HEAT TRANSFER DUE TO A SPHERE STEADILY ROTATING IN AN INFINITELY EXTENDING NON-NEWTONIAN FLUID

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The problem of heat transfer due to a uniformly or non-uniformly heated or insulated sphere steadily rotating in an infinitely extending non-Newtonian fluid characterized by the constitutive equations proposed by Oldroyd, Rivlin-Ericksen and Walters has been investigated. It is observed that the effect of the non-Newtonian parameter m is to increase the temperature in comparison to that for the Newtonian fluids up to a certain critical m_c of m above which it starts decreasing with further increase of m and even becomes less than the temperature for the Newtonian fluids for some $m > m_c$. When the sphere is uniformly or non-uniformly heated, the temperature attains its maximum in a region close to the boundary due to maximum viscous dissipation in this region. Even when sphere is insulated the effect of non-Newtonian parameter on the temperature distribution is similar to the one as stated above except that now the maximum temperature is attained on the boundary as it should be.

1. Introduction

Recently Banks (1965) has studied the heat transfer due to a steadily rotating sphere in an infinitely extending incompressible Newtonian viscous fluid using the boundary layer approximation for the equations of momentum and energy without taking into account the dissipation by viscosity. The usual perturbation method has been used to solve these equations taking the rotation of the sphere to be small and neglecting the dissipation terms in the solution of the energy equation for the two types of thermal boundary conditions, namely the surface of the sphere is either maintained at a given temperature or it is heat-insulated.

The aim of the present investigation is to study the problem of heat transfer in non-Newtonian fluids due to the steady rotation of a uniformly or non-uniformly heated sphere and to see as to how the various non-Newtonian parameters affect the temperature distribution in the flow field. We have studied the effect of secondary flow on the temperature distribution taking into account all the dissipation effects. The problem of heat transfer when

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suction is imposed on the boundary of the sphere in any arbitrary direction will be taken up in a separate paper.

In accordance with the theorem proved by Bhatnagar (1967) we include in our present discussion all the classes of Oldroyd (1950), Rivlin and Ericksen (1955) and Walters (1962) fluids. The equations determining the primary motion, secondary motion and the zeroth and first order temperature fields are identical for the above-mentioned fluids but with the proper interpretation of non-Newtonian parameters in each case. However, the nature of the secondary flow field with or without arbitrary suction and injection for Rivlin-Ericksen fluids has already been discussed by Bhatnagar (1964). We note that for Rivlin-Ericksen fluids the stress components differ for the case $(2\phi_2+\phi_3) \neq 0$, ϕ_2 and ϕ_3 being the coefficients of visco-elasticity and cross-viscosity respectively. Similar results have been obtained for sphere-sphere and cone-cone geometries by Bhatnagar *et al.* (*in press*).

2. Formulation of the Problem and Solution for the Velocity Field

Consider a sphere of radius a rotating with a constant angular velocity Ω about the axis $\theta = 0$ in an infinitely extending non-Newtonian fluid which is otherwise at rest. Choose a spherical polar coordinate system (r, θ, ϕ) with origin at the centre of the sphere and the polar angle θ and azimuthal angle ϕ being measured from the axis of rotation and some convenient meridian plane respectively.

Let u, v and w represent the components of velocity in the increasing directions of r, θ, ϕ respectively. The boundary conditions determining the velocity field can be written as

$$u = 0,\ v = 0,\ w = a\Omega \sin\theta \text{ on } r = a\,;\ u = 0,\ v = 0,\ w = 0, \text{ as } r \to \infty. \qquad (2.1)$$

The constitutive equation for a class of elastico-viscous fluids has been given by Oldroyd in the form

$$S_{ik} = p_{ik} - pg_{ik} \qquad (2.2)$$

$$E_{ik} = \tfrac{1}{2}(u_{i,k}+u_{k,i}) \qquad (2.3)$$

$$p^{ik}+\lambda_1 \frac{\delta p^{ik}}{\delta t}+\mu_0 p^j_j E^{ik} = 2\eta_0\left(E^{ik}+\lambda_2 \frac{\delta E^{ik}}{\delta t}\right) \qquad (2.4)$$

$$\frac{\delta p^{ik}}{\delta t} = \frac{\partial p^{ik}}{\partial t}+u^j p^{ik}_{,j}+\Omega^i_{.m} p^{mk}+\Omega^k_{.m} p^{im}-E^i_m p^{mk}-E^k_m p^{im} \qquad (2.5)$$

$$\Omega_{ik} = \tfrac{1}{2}(u_{k,i}-u_{i,k}) \qquad (2.6)$$

where p_{ik} is the stress tensor and S_{ik} is the deviatoric stress tensor related to the change of shape of the material element, g_{ik} the metric tensor, E_{ik} the rate of strain tensor, p an isotropic pressure and u_i the velocity vector; a suffix following a comma represents the covariant derivative. η_0 is treated as a constant having dimensions of viscosity and the constant parameters λ_1, λ_2, μ_0 have dimensions of time.

The constitutive equations for Rivlin-Ericksen and Walters fluids have been recorded in respective references (Bhatnagar 1964 and Bhatnagar *et al.* 1967).

We render all the physical and dynamical quantities dimensionless with the help of radius a as the characteristic length, $a\Omega$ as the characteristic velocity so that the isotropic pressure is given by $\eta_0\Omega p$ and any stress component by $\eta_0\Omega p_{ik}$.

The equations of continuity and momentum for steady motion in a general curvilinear coordinate system can be written as

$$g^{ii}u_{i,i} = 0 \qquad \ldots \quad (2.7)$$

$$Ru^j u_{i,j} = -p_{,i} + g^{ii}p_{ij,j} \qquad \ldots \quad (2.8)$$

where $R = a^2\Omega\rho/\eta_0$ represents the Reynolds number for the flow.

