VECTOR FIELDS IN GENERALIZED KÄHLERIAN MANIFOLD

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The almost complex general manifold equipped with the almost complex structure was studied by the present authors (1967). In the present paper K-generalized manifold and $*O$-generalized manifold have been introduced and results relating almost-Hermitian general manifold, Hermitian general manifold, almost Kähler general manifold, almost semi-Kähler general manifold, generalized Kählerian manifold, K-generalized manifold and $*O$-generalized manifold have been obtained.

§ 1. Let M be a C^{∞} differentiable general Riemannian manifold, $F(M)$ the ring of real-valued differentiable functions on M and $\mathfrak{X}(M)$ the module of derivations of $F(M)$. Then all the elements of $\mathfrak{X}(M)$ are vector fields (Gray 1966; Helgason 1962).

Every Riemannian metric g associated with the general Riemannian manifold M defines an inner product in $\mathfrak{X}(M)$, which we write as $g(X, Y)$ for $X, Y \in \mathfrak{X}(M)$. Let $\underline{g}$ denote the symmetric part of g, which defines a positive definite inner product $\underline{g}(X, Y)$ for $X, Y \in \mathfrak{X}(M)$.

Let $x^1, x^2, \ldots x^n$ be the local coordinates of a point P in M which induces the natural basis $\left(\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \ldots \frac{\partial}{\partial x^n}\right)$ at P. Further we define $\bar{g}(X, Y)$ to be such that its jkth component $\bar{g}^{jk} = \bar{g}(dx^j, dx^k)$ is related to the ijth component $\underline{g}_{ij} = \underline{g}\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^g}\right)$ of $\underline{g}(X, Y)$ by $\underline{g}_{ij}\bar{g}^{jk} = \delta^k_i$ and use quantities $\bar{g}^{jk}$ and $\underline{g}_{jk}$ to raise and lower the indices.

Corresponding to each vector field $X \in \mathfrak{X}(M)$ we define the covector X_c of X (Kobayashi and Nomizu 1963) by the relation $\left\langle Y, X_c \right\rangle = \underline{g}(X, Y)$ where Y is any member of $\mathfrak{X}(M)$.

If ξ^i are the components of a vector or vector field at a point P with respect to $x^1, x^2, \ldots x^n$, that is $X = \Sigma\, \xi^i \frac{\partial}{\partial x^i}$, then the components ξ_i of the corresponding covector or the corresponding 1-form $\alpha = \psi(X)$ (Kobayashi and Nomizu 1963) are related to ξ^i by

$$\xi^i = \Sigma\, \bar{g}^{ij}\xi_j \text{ or } \xi_i = \Sigma\, \underline{g}_{ij}\xi^j.$$

An almost complex general manifold is a differentiable manifold M equipped with a (1, 1) tensor H which satisfies

$$H^2 = -I \quad \ldots \quad (1.1)$$

where I is the identity map [H can also be regarded as $F(M)$—0 linear map $H: \mathfrak{X}(M) \rightarrow \mathfrak{X}(M)$ (Gray 1966; Goldberg 1962)].

From (1.1) it follows that the general manifold M is even dimensional.

M is almost-Hermitian general manifold if it is almost complex and has a Riemannian metric for which

$$\langle X, Y \rangle = \langle HX, HY \rangle, \; X, Y \in \mathfrak{X}(M). \quad \ldots \quad (1.2)$$

An almost-Hermitian general manifold M is called generalized Kählerian manifold if

$$\nabla_X H = 0 \quad \ldots \quad (1.3)$$

where ∇_X denotes the covariant derivative with respect to X and connection $\mathcal{L}$ of M whose components are not necessarily symmetric (i.e. $\mathcal{L}^k_{ij} \neq \mathcal{L}^k_{ji}$) (Eisenhart 1964).

