SIMILAR SOLUTIONS OF FREE CONVECTION LAMINAR POWER LAW FLUIDS FLOW

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Possible similar solutions of unsteady free convection laminar power law fluids flow past a porous vertical wall have been discussed by group transformations. It is found that if the wall temperature and suction velocity vary as $\bar{t}^{\frac{1}{2}}$, then the similarity solution is possible. For the wall temperature varying as exp (pt) and suction at the wall constant, the similar solution does not exist (in case of power law fluids flow), p being a constant.

NOTATIONS

$\bar{x}$, $\bar{y}$ = distances parallel and perpendicular to the wall
$\bar{u}$, $\bar{v}$ = velocity components in x-, y-directions
$\bar{t}$ = time
g = acceleration due to gravity
β = coefficient of volume expansion
ν = coefficient of kinematic viscosity
ρ = density
$\bar{p}_0$ = pressure
$\bar{T}$, $\bar{T}_1$ and $\bar{T}_\infty$ = temperature and equilibrium temperature respectively
k = thermal diffusivity
P_r = Prandtl number
L = reference length
y, t, u, v, T, p_0 = dimensionless quantities for $\bar{y}$, $\bar{t}$, $\bar{u}$, $\bar{v}$, $\bar{T}$, $\bar{p}_0$
$h = \frac{\nu^{n-1}}{L^{2(n-1)}}$ = constant
n = flow behaviour index with $n = 1$ for Newtonian fluids flow
$p = \beta_2/\beta_1$ = constant
$-v_0$ = constant, suction velocity at plate
$\eta = \frac{y}{\sqrt{t}}$, a parameter
F, θ = functions of η
f_1, f_2 and G = functions of t and y respectively
q, N, τ_0 = rate of heat transfer, Nusselt number, skin friction respectively

$$\left.\begin{array}{l}\alpha_1, \alpha_2, \alpha_3 \ldots, \beta_1, \beta_2, \ldots, \\ a_1, a_2, a_3 \ldots, b_1, b_2, b_3 \ldots, \\ A, b\end{array}\right\} = \text{certain constants}$$

Introduction

Recently Na (1964) has used the group theoretical methods to find the similar solutions of the flow of power law fluids near an accelerating plate. Two groups of transformations have been used and similar solutions are deduced. Possible similarity solutions of unsteady free convection flow past a vertical plate with suction has been discussed by Nanda and Sharma (1962) in the case of Newtonian fluids flow. Two possible cases have been found by them.

In this paper, the method of Na (1964) has been extended to the problem of Nanda and Sharma (1962) if the fluids flow be non-Newtonian. It has been found that, in this case, only one class of similar solution is possible. It has been shown (Nanda and Sharma 1962) that when the surface temperature varies as some power of t, the suction velocity varies as $\bar{t}^{\frac{1}{2}}$. It has been shown in the present study for the power law fluids flow that for similarity purposes both the wall temperature and suction velocity must vary as $\bar{t}^{\frac{1}{2}}$. However, if the wall temperature varies as exp (pt), similar solution is not possible, where p is a constant.

Basic Equations

If we consider the $\bar{x}$-axis vertical along the plate and $\bar{y}$-axis perpendicular to it, the basic equations which describe the unsteady free convection laminar flow of power law fluids are

$$\frac{\partial \bar{u}}{\partial \bar{t}}+\bar{v}\frac{\partial \bar{u}}{\partial \bar{y}} = g\beta(\bar{T}-\bar{T}_\infty)+\nu\frac{\partial}{\partial \bar{y}}\left\{\left|\frac{\partial \bar{u}}{\partial \bar{y}}\right|^{n-1}\frac{\partial \bar{u}}{\partial \bar{y}}\right\} \quad \ldots \quad \ldots \quad (1)$$

$$\frac{\partial \bar{v}}{\partial \bar{t}} = -\frac{1}{\rho}\frac{\partial \bar{p}_0}{\partial \bar{y}} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

$$\frac{\partial \bar{T}}{\partial \bar{t}}+\bar{v}\frac{\partial \bar{T}}{\partial \bar{y}} = k\frac{\partial^2 \bar{T}}{\partial \bar{y}^2} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (3)$$

$$\frac{\partial \bar{v}}{\partial \bar{y}} = 0. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (4)$$

We introduce non-dimensional quantities

$$\left.\begin{array}{l} y = \bar{y}/L,\ t = \nu\bar{t}/L^2,\ u = \bar{u}L/\nu,\ v = \bar{v}L/\nu \\ T = g\beta L^3(\bar{T}-\bar{T}_\infty)/\nu^2,\ p_0 = \bar{p}_0 L^2/\rho\nu^2. \end{array}\right\} \quad \ldots \quad \ldots \quad (5)$$