The boundary conditions (2.1) in dimensionless form reduce to

$$\left.\begin{array}{lll} u = 0, & v = 0, & w = \sin\theta \text{ on } r = 1 \\ u = 0, & v = 0, & w = 0 \quad \text{as } r \to \infty \end{array}\right\}. \qquad \ldots \quad (2.9)$$

We solve the velocity field approximately for the case when R is small so that it can be treated as a small perturbation parameter. The velocity components, stress components and the isotropic pressure can then be expressed in terms of power series in R in the form:

$$\left.\begin{aligned} u &= Ru_1(r, \theta) + R^2u_2(r, \theta) + \ldots \\ v &= Rv_1(r, \theta) + R^2v_2(r, \theta) + \ldots \\ w &= w_0(r)\sin\theta + R^2w_2(r, \theta) + \ldots \\ p_{rr} &= Rf_1(r, \theta) + R^2f_2(r, \theta) + \ldots \\ p_{\theta\theta} &= Rg_1(r, \theta) + R^2g_2(r, \theta) + \ldots \\ p_{\phi\phi} &= Rh_1(r, \theta) + R^2h_2(r, \theta) + \ldots \\ p_{r\theta} &= Rl_1(r, \theta) + R^2l_2(r, \theta) + \ldots \\ p_{r\phi} &= m_0(r, \theta) + R^2m_2(r, \theta) + \ldots \\ p_{\theta\phi} &= R^2n_2(r, \theta) + \ldots \\ p &= N_0 + RN_1(r, \theta) + R^2N_2(r, \theta) + \ldots \end{aligned}\right\} \qquad \ldots \quad (2.10)$$

Substituting these expressions in eqns. (2.2)–(2.8) and separating the various order terms in R, the zeroth and the first order equations are given by

$$m_0 = r\sin\theta \frac{d}{dr}\left(\frac{w_0}{r}\right) \qquad \ldots \quad (2.11)$$

$$o = \frac{1}{r^3}\frac{\partial}{\partial r}(r^3 m_0) \qquad \ldots \quad (2.12)$$

$$N_0 = \text{constant} \qquad \ldots \quad (2.13)$$

$$f_1 = 2\frac{\partial u_1}{\partial r} \qquad \ldots \quad (2.14)$$

$$g_1 = \frac{2}{r}\left(u_1+\frac{\partial v_1}{\partial \theta}\right) \qquad \ldots \quad (2.15)$$

$$h_1 = \frac{2}{r}(u_1+v_1 \cot\theta)+2mr^2 \sin^2\theta \left\{\frac{d}{dr}\left(\frac{w_0}{r}\right)\right\}^2 \qquad \ldots \quad (2.16)$$

$$l_1 = \frac{1}{r}\frac{\partial u_1}{\partial \theta}+r\frac{\partial}{\partial r}\left(\frac{v_1}{r}\right) \qquad \ldots \quad (2.17)$$

$$-\frac{w_0^2 \sin^2\theta}{r} = -\frac{\partial N_1}{\partial r}+\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 f_1)+\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(l_1\sin\theta)-\frac{g_1+h_1}{r} \qquad \ldots \quad (2.18)$$

$$-\frac{w_0^2 \cos\theta\sin\theta}{r} = -\frac{1}{r}\frac{\partial N_1}{\partial \theta}+\frac{1}{r^3}\frac{\partial}{\partial r}(r^3 l_1)+\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(g_1\sin\theta)-\frac{h_1\cot\theta}{r} \qquad (2.19)$$

$$\frac{\partial}{\partial r}(r^2 u_1\sin\theta)+\frac{\partial}{\partial\theta}(rv_1\sin\theta) = 0 \qquad \ldots \quad (2.20)$$

where

$$m' = mR = (\lambda_1-\lambda_2)\Omega \quad \text{for Oldroyd fluids}$$

$$= \frac{\phi_3\Omega}{2\eta_0} \quad \text{for Rivlin-Ericksen fluids with } (2\phi_2+\phi_3) = 0$$

$$= \frac{\int_0^\infty \tau N(\tau)\,d\tau}{\int_0^\infty N(\tau)\,d\tau}\,\Omega \quad \text{for Walters fluids.} \qquad \ldots \quad (2.21)$$

If $(2\phi_2+\phi_3) \neq 0$, then

$$f_1 = 2\frac{\partial u_1}{\partial r}+\frac{(2\phi_2+\phi_3)}{a^2\rho}\left\{r\frac{d}{dr}\left(\frac{w_0}{r}\right)\right\}^2. \qquad \ldots \quad (2.22)$$

(a) Primary Motion

Eliminating m_0 from eqns. (2.11) and (2.12) the zeroth order flow, which for the sake of simplicity we call the primary motion, is given by

$$r^2\frac{d^2 w_0}{dr^2}+2r\frac{dw_0}{dr}-2w_0 = 0 \qquad \ldots \quad (2.23)$$

which on using the boundary conditions (2.9) gives

$$w = w_0(r)\sin\theta = \frac{\sin\theta}{r^2}. \qquad \ldots \quad (2.24)$$

The streamlines in the primary motion are concentric circles with centres lying on the axis of rotation.

(b) Secondary Motion

The secondary motion is represented by eqns. (2.14)–(2.20) and the boundary conditions satisfied by u_1, v_1 are

$$\left.\begin{aligned} u_1 &= 0, \quad v_1 = 0 \quad \text{on } r = 1 \\ u_1 &= 0, \quad v_1 = 0 \quad \text{as } r \to \infty \end{aligned}\right\}. \qquad \text{(2.25)}$$

Introducing the stream function $\psi(r, \theta)$ for the secondary flow through

$$u_1 = -\frac{1}{r^2 \sin\theta}\frac{\partial\psi}{\partial\theta}, \quad v_1 = \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r} \qquad \text{(2.26)}$$

and eliminating f_1, g_1, h_1, l_1 and N_1 from eqns. (2.14)–(2.19), the equation determining $\psi(r, \theta)$ has the form

$$\left[\frac{\partial^2}{\partial r^2} + \frac{\sin^2\theta}{r^2}\left(\frac{\partial}{\partial\cos\theta}\right)^2\right]^2 \psi = \left[\frac{6}{r^5} - \frac{144m}{r^7}\right]\sin^2\theta\cos\theta. \qquad \text{(2.27)}$$