In the almost-Hermitian general manifold M we define a skew-symmetric tensor $\langle HX, Y \rangle$, which in turn defines a 2-form $B(X, Y)$.

$$B(X, Y) = \langle HX, Y \rangle \text{ for } X, Y \in \mathfrak{X}(M). \quad \ldots \quad (1.4)$$

As every skew-symmetric tensor of type (o, p) defines a unique p-form, we make no distinction between a form and the tensor which defines it.

Further, on the almost-Hermitian general manifold M the general Nijenhuis tensor $N(X, Y)$ is defined by

$$N(X, Y) = [X, Y] + H[HX, Y] + H[X, HY] - [HX, HY]. \quad \ldots \quad (1.5)$$

From which it can easily be verified that

$$N(X, Y) = -N(Y, X), \; N(HX, Y) = N(X, HY) = -HN(X, Y).$$

If the general Nijenhuis tensor vanishes then the almost-Hermitian general manifold is called Hermitian general manifold.

It is known that exterior derivative of a p-form is a $(p+1)$-form, clearly exterior derivative of $B(X, Y)$ will be the following

$$dB(X, Y, Z) = \nabla_X B(Y, Z) + \nabla_Y B(Z, X) + \nabla_Z B(X, Y) - B(Z, T(Y, X)) - B(X, T(Z, Y)) - B(Y, T(X, Z)) \quad \ldots \quad (1.6)$$

where d denotes the exterior derivative and $T(X, Y)$ the torsion tensor.

An almost-Hermitian general manifold M is called almost-Kählerian general manifold if $dB = 0$ (i.e. if the associated differential form is closed), almost semi-Kählerian if

$$H_j = -\nabla_i H^i = 0 \quad \ldots \quad (1.7)$$

and K-generalized manifold if

$$\nabla_X H(Y) + \nabla_Y H(X) = 0. \quad \quad (1.8)$$

In the generalized manifold M we define operators O and $*O$ whose components are given by

$$O_{ij}^{hl} = \tfrac{1}{2}(\delta_i^h \delta_j^l - H_i^h H_j^l), \; *O_{ij}^{hl} = \tfrac{1}{2}(\delta_i^h \delta_j^l + H_i^h H_j^l) \quad \quad (1.9)$$

which satisfies the following conditions

$$\left.\begin{array}{lll} O \,.\, O = O, & *O \,.\, O = 0, & O + *O = \delta \\ *O \,.\, *O = *O, & O \,.\, *O = 0 & \end{array}\right\} \quad \quad (1.10)$$

where δ denotes the kronecker symbol (Yano 1965).

Consider a tensor $T(\alpha_1, \alpha_2, \ldots \alpha_r)$ of the type $(r, 0)$ and the mapping O and $*O$ such that

$$O_{\alpha_p \alpha_q} T(\alpha_1, \alpha_2 \ldots \alpha_p \ldots \alpha_q \ldots \alpha_r) = 0 \quad \quad (1.11)$$

then the tensor $T(\alpha_1, \alpha_2 \ldots \alpha_r)$ is called hybrid in pth and qth places and

$$*O_{\alpha_p \alpha_q} T(\alpha_1 \ldots \alpha_p \ldots \alpha_q \ldots \alpha_r) = 0 \quad \quad (1.12)$$

then the tensor $T(\alpha_1 \ldots \alpha_r)$ is called pure in pth and qth places.

In other words, the tensor T with components $T^{i_1 \ldots i_p \ldots i_q \ldots i_r}$ is called hybrid or pure in indices i_p and i_q according as

$$O_{i_p i_q}^{j_p j_q} T^{i_1 \ldots i_p \ldots i_q \ldots i_r} = 0 \;\text{ or }\; *O_{i_p i_q}^{j_p j_q} T^{i_1 \ldots i_p \ldots i_q \ldots i_r} = 0$$

respectively.

