ON f-INJECTIVE MODULES AND SEMI-HEREDITARY RINGS

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Let M be a unital right R-module over a ring R with unity. Let I be a right ideal of R. M is said to be I-injective if for every R-homomorphism $f: I \to M$ there exists $m \in M$ such that $f(x) = mx \; \forall \; x \in I$. A module M which is I-injective $\forall$ finitely generated right ideal I is called an f-injective module. The following results are proved: (a) A module M is f-injective if (i) $lr(p) = Mp \; \forall \; p \in R$, (ii) $l(I \cap J) = l(I)+l(J)$ for all finitely generated right ideals I and J, where $l(S) = \{m \in M : mS = 0\}$ for subsets S of R; (b) a ring R is right Noetherian if every f-injective module over R is injective; (c) a ring R is semi-hereditary if each quotient module of every injective module is f-injective; and (d) a ring R is semi-hereditary self f-injective if it is regular.

§ 1. INTRODUCTION

All rings are assumed to possess unity and all modules considered are unital right modules throughout this paper. Let M be a module over a ring R. Let I be a module over a ring R. Let I be a right ideal of R. M is said to be I-injective if and only if every R-homomorphism f : I into M can be extended to a R-homomorphism of R into M. In other words M is I-injective if and only if for every R-homomorphism $f : I \to M$, there exists $m \in M$ such that $f(x) = mx$ for every $x \in I$. A module M which is I-injective for each right ideal I is said to be an injective module. Baer (1940, Theorem 2) who introduced this concept with a slightly different terminology proved that if $p \in R$, then M is pR-injective if and only if $lr(p) \subset Mp$. Since $lr(p) \supset Mp$ trivially, one can say that: M is pR-injective if and only if $lr(p) = Mp$.

In Section 2 necessary and sufficient conditions for a module M to be I-injective for all finitely generated right ideals I have been obtained. There emerges a criterion (2.2) for injectivity of modules over Noetherian Rings. Call a module M which is I-injective for all finitely generated right ideals I an f-injective module. Trivially an f-injective module over a Noetherian ring is injective. Conversely, a ring R over which every f-injective module is injective is proved to be a Noetherian ring.

Osofsky (1964) proved that a self-injective hereditary ring (Cartan and Eilenberg 1956, p. 12) is semi-simple Artinian. A related question is 'to

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determine the class of semi-hereditary self-injective rings'. It is proved in Section 3 that the class of semi-hereditary self-injective rings coincides with the class of regular self-injective rings.

Three new characterizations of semi-hereditary rings (Theorems 3.1 and 3.2) have been obtained, one amongst them states: A ring R is semi-hereditary if and only if every quotient module of an injective R module is f-injective. This characterization resembles that of hereditary rings: Every quotient module of an injective module is injective (Cartan and Eilenberg 1956, p. 14).

Throughout this note $r(S) = \{r \in R : Sr = 0\}$ where S is a subset of M. $l(x) = \{m \in M : mX = 0\}$ for subsets X of R.

§2. Conditions for I-Injectivity

In this section necessary and sufficient conditions for a module M to be I-injective for all finitely generated right ideals I have been obtained.

Theorem 2.1—A module M is I-injective for all finitely generated right ideals I if and only if (i) $lr(p) = Mp$ for all $p \in R$, and (ii) $l(A \cap B) = l(A) + l(B)$ for all finitely generated right ideals A and B.

Proof: Let M be I-injective for all finitely generated right ideals I. Using the result of Baer (1940) it follows that $lr(p) = Mp$ for all $p \in R$. Now let A, B be any finitely generated right ideals. Trivially $l(A)+l(B) \subset l(A \cap B)$. Let $m \in l(A \cap B)$. Let m_1 be a fixed element of M. Define $f_1 : A \to M$ and $f_2 : B \to M$ such that $f_1(a) = m_1 a \ \forall \ a \in A$ and $f_2(b) = (m+m_1)\, b \ \forall \ b \in B$. Clearly f_1, f_2 coincide on $A \cap B$. Therefore, $f : A+B \to M$ such that $f(a+b) = f_1(a)+f_2(b) \ \forall \ a \in A,\ b \in B$ is a well-defined mapping of $A+B$ into M. Therefore, there exists $n \in M$ such that $n(a+b) = f(a+b) \forall \ a \in A,\ b \in B$. Therefore, $na = f(a) = m_1 a \ \forall \ a \in A$. $(n-m_1)\, A = 0$. Also for every $b \in B$, $nb = f(b) = f_2(b) = (m+m_1)b$. $(n-m-m_1)\, B = 0$. Therefore, $m = (n-m_1) + (m+m_1-n) \in l(A)+l(B)$.

Conversely assume (i) and (ii). Then by using the result of Baer (1940) it is clear that M is pR-injective for every $p \in R$. Now assume that M is I-injective for all right ideals I which are generated by $(n-1)$ elements. Let $f : I \to M$ where $I = a_1R + a_2R + \ldots + a_nR$. Set $J = a_1R + \ldots + a_{n-1}R$ and $K = a_nR$. Let f_1, f_2 denote the restriction of f to J and K respectively, then there exist m_1, m_2 such that $f_1(x) = m_1 x \ \forall \ x \in J$ and $f_2(y) = m_2 y \ \forall \ y \in k$. Since f_1, f_2 coincide (with f) on $J \cap K$, therefore, $(m_1 - m_2)(J \cap K)0$. There exist n_1, n_2 in $l(J)$ and $l(K)$ respectively such that $m_1 - m_2 = n_1 - n_2$. Let $m = m_1 - n_1 (= m_2 - n_2)$. Then $mx = (m_1 - n_1)x = m_1 x = f_1(x) = f(x) \ \forall \ x \in J$ and similarly $my = f(y) \ \forall \ y \in K$. Hence $mx = f(x) \ \forall \ x \in I$.