If $(2\phi_2+\phi_3) \neq 0$ in the case of Rivlin-Ericksen fluids then

$$m = \frac{1}{a^2\rho}(\phi_2+\phi_3). \qquad \text{(2.28)}$$

The boundary conditions satisfied by ψ are

$$\left.\begin{aligned} \frac{\partial\psi}{\partial r} &= 0, \; \frac{\partial\psi}{\partial\theta} = 0 \quad \text{on} \quad r = 1 \\ \frac{\partial\psi}{\partial r} &= 0, \; \frac{\partial\psi}{\partial\theta} = 0 \quad \text{as} \quad r \to \infty \end{aligned}\right\}. \qquad \text{(2.29)}$$

Solving eqn. (2.27) with the help of boundary conditions (2.29), we get

$$\psi(r, \theta) = \frac{(r-1)^2}{8r^2}\left[1 - 4m\left(1+\frac{2}{r}\right)\right]\sin^2\theta\cos\theta. \qquad \text{(2.30)}$$

We note that for $0 < \theta < \frac{\pi}{2}$,

$\psi \gtrless 0$ according as

$$r \gtrless \frac{8m}{1-4m}. \qquad \text{(2.31)}$$

Also $\psi = 0$ if

$$m = \frac{r}{4(r+2)}. \qquad \text{(2.32)}$$

Therefore, the circle $r = r_0 = 8m/(1-4m)$ in the meridian plane divides the flow field into two distinct regions: In the inner region, the streamlines form closed loops whereas, in the outer region, they have the same patterns as the streamlines for a Newtonian fluid, i.e. the fluid is drawn in at the pole and thrown out at the equator. The inequality (2.31) is important from the point of breaking of the flow. If $r_0 < 1$, i.e. $m_c < 1/12$, there cannot be any breaking of the secondary flow. Similar results have been obtained by Bhatnagar (1964) for Rivlin-Ericksen fluids and by Thomas and Walters (1964) for Walters fluids.

When the non-Newtonian parameter m is sufficiently small ($m < 1/12$), the secondary flow in each quadrant resembles that of a Newtonian fluid. For still large values of m, the flow field in each quadrant is broken into two distinct domains as described above, the sense of flow in adjacent quadrants being opposite and in opposite quadrants being similar (Fig. 1, for $m = 0{\cdot}115$). For sufficiently larger values of m, r_0 increases and for a critical value of m, $r_0 \to \infty$ and then the flow in the entire field becomes similar to that of Newtonian fluid except the direction of flow being opposite.

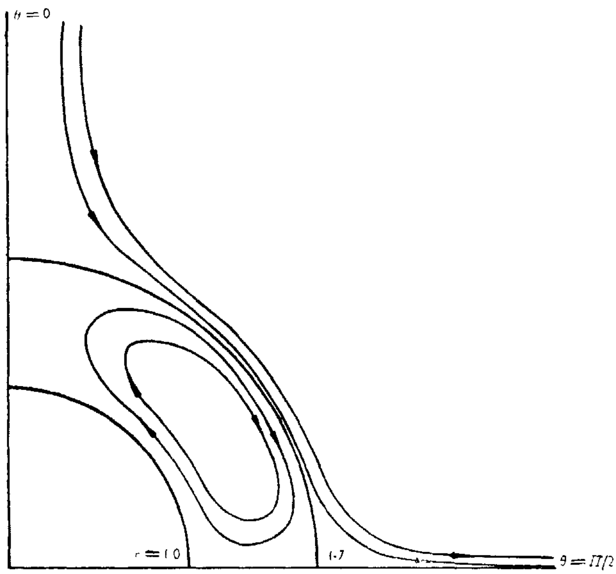

FIG. 1. Secondary flow for a non-Newtonian fluid $m = 0{\cdot}115$ depicting the separation in a meridian plane.

3. Solution of the Energy Equation

We shall now investigate the nature of the temperature field under various types of thermal boundary conditions to be given later.

In the case of steady motion, the energy equation in spherical polar coordinate system (r, θ, ϕ) has the form

$$R\left[u\frac{\partial T}{\partial r}+\frac{v}{r}\frac{\partial T}{\partial \theta}\right]=\frac{1}{\sigma}\nabla^2 T+\beta\left[\frac{\partial u}{\partial r}p_{rr}+\frac{1}{r}\left(u+\frac{\partial v}{\partial \theta}\right)p_{\theta\theta}\right.$$
$$+\frac{1}{r}(u+v\cot\theta)p_{\phi\phi}+\frac{1}{r}\left(\frac{\partial u}{\partial \theta}+r\frac{\partial v}{\partial r}-v\right)p_{r\theta}+\left(\frac{\partial w}{\partial r}-\frac{w}{r}\right)p_{r\phi}$$
$$\left.+\frac{1}{r}\left(\frac{\partial w}{\partial \theta}-w\cot\theta\right)p_{\theta\phi}\right] \quad \ldots \quad (3.1)$$

where

$$\beta = \frac{E}{R^2}, E = \frac{a^2\Omega^2}{c_p T_c} \text{ (Eckert number)}$$

$$\sigma = \frac{\eta_0 c_p}{R} \text{ (Prandtl number)}$$

and the dimensionless temperature $T = (T - T_\infty)/T_c$, T_c being constant temperature at the pole $\theta = 0$.