Further, if the mapping O and $*O$ of the tensor $T(\alpha_1 \ldots \alpha_r, \beta_1 \ldots \beta_s)$ of type (r, s) satisfies

$$O_{\alpha_p \beta_q} T(\alpha_1 \ldots \alpha_p \ldots \alpha_r, \beta_1, \ldots \beta_q \ldots \beta_s) = 0 \quad \quad (1.13)$$

then T is called hybrid in pth and qth places and

$$*O_{\alpha_p \beta_q} T(\alpha_1 \ldots \alpha_p \ldots \alpha_r, \beta_1 \ldots \beta_q \ldots \beta_s) = 0 \quad \quad (1.14)$$

then T is called pure in pth and qth places.

In an almost-Hermitian general manifold M if

$$\nabla_X H(Y) + \nabla_{HX} H(HY) = 0 \text{ for all } X, Y \in \mathfrak{X}(M) \quad \quad (1.15)$$

then the almost-Hermitian general manifold is called $*O$-generalized manifold. From (1.3) and (1.8) we obtain:

Theorem (1.1)—Every generalized Kählerian manifold is K-generalized manifold.

From (1.7) and (1.8) we have (Koto 1960):

Theorem (1.2)—Every K-generalized manifold is an almost semi-Kählerian general manifold.

From (1.2) and (1.5) we have:

An almost-Hermitian general manifold is a Hermitian manifold if $N(X, Y)$ vanishes.

From (1.3) and (1.15) it follows that:

Theorem (1.3)—Every generalized Kählerian manifold is a $*O$-generalized manifold.

§ 2. In an almost-Hermitian general manifold a covector field U is covariant almost analytic (Prakash and Mathai 1967; Yano 1965) if it satisfies.

$$\nabla_X \left\langle H(Y), U \right\rangle = \left\langle Y, \nabla_{HX} U \right\rangle + \left\langle \nabla_Y H(X), U \right\rangle - \left\langle T(HX, Y), U \right\rangle - \left\langle HT(Y, X), U \right\rangle$$

or

$$\left\langle HY, \nabla_X U \right\rangle + \left\langle \nabla_X H(Y), U \right\rangle = \left\langle Y, \nabla_{HX} U \right\rangle + \left\langle \nabla_Y H(X), U \right\rangle - \left\langle T(HX, Y), U \right\rangle - \left\langle HT(Y, X), U \right\rangle. \quad \ldots \quad \ldots \quad (2.1)$$

In a general manifold M a covector field U is closed if it satisfies

$$\left\langle Y, \nabla_X U \right\rangle - \left\langle X, \nabla_Y U \right\rangle + \left\langle T(Y, X), U \right\rangle = 0. \quad \ldots \quad (2.2)$$

We define a new covector field $\tilde{U}$ in terms of the covector field U by

$$\left\langle X, \tilde{U} \right\rangle = \left\langle HX, U \right\rangle \text{ for all } X \in \aleph(M). \quad \ldots \quad \ldots \quad (2.3)$$

Equation (2.1) can be written as

$$\nabla_X \left\langle H(Y), U \right\rangle - \nabla_Y \left\langle H(X), U \right\rangle = \left\langle Y, \nabla_{HX} U \right\rangle - \left\langle HX, \nabla_Y U \right\rangle - \left\langle T(HX, Y), U \right\rangle - \left\langle HT(Y, X), U \right\rangle. \quad \ldots \quad \ldots \quad (2.4)$$

Accordingly, using (2.3) eqn. (2.4) can be written as

$$\left\langle X, \nabla_Y \tilde{U} \right\rangle - \left\langle Y, \nabla_X \tilde{U} \right\rangle + \left\langle T(X, Y), \tilde{U} \right\rangle = \left\langle X, \nabla_{HY} U \right\rangle - \left\langle HY, \nabla_X U \right\rangle + \left\langle T(X, HY), U \right\rangle.$$

Hence we obtain:

Theorem (2.1)—In an almost-Hermitian general manifold M if a covariant almost analytic covector field U is closed then $\tilde{U}$ is closed.

