ON THE RE-ENTRY OF VARIABLE AND CONSTANT GEOMETRY HEAVY BALLISTIC MISSILES

by Ashis Kumar Pal, *Department of Mathematics, Jadavpur University, Calcutta 32*

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A theory of re-entry of ballistic missiles under certain conditions into the earth's atmosphere together with a detailed study of the phenomena connected with the re-entry of a light ballistic missile was given by Angelo Miele (1959). In the present paper the phenomena connected with the re-entry of a heavy missile have been studied by the same method. The graphical and numerical results were compared with those for a light missile.

Assumptions

The basic assumptions made in the present note are:

(i) The re-entering missile is regarded as a particle of constant mass.

(ii) The earth is flat and non-rotating and the acceleration due to gravity is constant.

(iii) The missile flies with zero angle of attack so that the total drag is identical with the zero-lift drag. The curvature of the trajectory is disregarded and the flight path is approximated by a straight line.

(iv) The atmosphere is isothermal so that both the density and the pressure are exponential functions of the altitude.

The missile is constructed with a variable geometry apparatus so that the zero-lift drag can be controlled in flight.

Introduction

The equations of motion as obtained by Miele (1959) are

$$\left.\begin{aligned} \dot{\Pi} - r\pi M &= 0 \\ \dot{M} - K + \epsilon\pi M^2 &= 0 \end{aligned}\right\} \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

where

$$\Pi = \frac{P}{P_0}$$

$$= e^{-r\eta} \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

P and P_0 are the pressures at any altitude and at the sea-level.

$$\left.\begin{aligned} r &= \frac{C_p}{C_v} \text{ (ratio of specific heats)} \\ \eta &= \frac{hg}{a^2} \end{aligned}\right\} \quad \ldots \quad \ldots \quad \ldots \quad (3)$$

where a is the speed of sound and M is the Mach number, K is a non-dimensional constant ($K = 1$ in case of a heavy missile and $K = 0$ for a light missile). The ballistic parameter

$$\epsilon = -\frac{rP_0SC_{D_0}}{2mg \sin\theta}. \quad \ldots \quad \ldots \quad \ldots \quad (4)$$

The dot sign in eqn. (1) denotes derivatives with respect to the non-dimensional time τ
where

$$\tau = -t\,.\frac{g}{a}\sin\theta. \quad \ldots \quad \ldots \quad \ldots \quad (5)$$

Assuming the ballistic factor $\epsilon = \epsilon(\pi)$ the relationship between the Mach number and the pressure ratio as obtained by Miele (1959) is

$$\frac{M}{M_i} = \left(\frac{1+Q}{P}\right)^{\frac{1}{2}} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (6)$$

where

$$\left.\begin{aligned} P &= \exp\left[\frac{2}{r}\int_{\pi_i}^{\pi} \epsilon\, d\pi\right] \\ Q &= \frac{2K}{rM_i^2}\int_{\pi_i}^{\pi} \exp\left[\frac{2}{r}\int_{\pi_i}^{\pi} \epsilon\, d\pi\right]\frac{d\pi}{\pi} \end{aligned}\right\} \quad \ldots \quad \ldots \quad \ldots \quad (7)$$

and M_i is the initial Mach number at the point of re-entry of the ballistic missile.

The relationship between the instantaneous deceleration α and the pressure ratio Π is also obtained by Miele (1959),

$$\alpha = \epsilon\pi\left[\frac{1+Q}{P}\right] - \frac{K}{M_i^2} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (8)$$

where

$$\alpha = -\frac{\dot{M}}{M_i^2}$$

$$= \frac{1}{M_i^2}\frac{dv/dt}{g\sin\theta}. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (9)$$

Heavy Missile Flying with Constant Ballistic Factor

In this case $K = 1$, $\epsilon =$ constant, P and Q are as follows:

$$\left.\begin{aligned} P &= \exp(z-z_i) \\ Q &= \frac{2}{rM_i^2 \exp(z_i)}\,[E_i(z)-E_i(z_i)] \end{aligned}\right\} \quad \cdots \quad \cdots \quad (10)$$

where

$$z = \frac{2\epsilon\pi}{r}$$

and

$$\left.\begin{aligned} E_i(z) &= \int_{-\infty}^{z} \frac{e^t}{t}\,dt \\ &= r_E + \log z + \sum_{n=1}^{\infty} \frac{z^n}{n\,\underline{|n}} \end{aligned}\right\} \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad (11)$$

(Jhanke-Emde-Losch 1960).