In terms of the dimensionless variables we consider two types of thermal boundary conditions, namely

(i)
$$\left.\begin{aligned} T &= 1 + \alpha \sin^2 \theta \text{ on } r = 1 \\ T &= 0 \qquad \text{as } r \to \infty \end{aligned}\right\} \quad \ldots \quad \ldots \quad \ldots \quad (3.2)$$

α being some positive constant, and

(ii)
$$\left.\begin{aligned} \frac{\partial T}{\partial r} &= 0 \quad \text{on } r = 1 \\ T &= 0 \quad \text{as } r \to \infty \end{aligned}\right\}. \quad \ldots \quad \ldots \quad \ldots \quad (3.3)$$

As for the velocity field, here also we take

$$T(r, \theta) = T_1(r, \theta) + R^2 T_2(r, \theta). \quad \ldots \quad \ldots \quad \ldots \quad (3.4)$$

Substituting (3.4), (2.10) in (3.1) and equating various order terms in R, the zeroth and first order temperature are determined by

$$\nabla^2 T_1 = -\frac{9\beta\sigma}{r^6} \sin^2 \theta \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (3.5)$$

and

$$\nabla^2 T_2 = \sigma \left[u_1 \frac{\partial T_1}{\partial r} + \frac{v_1}{r} \frac{\partial T_1}{\partial \theta} \right] - \beta\sigma \left[f_1 \frac{\partial u_1}{\partial r} + \frac{g_1}{r} \left(u_1 + \frac{\partial v_1}{\partial \theta} \right) + \frac{h_1}{r} (u_1 + v_1 \cot \theta) \right.$$
$$\left. + \frac{l_1}{r} \left(\frac{\partial u_1}{\partial \theta} + r \frac{\partial v_1}{\partial r} - v_1 \right) - \frac{3 \sin \theta}{r^3} \left\{ m_2 + r \frac{\partial}{\partial r} \left(\frac{w_2}{r} \right) \right\} \right] \quad \ldots \quad \ldots \quad (3.6)$$

where

$$\nabla^2 \equiv \frac{\partial^2}{\partial r^2} + \frac{2}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + \frac{\cot \theta}{r^2} \frac{\partial}{\partial \theta} \quad \ldots \quad \ldots \quad \ldots \quad (3.7)$$

and

$$m_2 = r \frac{\partial}{\partial r} \left(\frac{w_2}{r} \right) - m \left[\frac{12 u_1 \sin \theta}{r^4} + \frac{9 \sin \theta}{r^3} \frac{\partial u_1}{\partial r} \right]. \quad \ldots \quad (3.8)$$

The equation determining $w_2(r, \theta)$ is of the form

$$\frac{\partial^2 w_2}{\partial r^2} + \frac{\sin \theta}{r^2} \frac{\partial^2}{\partial \theta^2} (w_2/\sin \theta) + 2 \frac{\partial}{\partial r} \left(\frac{w_2}{r} \right) + \frac{3 \cos \theta}{r^2} \frac{\partial}{\partial \theta} (w_2/\sin \theta)$$
$$= \frac{1}{r^3} \left[2v_1 \cos \theta - u_1 \sin \theta + m \left[\sin \theta \left\{ 9 \frac{\partial^2 u_1}{\partial r^2} + \frac{12}{r} \frac{\partial u_1}{\partial r} - \frac{12 u_1}{r^2} + \frac{3}{r^2} \frac{\partial^2 u_1}{\partial \theta^2} \right. \right. \right.$$
$$\left. \left. \left. + 6 \frac{\partial^2}{\partial r \partial \theta} \left(\frac{v_1}{r} \right) \right\} + \frac{9 \cos \theta}{r} \left\{ \frac{1}{r} \frac{\partial u_1}{\partial \theta} + 2r \frac{\partial}{\partial r} \left(\frac{v_1}{r} \right) \right\} \right] \right]. \quad \ldots \quad \ldots \quad (3.9)$$

The boundary conditions satisfied by $w_2(r, \theta)$ are

$$w_2 = 0 \text{ on } r = 1 \text{ and } w_2 \to 0 \text{ as } r \to \infty. \quad \ldots \quad \ldots \quad (3.10)$$

Substituting for u_1, v_1 from (2.26) and (2.30) and solving (3.9) using the conditions (3.10), $w_2(r, \theta)$ is found to have the following form

$$w_2(r, \theta) = \left[-\frac{70}{33}\frac{m^2}{r^8}+\frac{1}{4r^7}(m+12m^2)-\frac{5}{14}\frac{m}{r^6}+\frac{1}{6r^5}\left(-\frac{1}{8}+\frac{3m}{2}-12m^2\right)\right.$$
$$+\frac{1}{r^4}\left(-\frac{69}{55}m^2-\frac{2m}{5}+\frac{33}{400}\right)+\frac{\log r}{35r^4}+\frac{1}{4r^3}\left(m-\frac{1}{4}\right)+\frac{1}{r^2}$$
$$\left.\times\left(\frac{1}{1200}+\frac{m}{140}-\frac{2m^2}{15}\right)\right]\sin\theta+\left[\frac{9m^2}{11r^8}-\frac{1}{2r^7}\left(\frac{m}{4}+3m^2\right)+\frac{m}{4r^6}+\frac{1}{8r^5}\right.$$
$$\left.\times\left(\frac{1}{8}-3m+18m^2\right)+\frac{1}{r^4}\left(-\frac{5}{64}+\frac{m}{2}-\frac{69}{44}m^2\right)-\frac{\log r}{28r^4}+\frac{1}{2r^3}\left(\frac{1}{8}-\frac{m}{2}\right)\right]\sin^3\theta. \quad (3.11)$$

We shall now discuss cases (i) and (ii) in turn.