If the covector field U and $\tilde{U}$ are both closed then

$$\left\langle X, \nabla_Y U \right\rangle - \left\langle Y, \nabla_X U \right\rangle + \left\langle T(X, Y), U \right\rangle = 0$$

and

$$\left\langle X, \nabla_Y \tilde{U} \right\rangle - \left\langle Y, \nabla_X \tilde{U} \right\rangle + \left\langle T(X, Y), \tilde{U} \right\rangle = 0.$$

The second equation can be written as

$$\nabla_Y \langle H(X), U \rangle = \nabla_X \langle H(Y), U \rangle - \langle HT(X, Y), U \rangle .$$

Since U is closed, the above equation reduces to

$$\nabla_Y \langle H(X), U \rangle = \langle X, \nabla_{HY} U \rangle + \langle \nabla_X H(Y), U \rangle - \langle T(HY, X), U \rangle - \langle HT(X, Y), U \rangle$$

which is same as equation (2.1)
Hence we obtain:

Theorem (2.2)—In an almost-Hermitian general manifold M if both the covector fields U and $\tilde{U}$ are closed, the covector field U is covariant almost analytic.

It is known that in an almost-Hermitian general manifold M if a covector field U is covariant almost analytic then the covector field $\tilde{U}$ is covariant almost analytic (Prakash and Mathai 1967).

Operating eqn. (1.8) by $\bar{g}^{hk}$ we have

$$-\sum_{h,k} \bar{g}^{hk} \nabla_k(B)(Y, X) \frac{\partial}{\partial x^h} = \nabla_Y H(X) - \nabla_X H(Y) - \sum \bar{g}^{hk} B(T(X, X_k), Y) \frac{\partial}{\partial x^h} - \sum \bar{g}^{hk} B(X, T(Y, X_k)) \frac{\partial}{\partial x^h} - HT(Y, X). \quad .. \quad (2.5)$$

Now

$$\nabla_Y \langle H(X), U \rangle - \nabla_X \langle H(Y), U \rangle = \langle \nabla_Y H(X), U \rangle - \langle \nabla_X H(Y), U \rangle + \langle HX, \nabla_Y U \rangle - \langle HY, \nabla_X U \rangle .$$

In view of eqn. (2.5) this reduces to

$$\nabla_Y \langle HX, U \rangle - \nabla_X \langle HY, U \rangle = -L_u(B)(Y, X) + \langle HT(Y, X), U \rangle$$

where L_u the Lie-derivative (Yano 1957) with respect to the covector U is defined as

$$L_u(B)(Y, X) = \Sigma \bar{g}^{hk} U_h \nabla_k(B)(Y, X) + \langle HY, \nabla_X U \rangle - \langle HX, \nabla_Y U \rangle - \Sigma \bar{g}^{lr} B(T(X, X_l), Y) U_r - \Sigma \bar{g}^{lr} B(X, T(Y, X_l)) U_r .$$

Further if the covector field U is covariant almost analytic, then using eqn. (2.4) the above eqn. reduces to

$$-L_u(B)(Y, X) = \langle X, \nabla_{HY} \tilde{U} \rangle - \langle HY, \nabla_X U \rangle + \langle T(X, HY), U \rangle . \quad (2.6)$$

Theorem (2.3)—In an almost-Kählerian general manifold, the covariant almost analytic vector U satisfies

$$L_u(B)(Y, X)+\langle X, \nabla_{HY}U\rangle-\langle HY, \nabla_X U\rangle+\langle T(X, HY), U\rangle = 0.$$

Theorem (2.4)—If in an almost-Kähler general manifold the covariant almost analytic vector U is closed, then Lie-derivative of form B with respect to U is zero throughout (i.e. $L_u(B)\ (Y, X) = 0$).