The relationship between the Mach number and z for both the light ($K = 0$) and heavy ($K = 1$) missiles obtained by Miele (1959) is:

$$\frac{M}{M_i} = \exp\left(-\frac{z}{2}\right), \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad (12)$$

$$\frac{M}{M_i} = \left[\frac{\exp(z_i) + \frac{2}{rM_i^2}\{E_i(z)-E_i(z_i)\}}{\exp(z)}\right]^{\frac{1}{2}}. \quad \cdots \quad \cdots \quad (13)$$

These relations are represented graphically for $z_i = 0{\cdot}01$, $M_i = 6$, $r = 1{\cdot}4$ by the two lowest curves in Fig. 1.

The deceleration for the heavy and light missiles as obtained by Miele (1959) is

$$\left.\begin{aligned} \alpha &= \frac{r}{2} z \frac{\exp(z_i) + \frac{2}{rM_i^2}\{E_i(z)-E_i(z_i)\}}{\exp(z)} - \frac{1}{M_i^2} \\ \alpha &= \frac{r}{2} \,.\, z \exp(-z) \end{aligned}\right\}. \quad \cdots \quad \cdots \quad (14)$$

For the same numerical data these relations are represented by the two lowest curves in Fig. 2.

The maximum deceleration in the case of a heavy missile occurs at the altitude where

$$\frac{2}{rM_i^2} \times \frac{\exp(z)}{\exp(z_i) + \frac{2}{rM_i^2}\{E_i(z)-E_i(z_i)\}} + (1-z) = 0. \quad \cdots \quad (15)$$

This is obtained by Miele (1959) from

$$\frac{d\alpha}{d\pi} = 0$$

where α is given by eqn. (8).

Since eqn. (15) cannot be solved analytically, the solution is obtained by numerical methods.