Case (*i*)

(*a*) *Zeroth order solution.*—Equation (3.5) is to be solved under the following boundary conditions

$$\left.\begin{array}{ll} T_1 = 1+\alpha\sin^2\theta \text{ on } r=1 \\ T_1 = 0 \quad \text{as } r\to\infty \end{array}\right\}. \quad (3.12)$$

Solving (3.5) and using (3.12),

$$T_1(r, \theta) = \frac{1}{r}+\frac{\alpha(2r^2+1)}{3r^3}+\frac{\beta\sigma}{2r^4}(r^3+r-2)$$
$$-\cos^2\theta\left[\frac{\alpha}{r^3}+\frac{3\beta\sigma}{2r^4}(r-1)\right]. \quad (3.13)$$

(*b*) *First order solution.*—$T_2(r, \theta)$ satisfies the boundary conditions

$$\left.\begin{array}{l} T_2 = 0 \text{ on } r=1 \\ T_2 = 0 \text{ as } r\to\infty \end{array}\right\}. \quad (3.14)$$

Substituting for u_1, v_1, f_1, g_1, h_1, l_1, m_2, w_2 and T_1 from eqns. (2.26), (2.30), (2.14)-(2.17), (3.8), (3.11) and (3.13) respectively, and solving (3.6) with the help of the boundary conditions (3.14), we get

$$T_2(r, \theta) = \frac{C_1}{r}+\left[\frac{F_1}{12r^4}+\frac{F_2}{20r^5}+\frac{F_3}{30r^6}+\frac{F_4}{42r^7}+\frac{F_5}{56r^8}\right.$$
$$\left.+\frac{F_6}{72r^9}+\frac{F_7}{90r^{10}}\right]+\left[\frac{C_2}{r^3}+\frac{Q_1}{4r^2}+\frac{Q_2\log r}{5r^3}-\frac{Q_3}{6r^4}-\frac{Q_4}{14r^5}\right.$$
$$-\frac{Q_5}{24r^6}-\frac{\beta\sigma}{98r^6}\left(\log r+\frac{11}{24}\right)-\frac{Q_6}{36r^7}-\frac{Q_7}{50r^8}-\frac{Q_8}{66r^9}$$
$$\left.-\frac{Q_9}{84r^{10}}\right]P_2(\mu)+\left[\frac{C_3}{r^5}+\frac{M_1}{8r^4}-\frac{M_2\log r}{9r^5}+\frac{M_3}{10r^6}\right.$$
$$\left.+\frac{12\beta\sigma}{490r^6}\left(\log r+\frac{11}{10}\right)+\frac{M_4}{22r^7}+\frac{M_5}{36r^8}+\frac{M_6}{52r^9}+\frac{M_7}{70r^{10}}\right]P_4(\mu) \quad (3.15)$$

where $P_2(\mu)$ and $P_4(\mu)$ are the usual Legendre Polynomials and

$$C_1 = -\left[\frac{F_1}{12}+\frac{F_2}{20}+\frac{F_3}{30}+\frac{F_4}{42}+\frac{F_5}{56}+\frac{F_6}{72}+\frac{F_7}{90}\right] \quad \ldots \quad (3.16)$$

$$C_2 = -\frac{Q_1}{4}+\frac{Q_3}{6}+\frac{Q_4}{14}+\frac{Q_5}{24}+\frac{11\beta\sigma}{2352}+\frac{Q_6}{36}+\frac{Q_7}{50}+\frac{Q_8}{66}+\frac{Q_9}{84} \quad \ldots \quad (3.17)$$

$$C_3 = \frac{M_1}{8} - \left[\frac{M_3}{10}+\frac{132\beta\sigma}{4900}+\frac{M_4}{22}+\frac{M_5}{36}+\frac{M_6}{52}+\frac{M_7}{70}\right] \quad \ldots \quad (3.18)$$

$$\left.\begin{aligned}
F_1 &= \sigma\left[-\frac{\alpha}{10}-\frac{3\beta\sigma}{20}+\frac{m}{5}(2\alpha+3\beta\sigma)-\beta\left(\frac{47}{200}-\frac{17m}{14}+2m^2\right)\right]\\
F_2 &= \sigma\left[\frac{4\alpha}{15}+\frac{\beta\sigma}{5}(3-4m)-\beta\left(\frac{49m}{10}-\frac{24}{15}\right)\right]\\
F_3 &= \sigma\left[-\frac{\alpha}{6}-\frac{3\beta\sigma}{4}-m(2\alpha+3\beta\sigma)-\beta\left(\frac{217}{60}+8m-48m^2\right)\right]\\
F_4 &= \sigma\left[\frac{3\beta\sigma}{10}+m\left(\frac{8\alpha}{5}-\frac{4}{3}-6\beta\sigma\right)-\beta\left(\frac{192m^2}{5}-\frac{252m}{5}-\frac{17}{5}\right)\right]\\
F_5 &= \sigma\left[-\frac{14m\beta\sigma}{5}-\beta\left(-\frac{53}{6}+\frac{354m}{5}+168m^2\right)\right]\\
F_6 &= \frac{\beta\sigma}{5}[116m+1392m^2]\\
F_7 &= -\frac{588m^2\beta\sigma}{5}
\end{aligned}\right\} \quad \ldots \quad (3.19)$$

$$\left.\begin{aligned}
Q_1 &= \sigma\left[-\frac{1}{4}-\frac{\alpha}{6}+m\left(1+\frac{2\alpha}{3}\right)-\frac{\beta\sigma}{8}(1-4m)\right]\\
Q_2 &= \sigma\left[\frac{1}{2}+\frac{\alpha}{3}+\frac{\beta\sigma}{4}\right]\\
Q_3 &= \sigma\left[-\left(\frac{\alpha}{42}+\frac{1}{4}\right)-3m\left(1+\frac{6\alpha}{7}\right)+\frac{\beta\sigma}{56}(5-132m)\right.\\
&\qquad \left.-\beta\left\{-\frac{361}{1400}+\frac{78m}{35}-\frac{206m^2}{35}\right\}\right]\\
Q_4 &= \sigma\left[-\frac{\alpha}{3}(1-4m)+\frac{\beta\sigma}{7}(m-2)+2m-\beta\left\{\frac{25}{14}-\frac{50m}{7}\right\}\right]\\
Q_5 &= \sigma\left[\frac{4\alpha}{21}(1+12m)-\frac{\beta\sigma}{14}(1-48m)-\beta\left\{-\frac{2179}{588}-\frac{40m}{7}+\frac{3396m^2}{77}\right\}\right]\\
Q_6 &= \sigma\left[\frac{\beta\sigma}{7}(1-126m)-\frac{4m}{7}\left(3\alpha+\frac{10m}{3}\right)-\beta\left\{\frac{47}{14}+\frac{342m}{7}-\frac{240m^2}{7}\right\}\right]\\
Q_7 &= \sigma\left[-\frac{6m\beta\sigma}{7}+\beta\left\{\frac{235}{21}+\frac{354m}{7}+\frac{1200m^2}{7}\right\}\right]\\
Q_8 &= -\frac{\beta\sigma}{7}[158m+1896m^2]\\
Q_9 &= \frac{8040m^2\beta\sigma}{77}
\end{aligned}\right\} \quad (3.20)$$