Equations from (1) to (4) are then reduced to

$$\frac{\partial u}{\partial t}+v\frac{\partial u}{\partial y}=T+h\frac{\partial}{\partial y}\left\{\left|\frac{\partial u}{\partial y}\right|^n\right\} \qquad (6)$$

$$\frac{\partial v}{\partial t}=-\frac{\partial p_0}{\partial y} \qquad (7)$$

$$\frac{\partial T}{\partial t}+v\frac{\partial T}{\partial y}=\frac{1}{P_r}\frac{\partial^2 T}{\partial y^2} \qquad (8)$$

$$\frac{\partial v}{\partial y}=0 \qquad (9)$$

where P_r is the Prandtl number. Using the boundary conditions for u, we find that $\frac{\partial u}{\partial y}$ is positive inside the boundary layer and thus we have substituted

$$h=\frac{\nu^{n-1}}{L^{2(n-1)}}=\text{constant.}$$

The boundary conditions are

$$\left.\begin{array}{l} y\geqslant 0,\quad t<0: u=v=T=0 \\ y=0,\quad t\geqslant 0: u=0,\ v=-v_0 f_1(t),\ T=f_2(t) \\ y=\infty, t\geqslant 0: u=0,\ T=0 \end{array}\right\} \qquad (10)$$

where $f_i(t)$ $(i=1, 2, \ldots)$ is a function of t.

In a thin boundary layer, the pressure gradient at the wall is small. Thus, from eqns. (7) and (9), we take for v, as varying with time or a constant quantity.

SIMILARITY SOLUTIONS

Following Na (1964), we introduce two group transformations to find the similar solutions.

(1) *Linear Group*

Substituting one parameter group of transformations

$$t=A^{\alpha_1}\bar{t},\ u=A^{\alpha_2}\bar{u},\ v=A^{\alpha_3}\bar{v},\ y=A^{\alpha_4}\bar{y},\ T=A^{\alpha_5}\bar{T}_1 \qquad (11)$$

into eqns. (6) and (8), we have the following relations between α_i, $i=1$, 2, 3, 4, 5, for the invariance purposes

$$\left.\begin{array}{l} \alpha_2-\alpha_1=\alpha_2+\alpha_3-\alpha_4=\alpha_5=n(\alpha_2-\alpha_4)-\alpha_4 \\ \alpha_5-\alpha_1=\alpha_5+\alpha_3-\alpha_4=\alpha_5-2\alpha_4 \end{array}\right\} \qquad (12)$$

where $\alpha_1, \alpha_2, \ldots$ and A are certain constants.
Thus for $n\neq 1$ and $n=1$, we have

$$\alpha_2=\frac{\alpha_1}{2},\ \alpha_3=-\frac{\alpha_1}{2},\ \alpha_4=\frac{\alpha_1}{2},\ \alpha_5=-\frac{\alpha_1}{2} \qquad (13)$$

and

$$\left.\begin{array}{l} \dfrac{y}{\sqrt{t}} = \dfrac{\bar{y}}{\sqrt{\bar{t}}},\ \dfrac{u}{\sqrt{t}} = \dfrac{\bar{u}}{\sqrt{\bar{t}}} \\ vt^{\frac{1}{2}} = \bar{v}\bar{t}^{\frac{1}{2}},\ Tt^{\frac{1}{2}} = \bar{T}_1\bar{t}^{\frac{1}{2}}. \end{array}\right\} \qquad \ldots \quad (14)$$

Thus the absolute invariants for this group of transformations are

$$\left.\begin{array}{l} \eta = \dfrac{y}{\sqrt{t}} \\ u = F(\eta)t^{\frac{1}{2}} \\ v = -v_0 t^{-\frac{1}{2}} \\ T = \theta(\eta)t^{-\frac{1}{2}} \end{array}\right] \qquad \ldots \quad (15)$$

Using equations of set (15) into eqns. (6) and (8), we have

$$hnF'^{(n-1)}F''+F'\left(\frac{\eta}{2}+v_0\right)-\frac{1}{2}F+\theta = 0 \qquad \ldots \quad (16)$$

$$\frac{1}{P_r}\theta''+\theta'\left(\frac{\eta}{2}+v_0\right)+\theta = 0. \qquad \ldots \quad (17)$$

The boundary conditions for $t \geqslant 0$, give

$$\left.\begin{array}{l} \eta = 0: \quad F = 0,\ \theta = 1 \\ \eta = \infty: F = 0,\ \theta = 0 \end{array}\right\} \qquad \ldots \quad (18)$$

Equation (17) is independent of n and its solution can be easily obtained either in parabolic cylindrical function or in series. The constant α of Nanda and Sharma (1962) in our case is $-\frac{1}{2}$. Nanda and Sharma (1962) have solved above equations in terms of parabolic cylindrical functions as well in series for large α. For $n \neq 1$, it would be difficult to solve eqn. (16) in series due to first term. However, if n is large, the first term in eqn. (16) is dropped as $F' \ll 1$ and the approximate solution is obtained, relaxing one of the boundary conditions.