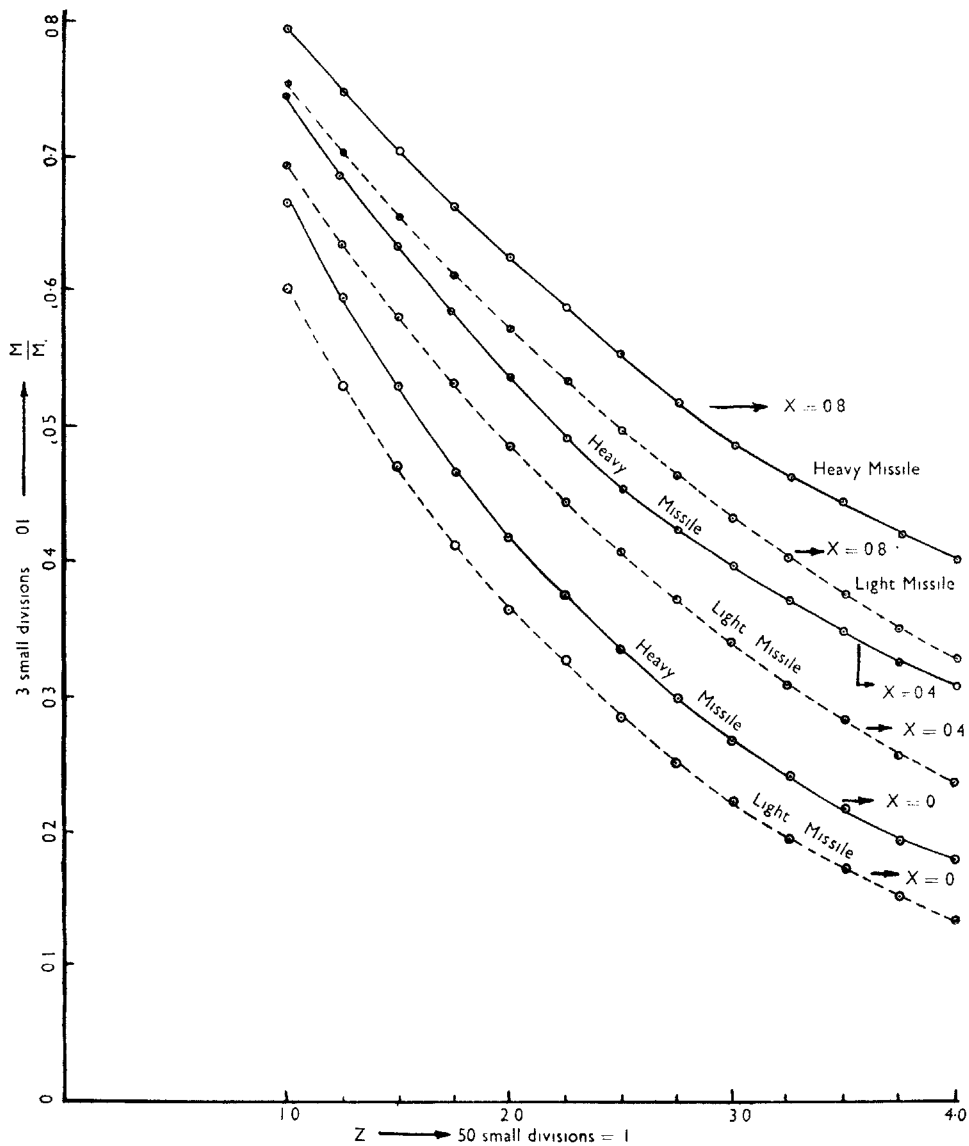

FIG. 1.

From eqns. (14) and (15), α for the heavy missile is found to be

$$\alpha = \frac{1}{(z-1)^2 M_i^2}. \qquad (16)$$

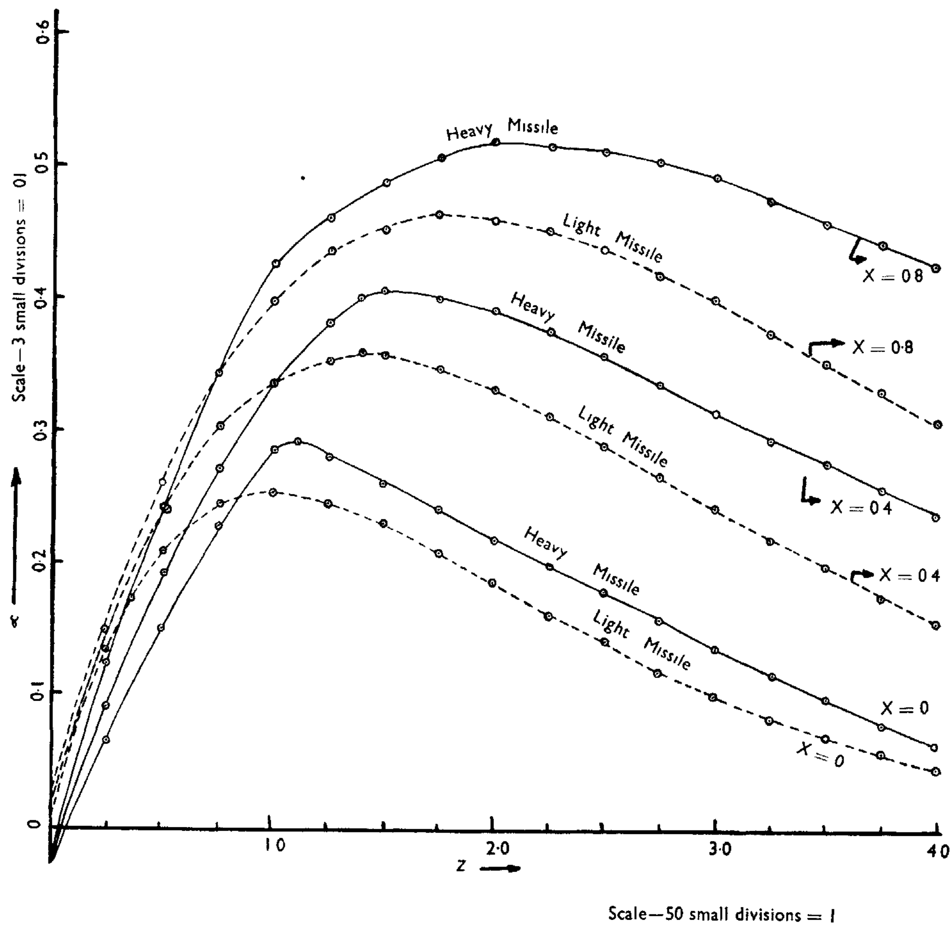

FIG. 2.