$$\left.\begin{aligned}
M_1 &= \sigma\left[-\frac{9\alpha}{35}(1-4m)-\frac{27\beta\sigma}{70}(1-4m)-\frac{\beta}{35}(9-72m+144m^2)\right]\\
M_2 &= \sigma\left[\frac{2\alpha}{5}+\frac{3\beta\sigma}{35}(13-24m)-\frac{\beta}{35}(-39+156m)\right]\\
M_3 &= \sigma\left[-\frac{\alpha}{7}(1+12m)-\frac{3\beta\sigma}{14}(5+12m)-\beta\left(\frac{2379}{980}+\frac{72m}{7}-\frac{3996m^2}{77}\right)\right]\\
M_4 &= \sigma\left[\frac{12\beta\sigma}{35}+\frac{m}{7}\left(\frac{24\alpha}{5}-4\right)-\beta\left(-\frac{157}{70}-\frac{1356m}{35}+\frac{1656m^2}{35}\right)\right]\\
M_5 &= \sigma\left[-\frac{72m\beta\sigma}{35}-\beta\left(\frac{9}{14}+\frac{1422m}{35}+\frac{648m^2}{7}\right)\right]\\
M_6 &= \frac{\beta\sigma}{35}[498m+5976m^2]\\
M_7 &= -\frac{30024m^2\beta\sigma}{385}
\end{aligned}\right\} \quad (3.21)$$

Case (ii)

(*a*) *Zeroth order solution.*—When the boundary of the sphere is insulated we solve eqn. (3.5) under the boundary conditions

$$\left.\begin{aligned}\frac{\partial T_1}{\partial r}&=0 \text{ on } r=1\\ T_1&=0 \text{ as } r\to\infty\end{aligned}\right\} \quad \text{.. .. .. (3.22)}$$

Thus $T_1(r, \theta)$ in this case is given by

$$T_1(r,\theta) = -\frac{\beta\sigma}{6}\left(\frac{6}{r^4}-\frac{4}{r^3}-\frac{12}{r}\right)-\frac{\beta\sigma}{2}\left(\frac{4}{r^3}-\frac{3}{r^4}\right)\cos^2\theta\,. \quad \text{.. (3.23)}$$

(*b*) *First order solution.*—Substituting the value of $T_1(r, \theta)$ from eqn. (3.23) in eqn. (3.6), u_1, v_1, f_1, g_1, h_1, l_1, m_2 and w_2 being the same as in case (i) and solving the resulting eqn. (3.6) using the boundary conditions

$$\left.\begin{aligned}\frac{\partial T_2}{\partial r}&=0 \text{ on } r=1\\ T_2&=0 \text{ as } r\to\infty\end{aligned}\right\} \quad \text{.. .. .. (3.24)}$$

we note that the first order temperature $T_2(r, \theta)$ in this case is also given by eqn. (3.15) but with the following values of the various coefficients and the constants

$$C_1 = -\left[\frac{F_1}{3}+\frac{F_2}{4}+\frac{F_3}{5}+\frac{F_4}{6}+\frac{F_5}{7}+\frac{F_6}{8}+\frac{F_7}{9}\right] \quad \text{.. .. .. (3.25)}$$

$$C_2 = -\frac{Q_1}{6}+\frac{Q_2}{15}+\frac{2Q_3}{9}+\frac{5Q_4}{42}+\frac{Q_5}{12}+\frac{7Q_6}{108}+\frac{4Q_7}{75}+\frac{Q_8}{22}+\frac{5Q_9}{126}+\frac{\beta\sigma}{168} \quad \ldots (3.26)$$

$$C_3 = \frac{M_1}{10}-\frac{M_2}{45}-\frac{3M_3}{25}-\frac{7M_4}{110}-\frac{2M_5}{45}-\frac{9M_6}{260}-\frac{M_7}{35}-\frac{168\beta\sigma}{6125} \quad \ldots (3.27)$$

$$\left.\begin{aligned}
F_1 &= \sigma\left[\frac{\beta\sigma}{5}(4m-1)-\beta\left(\frac{47}{200}-\frac{17m}{14}+2m^2\right)\right]\\
F_2 &= \sigma\left[-\frac{\beta\sigma}{5}\left(4m-\frac{11}{3}\right)-\beta\left(\frac{49m}{10}-\frac{8}{5}\right)\right]\\
F_3 &= \sigma\left[-\beta\sigma\left(4m+\frac{5}{6}\right)-\beta\left(\frac{217}{60}+8m-48m^2\right)\right]\\
F_4 &= \sigma\left[\frac{\beta\sigma}{5}\left(34m+\frac{3}{2}\right)-\frac{\beta}{5}(-17-252m+192m^2)\right]\\
F_5 &= \sigma\left[-\frac{14m\beta\sigma}{5}-\beta\left(-\frac{53}{6}+\frac{354m}{5}+168m^2\right)\right]\\
F_6 &= \frac{\beta\sigma}{5}[116m+1392m^2]\\
F_7 &= -\frac{588m^2\beta\sigma}{5}
\end{aligned}\right\} \quad \ldots \quad \ldots (3.28)$$