§ 3. In an almost-Hermitian general manifold if the torsion tensor $T(X, Y)$ whose components T^s_{ij} are pure in indices s and j then

$$*O^{jr}_{ls}T^s_{ij} = 0$$

which implies that

$$T(X, Y)+HT(X, HY) = 0.$$

Operating this with H and using (1.1) we have

$$HT(X, Y)-T(X, HY) = 0. \quad .. \quad .. \quad .. \quad (3.1)$$

Thus if U is covariant almost analytic vector field in an almost-Hermitian general manifold with the torsion tensor $T(X, Y)$ whose components T^s_{ij} are pure in indices s and j, eqn. (2.4) becomes

$$\nabla_Y\langle H(X), U\rangle-\nabla_X\langle H(Y), U\rangle = \langle X, \nabla_{HY}U\rangle-\langle HY, \nabla_H U\rangle \quad (3.2)$$

which can be put down as

$$\langle X, \nabla_Y\tilde{U}\rangle-\langle Y, \nabla_X\tilde{U}\rangle = \langle X, \nabla_{HY}U\rangle-\langle HY, \nabla_X U\rangle. \quad (3.3)$$

If now the covector field U is assumed to be pseudo-harmonic (Yano and Bochner 1953), i.e. satisfying

$$\langle Y, \nabla_X U\rangle-\langle X, \nabla_Y U\rangle = 0 \text{ and } \nabla^i U_i = 0 \quad .. \quad .. \quad (3.4)$$

then in view of

$$\nabla^j\tilde{U}_j = \nabla^j(H^t_j U_t) = H^t_j\nabla^j U_t$$
$$= H^{jt}\nabla_j U_t$$
$$= \frac{H^{jt}}{2}(\nabla_j U_t-\nabla_t U_j)$$

and (3.3) we obtain that if the covector field U is pseudo-harmonic then $\tilde{U}$ is also pseudo-harmonic.

Operating eqn. (3.3) by H we get

$$\langle X, \nabla_Y U\rangle-\langle Y, \nabla_X U\rangle = -\langle X, \nabla_{HY}\tilde{U}\rangle+\langle HY, \nabla_X\tilde{U}\rangle.$$

From which it follows that if $\tilde{U}$ is pseudo-harmonic then U is also pseudo-harmonic.

Theorem (3.1)—In a generalized Kählerian manifold M if the torsion tensor $T(X, Y)$ is pure in its covariant and contravariant places (or with components T^{i}_{jk} pure in i and k) the covariant almost analytic vector field U is pseudo-harmonic if and only if $\tilde{U}$ is pseudo-harmonic.

Now in a generalized Kählerian manifold with the torsion tensor $T(X, Y)$ pure in covariant and contravariant places, the condition (2.4) (as satisfied by a covariant almost analytic vector field) reduces to

$$\langle X, \nabla_Y U\rangle - \langle Y, \nabla_X U\rangle = -\langle HX, \nabla_{HY} U\rangle - \langle Y, \nabla_X U\rangle$$

or

$$\langle X, \nabla_Y U\rangle + \langle HX, \nabla_{HY} U\rangle = 0$$

or

$$*O^{tr}_{ji} \nabla_t U_r = 0$$

which shows that $\nabla_t U_r$ is pure in indices t and r.

Hence we obtain:

Theorem (3.2)—In a generalized Kählerian manifold with the torsion tensor $T(X, Y)$ pure in contravariant and covariant places, the necessary and sufficient condition for a covector field U to be covariant almost analytic is that $\nabla_i U_j$ is pure in i and j.

§ 4. In an almost-Hermitian general manifold M a vector field W is contravariant almost analytic (Yano 1957, Yano 1965; Prakash and Mathai 1967) if it satisfies

$$L_W H = 0 \qquad .. \quad .. \quad .. \quad .. \quad (4.1)$$

where

$$L_W H(Y) = \nabla_W H(Y) - \nabla_{HY} W + H\nabla_Y W + HT(Y, W) - T(HY, W)$$

for $W, Y \in \mathfrak{X}(M)$.