By numerical methods the value of z is obtained from eqn. (15) for which the deceleration α is maximum and z is found to be 1·08,

$$\alpha_{\max} = \frac{1}{(1{\cdot}08-1)^2} \times \frac{1}{M_i^2}$$

$$= 0{\cdot}3472. \qquad (17)$$

Miele (1959) has obtained that the maximum deceleration occurs at $z = 1$, in the case of a light missile. The value of α_{max} for a light missile is

$$\alpha_{\max} = \frac{r}{2e}$$

$$= 0{\cdot}257. \qquad \ldots \quad \ldots \quad \ldots \quad (18)$$

So from this it is evident that the maximum deceleration in the case of a light missile occurs at a higher altitude than in the case of a heavy missile and is also less in magnitude.

Heavy Ballistic Missile with Variable Ballistic Factor ϵ

In this case $K = 1$, $\epsilon = A\pi^x$ where A and x are constants, $x \neq -1$.

$$P = \exp\left[\frac{2}{r}\int_{\pi_i}^{\pi} A\pi^x \, d\pi\right]$$

$$= \exp\left[\frac{1}{x+1}(z-z_i)\right], \qquad \ldots \quad \ldots \quad \ldots \quad (19)$$

where

$$z = \frac{2\epsilon\pi}{r}$$

$$= \frac{2A\pi^{x+1}}{r}, \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (20)$$

$$Q = \frac{2}{rM_i^2}\int_{\pi_i}^{\pi} \exp\left[\frac{2}{r}\int_{\pi_i}^{\pi} A\pi^x \, d\pi\right]\frac{d\pi}{\pi}$$

$$= \frac{2}{rM_i^2} \times \frac{[E_i(z)-E_i(z_i)]}{(x+1)\exp\left(\dfrac{z_i}{x+1}\right)} \qquad \ldots \quad \ldots \quad \ldots \quad (21)$$

$$\frac{M}{M_i} = \left[\frac{\exp\left(\dfrac{z_i}{x+1}\right) + \dfrac{2}{rM_i^2(x+1)}\{E_i(z)-E_i(z_i)\}}{\exp\left(\dfrac{z}{x+1}\right)}\right]^{\frac{1}{2}} \qquad \ldots \quad \ldots \quad (22)$$

is obtained by substituting the values for P and Q in eqn. (6).
Miele (1959) has obtained

$$\frac{M}{M_i} = \exp\left[-\frac{z}{2(x+1)}\right] \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (23)$$

in the case of a light missile with variable ballistic factor [$\epsilon = A\pi^x$].
For the same numerical data as before eqns. (22) and (23) are graphically represented in Figs. 1 and 3 for several values of x, both positive and negative.