$$\left.\begin{aligned}
Q_1 &= \beta\sigma^2(m-\tfrac{1}{2})\\
Q_2 &= \beta\sigma^2\\
Q_3 &= \sigma\left[-\frac{\beta\sigma}{7}(50m+\tfrac{3}{2})-\beta\left(-\frac{361}{1400}+\frac{78m}{35}-\frac{206m^2}{35}\right)\right]\\
Q_4 &= \sigma\left[\frac{\beta\sigma}{7}\left(22m-\frac{19}{6}\right)-\frac{\beta}{7}\left(\frac{25}{2}-50m\right)\right]\\
Q_5 &= \sigma\left[\frac{\beta\sigma}{7}(32m+\tfrac{1}{6})-\beta\left(-\frac{2179}{588}-\frac{40m}{7}+\frac{3396m^2}{77}\right)\right]\\
Q_6 &= \sigma\left[\frac{\beta\sigma}{7}(1-12m)-\beta\left(\frac{47}{14}+\frac{342m}{7}-\frac{240m^2}{7}\right)\right]\\
Q_7 &= \sigma\left[-\frac{6m\beta\sigma}{7}+\frac{\beta}{7}\left(\frac{235}{3}+354m+1200m^2\right)\right]\\
Q_8 &= -\frac{\beta\sigma}{7}[158m+1896m^2]\\
Q_9 &= \frac{8040m^2\beta\sigma}{77}
\end{aligned}\right\} \quad \ldots (3.29)$$

$$\left.\begin{aligned}
M_1 &= \sigma\left[\frac{18\beta\sigma}{35}(4m-1)-\frac{\beta}{35}(9-72m+144m^2)\right]\\
M_2 &= \sigma\left[\frac{2\beta\sigma}{35}(-36m+23)-\frac{\beta}{35}(-39+156m)\right]\\
M_3 &= \sigma\left[-\frac{8\beta\sigma}{7}(3m+1)-\beta\left(\frac{2379}{980}+\frac{72m}{7}-\frac{3996m^2}{77}\right)\right]\\
M_4 &= \sigma\left[\frac{12\beta\sigma}{35}(16m+1)-\frac{\beta}{35}\left(-\frac{157}{2}-1356m+1656m^2\right)\right]\\
M_5 &= \sigma\left[-\frac{72m\beta\sigma}{35}-\beta\left(\frac{9}{14}+\frac{1422m}{35}+\frac{648m^2}{7}\right)\right]\\
M_6 &= \frac{\beta\sigma}{35}[498m+5976m^2]\\
M_7 &= -\frac{30024m^2\beta\sigma}{385}
\end{aligned}\right\} \quad (3.30)$$

By properly interpreting m as in (2.21), the above solutions give the temperature distribution for all the classes of Oldroyd, Rivlin-Ericksen and Walters fluids.

4. Discussion of the Results

We now discuss the nature of the temperature field when the sphere is maintained at a constant temperature, variable temperature or it is insulated. In order to do this we fix the Reynolds number R in our calculations to have the value 0·2 and vary m so as to study the effect of elasticity of the fluid on the temperature distribution. We have also fixed Prandtl number $\sigma = 1$ and $\beta = 5$ for all the cases studied. We have tabulated the temperature distribution for various values of the elastic number m for $\theta = \pi/6$ and various types of thermal boundary conditions as stated above.

Table I

Temperature distribution for uniformly heated sphere at $\theta = \pi/6$ for $\sigma = 1$ and $\beta = 5$

r \ m	0	0·5	4	10	12
1·0	1·0	1·0	1·0	1·0	1·0
1·2	1·411282	1·417838	1·526204	2·020388	1·911638
1·4	1·531555	1·540693	1·670209	2·105991	1·938521
1·6	1·517743	1·527669	1·639073	1·934110	1·720811
1·8	1·452589	1·462864	1·539496	1·733805	1·494656
2·0	1·383999	1·393753	1·457163	1·568374	1·320936
4·0	0·820091	0·826600	0·854597	0·788516	0·614172
6·0	0·563551	0·568133	0·584027	0·519693	0·394658

(*a*) $\alpha = 0$.—In Table I, we have collected the values of the temperature against the radial distance r and the elastic parameter m at $\theta = \pi/6$. It is clear that the temperature goes on increasing with an increase in the elastic number m up to a certain critical value m_c of m and then starts decreasing (Fig. 2). Such a behaviour is demonstrated in Table I also for $m = 12$. The maximum temperature for each m is attained in the region near the

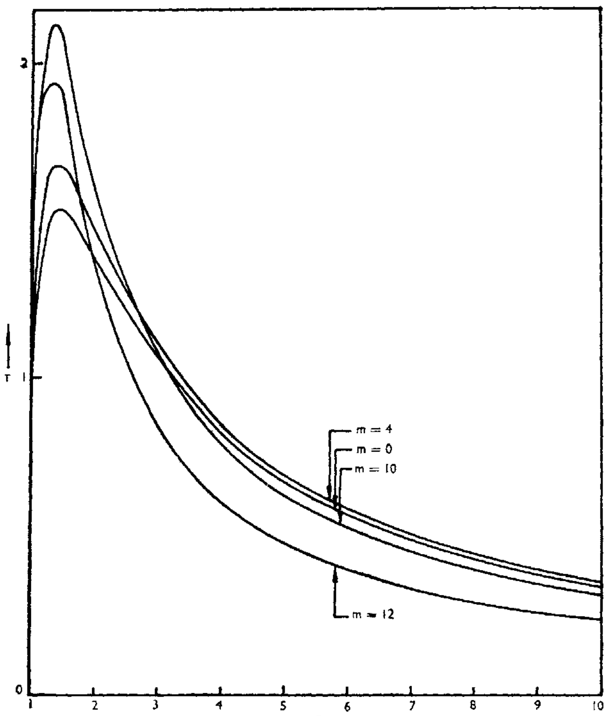

FIG. 2. Temperature distribution for a uniformly heated sphere showing the effect of elasticity.

boundary of the sphere outside which it gradually falls to zero. Thus there is a region near the boundary of the sphere in each case in which there is a large rise in temperature due to the fact that here the dissipation is large but the convection only takes place very slowly.