We solve eqns. (16) and (17) in series in terms of η at $n = 1$. If $P_r = 1$, we have on solving eqn. (17),

$$\theta = \phi(\eta)y_1 \qquad \ldots \quad (19)$$

where

$$y_1 = \exp\left\{-\frac{\eta^2}{8}-\frac{v_0\eta}{2}\right\}$$

and $\phi(\eta)$ satisfies the equation

$$\frac{d^2\phi}{d\eta^2}+\left[\left(\frac{3}{4}-\frac{v_0^2}{4}\right)-\frac{v_0\eta}{4}-\frac{\eta^2}{16}\right]\phi = 0. \qquad \ldots \quad (20)$$

To solve eqn. (20) in series, satisfying the boundary conditions, we assume

$$\phi = 1+a_1\eta+a_2\eta^2+a_3\eta^3+a_4\eta^4+. \qquad \ldots \quad (21)$$

Substituting (21) into (20) and comparing the various powers of η, we have the solution in terms of a_1, as

$$\phi = 1+a_1\left[\eta+\frac{B}{6}\eta^3+\frac{v_0}{48}\eta^4+\left(\frac{1}{320}+\frac{B^2}{120}\right)\eta^5+\ldots\right]$$

$$+\frac{B\eta}{2}+\frac{v_0\eta^3}{24}+\left(\frac{B^2}{24}+\frac{1}{192}\right)\eta^4+\left(\frac{v_0}{480}+\frac{v_0B}{160}\right)\eta^5+\ldots \quad .. \quad (22)$$

where we put, $B = \frac{v_0^2}{4}-\frac{3}{4}$ = constant, for convenience of calculations.

The rate of heat transfer from the wall to the fluid is

$$q = -k\left(\frac{\partial\bar{T}}{\partial\bar{y}}\right)_{y=0} = \frac{k\nu^2}{g\beta L^4}t^{-1}\left(\frac{v_0}{2}-a_1\right) \quad .. \quad .. \quad (23)$$

and the Nusselt number can be expressed as

$$N = \frac{Lq}{k(\bar{T}_w-\bar{T}_\infty)} = \bar{t}^{\frac{1}{2}}\left(\frac{v_0}{2}-a_1\right). \quad .. \quad .. \quad .. \quad (24)$$

Thus the undetermined constant a_1 in eqn. (22) is known in terms of the rate of heat transfer from the wall to the fluid and the Nusselt number.

It would be difficult to solve eqn. (16) for general value of n. However, if $n = 1$, let

$$F = (b_1\eta+b_2\eta^2+b_3\eta^3+b_4\eta^4+\ldots)\exp\left(-\frac{\eta^2}{8}-\frac{v_0\eta}{2}\right). \quad .. \quad (25)$$

Substituting the series (19) and (25) into eqn. (16) and comparing the various powers of η to get the constants b_i, we have the series solution as

$$F = \left[b_1\left(\eta+\frac{\eta^3}{8}+\frac{v_0^2\eta^3}{24}+\ldots\right)-\frac{a_1\eta^3}{6}-\frac{B}{12}\eta^3-\frac{\eta^2}{2}+\ldots\right]\exp\left\{-\frac{\eta^2}{8}-\frac{v_0\eta}{2}\right\}. \quad .. \quad (26)$$

The skin friction at the wall τ_0 can be easily seen directly proportional to the constant b_1, in terms of which other constants b_2, b_3, ... are determined.

(2) *Spiral Group*

Another one parameter group of transformations is chosen in the form

$$t = \bar{t}+\beta_1 b,\ u = e^{\beta_2 b}\bar{u},\ v = e^{\beta_3 b}\bar{v},\ y = e^{\beta_4 b}\bar{y},\ T = e^{\beta_5 b}\bar{T}_1 \quad .. \quad (27)$$

where β_1, β_2, ... and b are certain constants. Substituting equations from set (27) into eqns. (6) and (8) and equating the various powers of e^b on both the sides, we have for invariance purpose

$$\beta_2 = \beta_2+\beta_3-\beta_4 = \beta_5 = n(\beta_2-\beta_4)-\beta_4 \quad .. \quad .. \quad (28)$$

$$\beta_5 = \beta_5+\beta_3-\beta_4 = \beta_5-2\beta_4. \quad .. \quad .. \quad .. \quad .. \quad (29)$$

Thus for $n = 1$, i.e. Newtonian fluids flow, we have

$$\beta_5 = \beta_2,\ \beta_3 = \beta_4 = 0. \quad .. \quad .. \quad .. \quad (30)$$

Hence the following relations are established

$$\frac{u}{e^{pt}} = \frac{\bar{u}}{e^{p\bar{t}}}, \ \frac{T}{e^{pt}} = \frac{\bar{T}_1}{e^{p\bar{t}}}, \ v = \bar{v}, \ y = \bar{y} \quad \ldots \quad \ldots \quad \ldots \quad (31)$$

where $p = \beta_2/\beta_1 =$ constant.

Thus the absolute invariants are found to be

$$u = F(y)e^{pt}, \ v = -v_0 \text{ (const.)}, \ T = G(y)e^{pt}. \quad \ldots \quad \ldots \quad (32)$$

The suction velocity at the wall is constant and the wall temperature varies as e^{pt} which was concluded by Nanda and Sharma (1962) and several papers have appeared for this class of transformations to Newtonian fluid flow.

For $n \neq 1$, we have from eqns. (28) and (29)

$$\beta_2 = \beta_3 = \beta_4 = \beta_5 = 0.$$

Thus the determination of absolute invariants in this case is not possible and the similar solution does not exist.

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