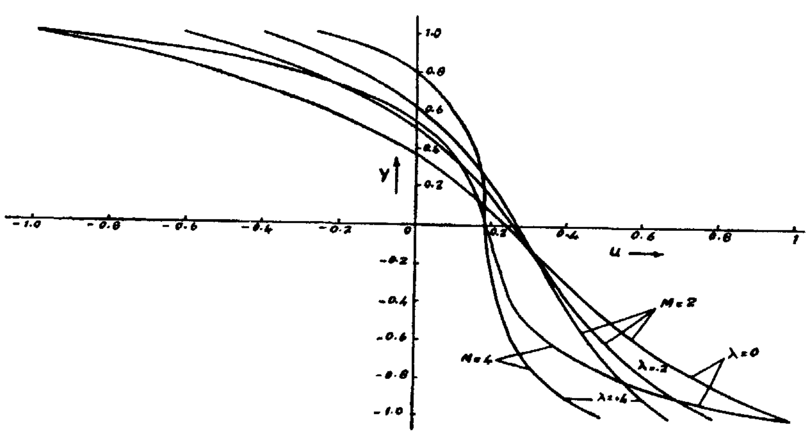

FIG. 2. Velocity profiles for $\phi_u = 0$, $\phi_l = 2$; $\lambda = 0, 0{\cdot}2, 0{\cdot}4$; $M = 2, 4$.

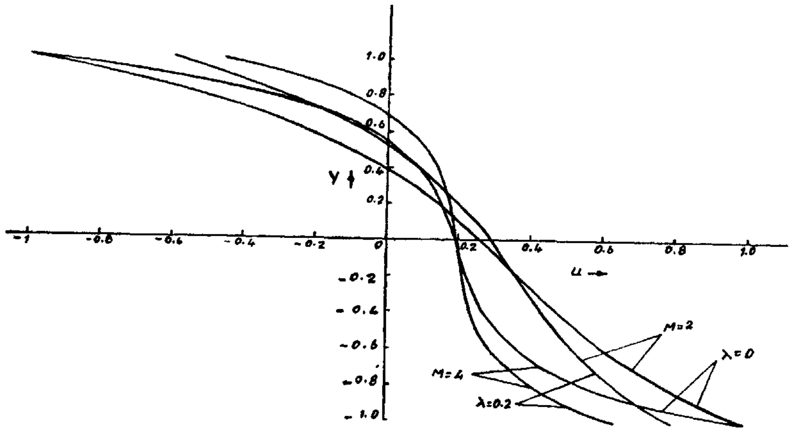

Fig. 3. Velocity profiles for $\phi_u = 0{\cdot}2$, $\phi_l = 4$, $\lambda = 0$, $0{\cdot}2 M = 2$, 4.

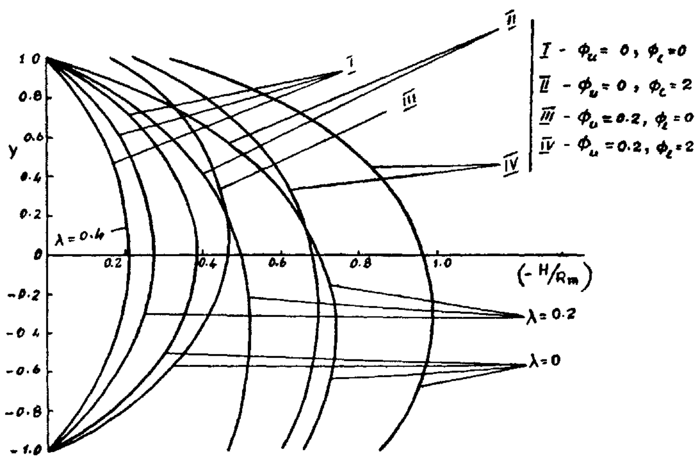

Fig. 4. Magnetic field profiles $M = 2$, $\lambda = 0$, $0{\cdot}2$, $0{\cdot}4$.

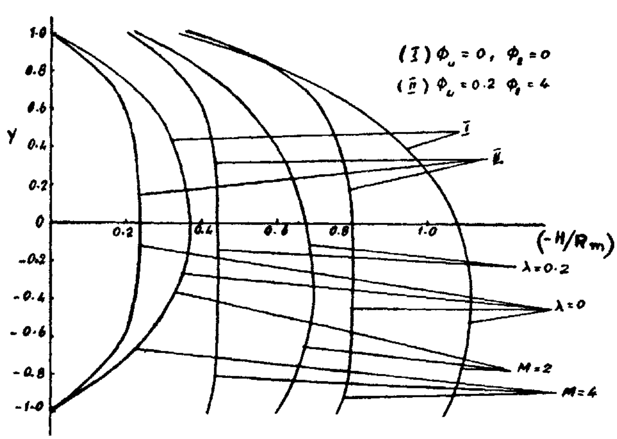

Fig. 5. Variation of magnetic field profiles with $M = 2$, 4; $\lambda = 0$, $0{\cdot}2$.