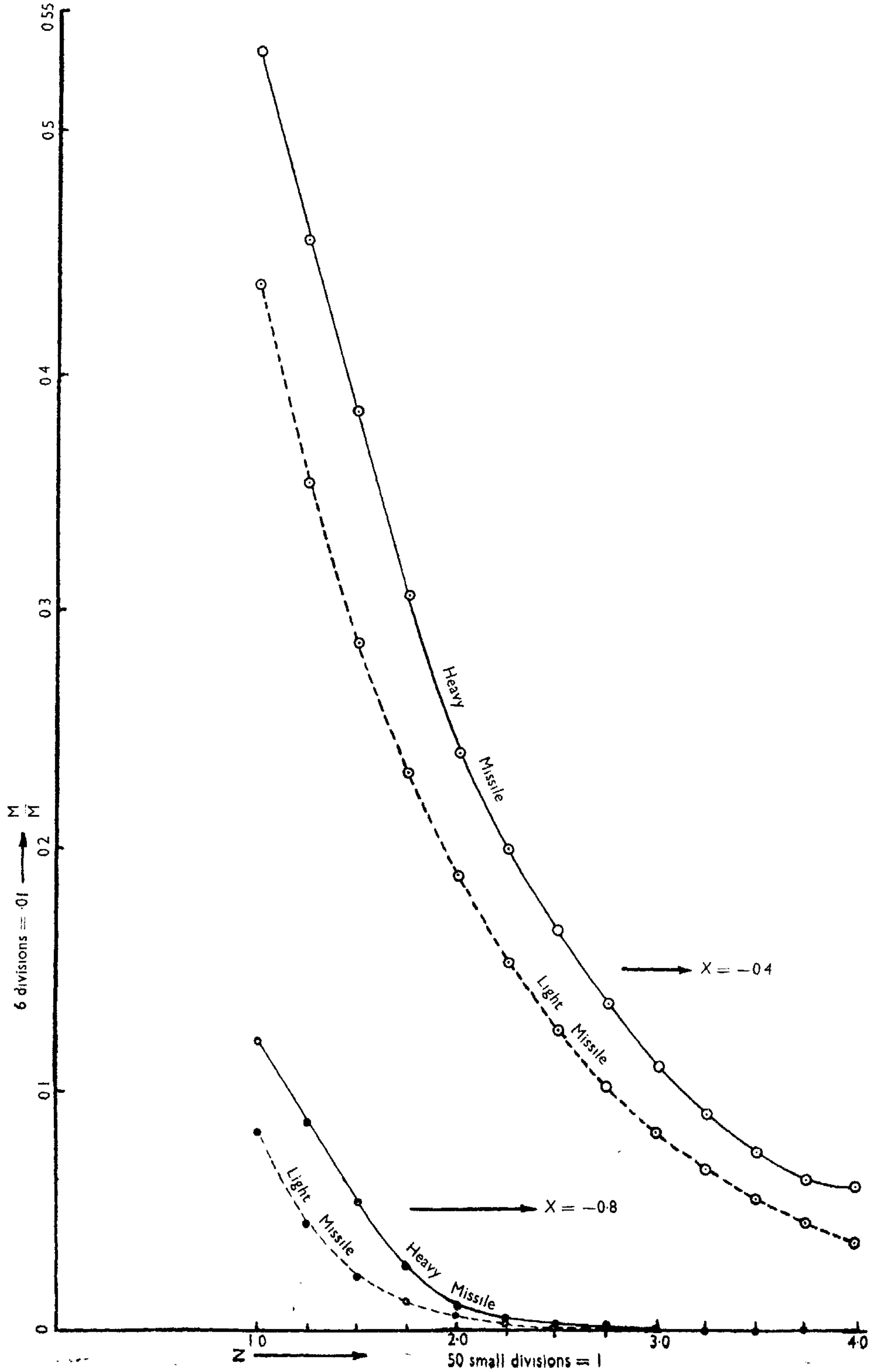
0.55
0.5
0.4
0.3
0.2
0.1
0
6 divisions = ·01 ⟶ M/M̄
Heavy Missile
Light Missile
X = −0·4
Light Missile
Heavy Missile
X = −0·8
1·0
2·0
3·0
4·0
Z ⟶
50 small divisions = 1

FIG. 3.

The deceleration α in the case of a heavy missile with variable ballistic factor is given by

$$\alpha = \frac{r}{2} \cdot z \frac{\exp\left(\dfrac{z_i}{x+1}\right) + \dfrac{2}{rM_i^2(x+1)}\{E_i(z) - E_i(z_i)\}}{\exp\left(\dfrac{z}{x+1}\right)} - \frac{1}{M_i^2}. \qquad .. \quad (24)$$

The corresponding relation for a light missile has already been obtained by Miele (1959)

$$\alpha = \frac{r}{2} \cdot z \exp\left[-\frac{z}{x+1}\right]. \qquad .. \quad .. \quad .. \quad (25)$$

The graphical representations of eqns. (24) and (25) are shown in Figs. 2 and 4 for several values of x, both positive and negative. The case of constant ballistic factor follows from these relations if zero is substituted for x.