(*b*) $\alpha = 1$.—When the surface of the sphere is maintained at a variable temperature given by the law $(1+\sin^2\theta)$, we note that the nature of the temperature field is similar to the one as discussed in the case when $\alpha = 0$. In Table II, we have recorded the values of the temperature at $\theta = \pi/6$ against the radial distance r and the elastic number m.

(*c*) $\left(\frac{\partial T}{\partial r}\right)_{r=1} = 0$.—When the sphere is thermally insulated, the maximum temperature is attained on the boundary as it should be, and therefore, for a fixed m, the temperature for all θ goes on decreasing as r increases finally tending to zero far away from the boundary (Table III). Here also we note that the temperature goes on increasing for a fixed θ, e.g. $\theta = \pi/6$ up to a certain value m_c of the elastic parameter m and for larger values of this critical parameter the temperature goes on decreasing as m increases and in fact becomes less than the temperature for the Newtonian fluid (Fig. 3) for some m.

TABLE II

Temperature distribution for a non-uniformly heated sphere at $\theta = \pi/6$ for $\alpha = 1$, $\sigma = 1$ and $\beta = 5$

r / m	0	0·5	4	10	12
1·0	1·250000	1·250000	1·250000	1·250000	1·250000
1·2	1·725680	1·732315	1·855594	2·370366	2·268659
1·4	1·8555871	1·865089	1·995790	2·474012	2·315226
1·6	1·832531	1·842633	1·955075	2·292812	2·088254
1·8	1·751489	1·761846	1·855194	2·074455	1·843638
2·0	1·664863	1·675093	1·753443	1·888233	1·648577
4·0	0·980132	0·986540	1·015061	0·969259	0·799071
6·0	0·672731	0·677335	0·713377	0·642608	0·520333

TABLE III

Temperature distribution for an insulated sphere at $\theta = \pi/6$ for $\sigma = 1$ and $\beta = 5$

r / m	0	0·5	4	10
1·0	6·458333	6·458333	6·458333	6·458333
1·2	6·026362	6·126001	6·380823	5·878770
1·4	5·607843	5·702294	5·891953	5·206017
1·6	5·165826	5·247051	5·387454	4·554751
1·8	4·753160	4·826062	4·929026	4·020314
2·0	4·383443	4·450001	4·536489	3·592361
4·0	2·366727	2·401958	2·414617	1·760030
6·0	1·600615	1·624297	1·628605	1·157847

(*d*) We note that the effect of the elastic parameter m up to a certain critical value m_c is to increase the rate of heat transfer at the sphere for the cases, above which it decreases, when the sphere is either maintained at constant or variable temperatures. When the sphere is insulated no heat transfer takes place from its boundary, which, therefore, is at higher temperature than any other region of the flow field.

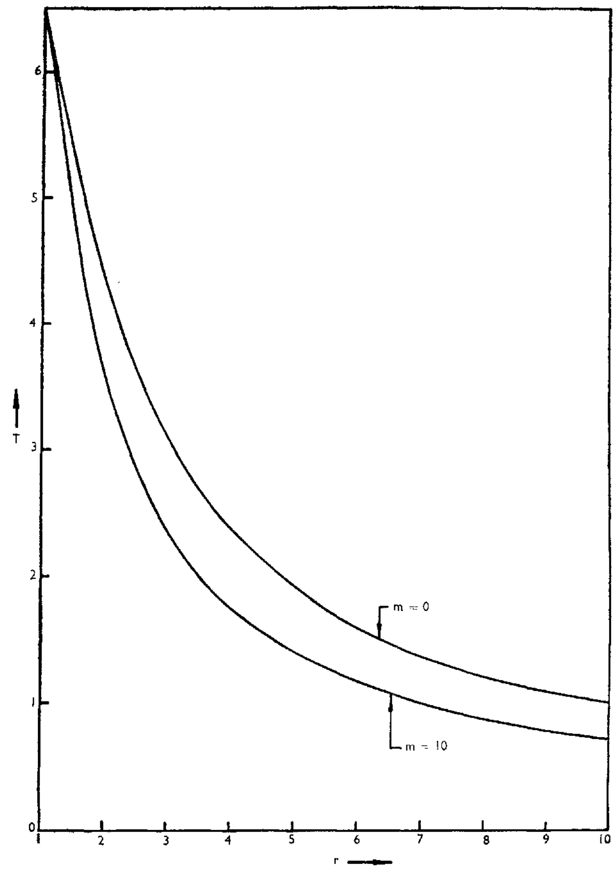

FIG. 3. Temperature distribution for an insulated sphere showing the effect of elasticity.

5. CONCLUSIONS

We sum up our conclusions as follows:

(i) All the three types of non-Newtonian fluids, namely the Oldroyd, Rivlin-Ericksen $\{(2\phi_2+\phi_3)=0\}$ and Walters fluids, behave identically for the same value of m and the secondary flow for each shows breaking and reversal of the flow which are the characteristic properties of all classes of non-Newtonian fluids, distinguishing them from the usual Newtonian fluids. Even when $(2\phi_2+\phi_3)\neq 0$ all types of fluids have identical flows and stresses except the component p_{rr} which differs from p_{rr} for the other two types of fluids by a term which is proportional to $(2\phi_2+\phi_3)$.

(ii) The effect of elasticity is to increase the temperature in comparison to that for the Newtonian fluid up to certain critical value m_c of the elastic parameter above which it starts decreasing and becomes less than the temperature for the Newtonian fluids for some elastic parameter $m > m_c$. The maximum temperature is attained in the region close to the boundary due to the maximum viscous dissipation in this region.

(iii) Even when the sphere is insulated the effect of elasticity on the temperature distribution is similar to the one stated in (ii).

(iv) The effect of elasticity is to increase the rate of heat transfer at the sphere up to a certain critical elastic parameter.

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