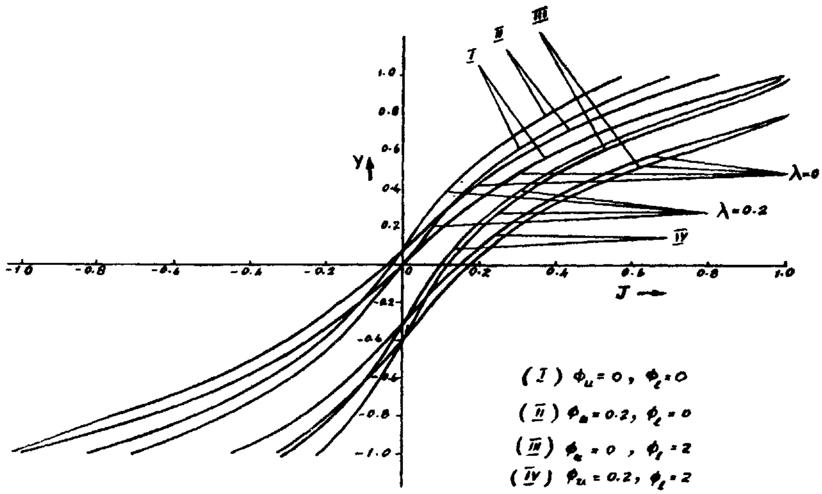

FIG. 6. Induced current $M = 2, \lambda = 0, 0{\cdot}2$.

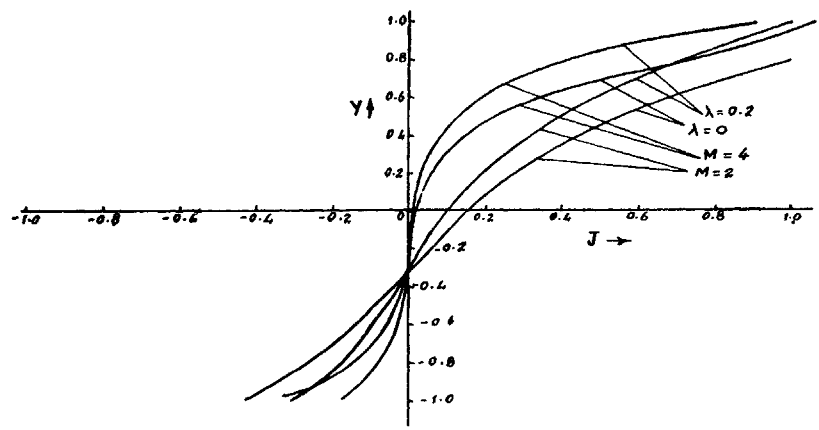

FIG. 7. Induced current $M = 2, 4\,;\ \phi_u = 0{\cdot}2,\ \phi_l = 2\,;\ \lambda = 0, 0{\cdot}2$.

The mass flow rate is given by

$$Q = \frac{Q_1}{2\rho UL}\int_{-1}^{1} u\,dy$$

$$= \frac{\Phi\left[\dfrac{1}{M} - (\coth M + \lambda M)\right]}{(1+\lambda M \coth M)} \qquad \ldots \quad \ldots \quad \ldots \quad (12)$$

Q is plotted in Figs. 8 and 9.

The viscous drag is defined as

$$\tau_v^* = -\mu\left(\frac{du_1}{dy_1}\right)_{y_1 = \pm L}$$

which in non-dimensional form is

$$R\tau = -\left(\frac{du}{dy}\right)_{y=\pm 1} \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (13)$$

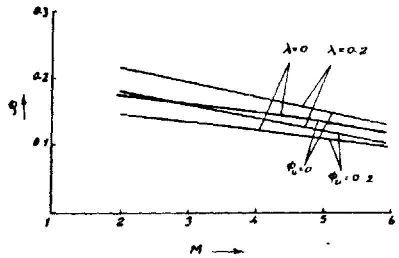

FIG. 8. Mass flow rate $\phi_l = 2$.

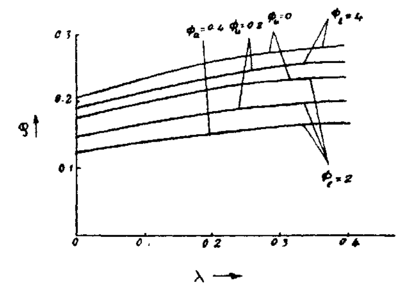

FIG. 9. Mass flow rate $M = 2$.