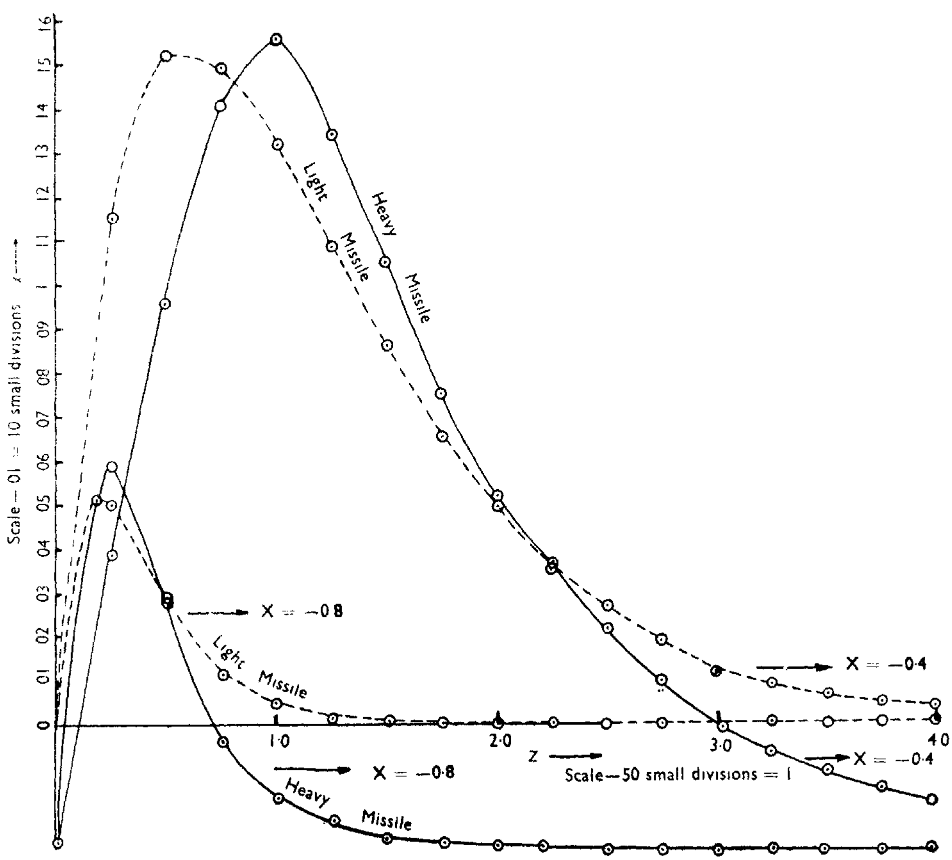

FIG. 4.

The maximum deceleration for heavy missile occurs at the altitude where

$$\frac{2}{rM_i^2} \times \frac{\exp\left(\frac{z}{x+1}\right)}{\exp\left(\frac{z_i}{x+1}\right) + \frac{2}{rM_i^2(x+1)}\{E_i(z) - E_i(z_i)\}} + (x+1) - z = 0. \qquad (26)$$

This is obtained from the equation

$$\frac{d\alpha}{d\pi} = 0$$

where α is given by eqn. (8) and P and Q are given by eqns. (19) and (21).

Since this equation cannot be solved analytically, the solution is obtained by numerical methods. From eqns. (24) and (26), we get

$$\alpha = \frac{1}{M_i^2} \times \frac{(x+1)}{[z-(x+1)]} . \qquad \ldots \quad \ldots \quad \ldots \quad (27)$$

For different values of x, the corresponding values of z, for which the deceleration α is maximum, are obtained by numerical solution of eqn. (26). Then substituting those values of z and the corresponding values of x in eqn. (27) the maximum decelerations are calculated and are as given below.

x	Light Missile		Heavy Missile	
	α_{max}	z	α_{max}	z
0·8	0·464	1·75	0·563	2
0·4	0·361	1·4	0·486	1·48
0	0·257	1	0·347	1·08
−0·4	0·154	0·5	0·157	1
−0·8	0·052	0·2	0·058	0·25

From analysis of the heavy and light ballistic missile performances, the following special features can be noticed for the heavy missile:

(i) For the first part of the re-entry trajectory a heavy missile decelerates more slowly than a light missile, but nearer the sea-level it decelerates more rapidly than a light missile.

(ii) For a heavy missile the maximum deceleration α_{max} is always greater than that in the case of a light missile. Moreover, the maximum deceleration occurs at a lower altitude than in the case of a light ballistic missile.

(iii) For different positive values of x there is a critical altitude in each case at which the deceleration α is the same for both light and heavy missiles.

(iv) For different negative values of x there exist two critical altitudes in each case at which the deceleration α is the same for both light and heavy missiles.

(v) For positive values of x the percentage of deceleration obtained is greater than the negative values in both the cases of light and heavy missiles.

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