Consider a generalized Kählerian manifold with the torsion tensor $T(X, Y)$ pure in its contravariant and covariant places, then equation (4.1) reduces to

$$\nabla_{HY} W = H\nabla_Y W. \qquad .. \quad .. \quad .. \quad .. \quad (4.2)$$

Vector field W defines a vector field $\tilde{W}$ given by

$$\tilde{W} = -HW. \qquad .. \quad .. \quad .. \quad .. \quad (4.3)$$

Consequently the component $\tilde{W}_i$ of the covector $\tilde{W}_c$ of W and the component W_i of the covector W_c of W are related by

$$\tilde{w}_i = w_j H^j_i$$

which can be written as

$$\langle X, \tilde{W}_c\rangle = \langle HX, W_c\rangle.$$

Thus (4.2) becomes

$$-\nabla_Y \tilde{W} = \nabla_{HY} W = \Sigma \bar{g}^{rh} \nabla_{HY} w_r \cdot \frac{\partial}{\partial x^h}.$$

Further in the generalized Kähler manifold if the covector W_c of W is pseudo-harmonic then above equation reduces to

$$\begin{aligned} -\nabla_Y \tilde{W} &= \Sigma \bar{g}^{rh} \left\langle HY,\ \nabla_r W_c \right\rangle \frac{\partial}{\partial x^h} \\ &= \Sigma \bar{g}^{rh} \left\langle Y,\ \nabla_r \tilde{W}_c \right\rangle \frac{\partial}{\partial x^h} \\ &= \Sigma \bar{g}^{rh} \nabla_r \tilde{W}_c(Y) \frac{\partial}{\partial x^h}. \end{aligned}$$

Alternatively

$$\left\langle Y,\ \nabla_X \tilde{W}_c \right\rangle + \left\langle X,\ \nabla_Y \tilde{W}_c \right\rangle = 0. \qquad \ldots \quad \ldots \quad (4.4)$$

Now covector field α is pseudo-killing (Yano and Bochner 1953) if it satisfies

$$\left\langle Y,\ \nabla_X \alpha \right\rangle + \left\langle X,\ \nabla_Y \alpha \right\rangle = 0.$$

Comparing this with (4.4) it follows that the covector field $\tilde{W}_c$ is pseudo-killing. Hence we obtain:

Theorem (4.1)—In a generalized Kählerian manifold with the torsion tensor $T(X, Y)$ pure in its contravariant and covariant places, if a vector field W is contravariant almost analytic and the covector of W is pseudo-harmonic then $\tilde{W}_c$ is a pseudo-killing vector.

In a generalized Kähler manifold with the torsion tensor $T(X, Y)$ pure in contravariant and covariant places, if a vector field V is both covariant and contravariant almost analytic then it satisfies

$$\left\langle HX,\ \nabla_Y V_c \right\rangle - \left\langle X,\ \nabla_{HY} V_c \right\rangle = 0 \qquad \ldots \quad \ldots \quad (4.5)$$

and

$$H\, \nabla_Y V - \nabla_{HY} V = 0. \qquad \ldots \quad \ldots \quad \ldots \quad (4.6)$$

Alternatively eqn. (4.6) can be written as

$$\left\langle HX,\ \nabla_Y V_c \right\rangle + \left\langle X,\ \nabla_{HY} V_c \right\rangle = 0. \qquad \ldots \quad \ldots \quad (4.7)$$

Adding (4.5) and (4.7) we get

$$\left\langle HX,\ \nabla_Y V_c \right\rangle = 0$$

$$\Rightarrow \nabla_Y V_c = 0 \quad \Rightarrow \nabla_Y V = 0.$$

Hence we obtain:

Theorem (4.2)—*In a generalized Kählerian manifold with the torsion tensor $T(X, Y)$ pure in contravariant and covariant places, if a vector field is covariant and contravariant almost analytic then it is covariant constant.*

Now (4.2) reduces to

$$\nabla_X W + H \nabla_{HX} W = 0$$

or

$$*O_{ir}^{sh} \nabla_s w^r = 0$$

which shows that $\nabla_s w^r$ is pure in r and s.

Hence we have:

Theorem (4.3)—In a generalized Kähler manifold with the torsion tensor $T(X, Y)$ pure in contravariant and covariant places, the necessary and sufficient condition for a vector field to be contravariant almost analytic is that $\nabla_i w^j$ is pure in j and i.

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