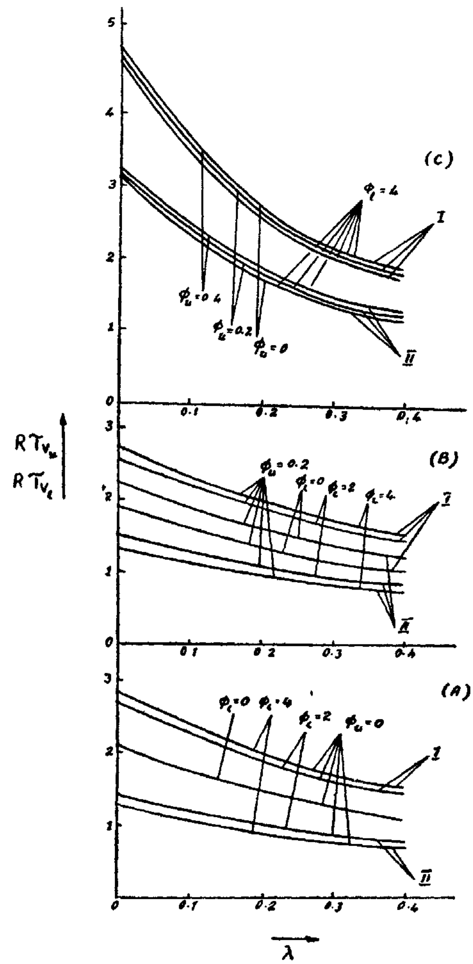

FIG. 10. Viscous drag coefficient. $M = 2$, I—$R\tau_{v_u}$; II—$R\tau_{v_l}$.

Hence from (8) and (13) we get

$$R\tau_{v_u} = M\left[\frac{\coth M - \Phi}{1+\lambda M \coth M}\right] \quad \ldots \quad \ldots \quad \ldots \quad (14)$$

and

$$R\tau_{v_l} = M\left[\frac{\coth M + \Phi}{1+\lambda \coth M}\right]. \quad \ldots \quad \ldots \quad \ldots \quad (15)$$

These are shown in Figs. 10 and 11.

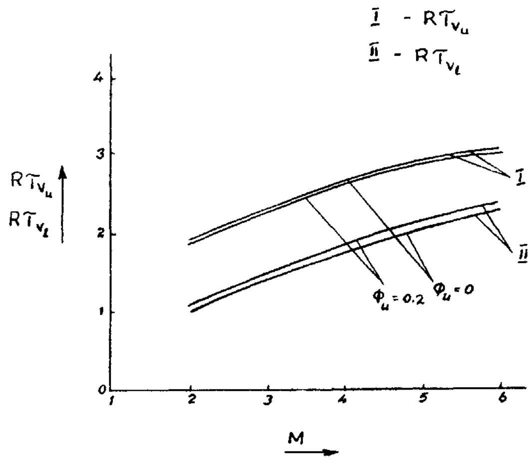

FIG. 11. Viscous drag coefficient. $\lambda = 0{\cdot}2$; $\phi_l = 2$.

The magnetic drag coefficient is given by

$$\tau_m^* = -\frac{\mu_c H_0 (H_x)_w}{4\pi\rho\nu(UL^{-1})}.$$

Hence in non-dimensional form, it is given by

$$\tau_m = -\frac{\mu_c H_0^2 L}{4\pi\rho\nu u}(H)_w. \quad \ldots \quad \ldots \quad \ldots \quad (16)$$

Hence, from (9), (10) and (16) we get

$$\tau_{m_u} = \frac{\phi_u M^2[1-\Phi M \coth M]}{1+\lambda M \coth M} \quad \ldots \quad \ldots \quad \ldots \quad (17)$$

$$\tau_{m_l} = \frac{\phi_l M^2[1+\Phi M \coth M]}{1+\lambda M \coth M}. \quad \ldots \quad \ldots \quad \ldots \quad (18)$$

The numerical values of τ_{m_u}, τ_{m_l} are entered in Table I.

The velocity gradient at the upper wall is given by

$$\left(\frac{du}{dy}\right)_{y=1} = \frac{M(\Phi - \coth M)}{1+\lambda M \coth M}. \quad \ldots \quad \ldots \quad \ldots \quad (19)$$

TABLE I

Values of magnetic drag coefficients

ϕ_u	ϕ_l	$\frac{M}{\lambda}$	τ_{mu} 2	τ_{mu} 4	τ_{ml} 2	τ_{ml} 4
0	0	0				
		0·2				
		0·4				
0	2	0			2·6019	6·3965
		0·2			1·8389	3·5525
		0·4			1·4219	2·4592
0	4	0			3·1074	7·1068
		0·2			2·1960	3·9470
		0·4			1·6980	2·7322
0·2	0	0	0·6625	2·2852		
		0·2	0·4682	1·2692		
		0·4	0·3620	0·8785		
0·2	2	0	1·2551	5·3336	3·4488	10·6640
		0·2	0·8870	2·9622	2·4374	5·9226
		0·4	0·6859	2·0505	1·8847	4·0998
0·2	4	0	1·3886	5·7874	4·2262	12·2516
		0·2	0·9814	3·2142	2·9869	6·8044
		0·4	0·7589	2·2250	2·3096	4·7102
0·4	0	0	1·1308	3·5545		
		0·2	0·7991	1·9741		
		0·4	0·6179	1·8665		
0·4	2	0	2·3609	9·9314	4·1950	14·3428
		0·2	1·6686	5·5158	2·9648	7·9658
		0·4	1·2902	3·8182	2·2925	5·5141
0·4	4	0	2·6738	11·1024	5·2618	16·9764
		0·2	1·8897	6·1661	3·7187	9·4285
		0·4	1·4612	4·2683	2·8755	6·5266

The slip-velocity at the upper wall is

$$\bar{u}_s = M\lambda\left[\frac{\coth M - \Phi}{1+\lambda M \coth M}\right]. \qquad \ldots \quad \ldots \quad \ldots \quad (20)$$

The centre-line velocity is given by

$$u_c = \frac{\Phi[1-\sinh M(\coth M+\lambda M)]}{\sinh M(1+\lambda M \coth M)}. \qquad \ldots \quad \ldots \quad (21)$$

The numerical values of $\left(\frac{du}{dy}\right)_{y=1}$, $\bar{u}_s$ u_c are entered in Table II for $M = 2$.

TABLE II

Values of $\left(\frac{du}{dy}\right)_{y=1}$, $\bar{u}_s$, u_c

ϕ_u	ϕ_l	$\frac{M}{\lambda}$	$\left(\frac{du}{dy}\right)_{y=1}$ 2	$\bar{u}_s$ 2	u_c 2
0	0	0	−2·0746	0	0
		0·2	−1·4662	0·2932	
		0·4	−1·1337	0·4535	
0	2	0	−2·7251	0	0·2477
		0·2	−1·9259	0·3851	0·2670
		0·4	−1·4892	0·5957	0·2775
0	4	0	−2·8514	0	0·2958
		0·2	−2·0152	0·4030	0·3188
		0·4	−1·5582	0·6233	0·3314
0·2	0	0	−1·9089	0	−0·0630
		0·2	−1·3491	0·2698	−0·0679
		0·4	−1·0432	0·4173	−0·0706
0·2	2	0	−2·6223	0	0·2088
		0·2	−1·8538	0·3707	0·2251
		0·4	−1·4334	0·5733	0·2340
0·2	4	0	−2·7840	0	0·2701
		0·2	−1·9676	0·3935	0·2911
		0·4	−1·5214	0·6085	0·3026
0·4	0	0	−1·7919	0	−0·1076
		0·2	−1·2664	0·2532	−0·1160
		0·4	−0·9792	0·3917	−0·1206
0·4	2	0	−2·5331	0	0·1746
		0·2	−1·7903	0·3580	0·1882
		0·4	−1·3843	0·5537	0·1956
0·4	4	0			
		0·2			
		0·4			

3. CONCLUSIONS

1. In general, the viscous drag at both the walls decreases as the value of the rarefaction parameter λ is on the increase, but increases with M, the Hartmann number.

From Table I, we have the following conclusions:

2. In general, the magnetic drag coefficient decreases with the increase in the rarefaction parameter λ.

3. It increases when the value of M is increased, for all values of λ.

From Table II, we have the following conclusions:

4. The value of the velocity-gradient at the upper wall, whether conducting or non-conducting, decreases as the value of λ is increased. Hence the force necessary to move this wall is less, when λ increases.

5. For the upper wall conducting or non-conducting the force necessary to move it is more as the value of the electrical conductance ratio ϕ_l increases.

6. For the same value of ϕ_l and in the presence or absence of the rarefaction of the medium, the force necessary to move the upper wall is less as the value of the electrical conductance ratio ϕ_u increases.

7. The slip-velocity at the wall increases with λ.

8. The slip-velocity also increases as ϕ_l is on the increase.

9. For the same value of ϕ_l, the value of $\bar{u}_s$ is more at the non-conducting wall than that at the conducting wall.

10. Also for the same value of ϕ_l, $\bar{u}_s$ decreases as ϕ_u is on the increase.

11. The centre-line velocity increases as λ and ϕ_l are on the increase.

12. For the same value of λ and ϕ_l, the centre-line velocity decreases as ϕ_u is on the increase.

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