Corollary.—A module M over a Noetherian ring R is injective if and only if (i) $lr(p) = M \, . \, p \ \forall \ p \in R$, and (ii) $l(I \cap J) = l(I) + l(J)$ for all right ideals I and J of R.

Theorem 2.2—If M is an injective right R-module, then (i) $l(A \cap B) = l(A)+l(B)$ for all right ideals A and B, and (ii) $lr(S) = MS$ for all finite subsets S of R.

PROOF: (i) can be proved exactly as the corresponding result in Theorem 2.1. To prove (ii), let $S = \{s_1, s_2, \ldots, s_n\}$ be a finite subset of R, then $lr(S) = l[r(s_1) \cap r(s_2) \ldots \cap r(s_n)] = lr(s_1)+lr(s_2)+ \ldots +lr(s_n) = Ms_1+ \ldots + Ms_n = MS$.

We call a module M an f-injective if M is I-injective for all finitely generated right ideals I.

Theorem 2.3—Let $\{M_i\}_{i \in I}$ be a family of R-modules, then $\prod_{i \in I} M_i$ is f-injective if and only if each module is f-injective.

PROOF: A proof of this theorem can be constructed on the lines of proposition 3.1 (Cartan and Eilenberg 1956, p. 8).

Theorem 2.4—Let $\{M_i\}_{i \in I}$ be a family of R modules, then the direct sum $\oplus \sum_{i \in I} M_i$ of the family of modules is f-injective if and only if each M_i is f-injective.

PROOF: Let each M_i be f-injective. Let $g: I \to \oplus \sum_{i \in I} M_i$ where I is a finitely generated right ideal of R. Clearly then $g(I) \subset \oplus \sum_{j \in J} M_j$ where J is a finite subset of I. By using Theorem 2.3, g can be realized by left multiplication by an element of $\oplus \sum_{j \in J} M_j$. The converse is clear from Theorem 2.3.

Theorem 2.5—A ring R is Noetherian if and only if any f-injective module is injective.

PROOF: If R is Noetherian, then clearly each f-injective module is injective. Conversely suppose that R is such that every f-injective module is injective then we prove that R is Noetherian by using the following theorem due to Bass: A ring R is Noetherian if and only if the direct sum of any family of injective modules is injective (Chase 1960). Let $\{M_i\}_{i \in I}$ be a family of injective modules, then by using Theorem 2.3, $\oplus \sum_{i \in I} M_i$ is f-injective. Therefore, $\oplus \sum_{i \in I} M_i$ is injective. Hence R is Noetherian.

§ 3. SEMI-HEREDITARY RINGS

In this section we study semi-hereditary rings.

A ring R is called (right) semi-hereditary if every finitely generated projective right ideal of R is a projective right R-module.

Theorem 3.1—The following conditions are equivalent.

(1) R is semi-hereditary;

(2) Every finitely generated submodule of a projective module is projective;

(3) Every diagram

$$\begin{array}{ccccc} O & \rightarrow & P & \xrightarrow{i} & P' \\ & & \downarrow g & & \\ A & \xrightarrow[\Pi]{} & B & \rightarrow & O \end{array} \qquad \cdots \qquad (D)$$

with P' projective, A injective and P finitely generated and exact rows can be embedded in a commutative diagram

$$\begin{array}{ccccc} O & \rightarrow & P & \xrightarrow{i} & P' \\ & & g \downarrow & \swarrow & \\ A & \xrightarrow[\Pi]{} & B & \rightarrow & O \end{array}$$

PROOF: (1) $\Leftrightarrow$ (2) is proved by Cartan and Eilenberg (1956, p. 15). (2) $\Rightarrow$ (3). Let the diagram (D) be given. By (2) P is projective. Therefore, there exists $h: P \rightarrow A$ such that $\Pi h = g$. Now as A is injective, there exists $h': P' \rightarrow A$ such that $h'i = h$. Let $h'' = (\Pi h')$. h'' is a map of P' into B and $(h'')\ i\ (\Pi h')\ i = \Pi(h'i) = \Pi h = g$.

(3) $\Rightarrow$ (2). Let P be a finitely generated submodule of a projective module P'. Then by proposition 5.1 of Cartan and Eilenberg (1956) in order to prove that P is projective it is sufficient that every diagram

$$\begin{array}{ccccc} & & P & & \\ & & \downarrow g & & \\ A & \xrightarrow[\Pi]{} & B & \rightarrow & O \end{array} \qquad \cdots \qquad (D')$$

in which the row is exact and A injective can be embedded in a commutative diagram

$$\begin{array}{ccccc} & \swarrow & P & & \\ & & \downarrow g & & \\ A & \xrightarrow[\Pi]{} & B & \rightarrow & O \end{array}$$

So let the diagram (D') be given. Embed it in the following diagram:

$$\begin{array}{ccccc} O & \rightarrow & P & \xrightarrow{i} & P' \\ & & \downarrow g & & \\ A & \xrightarrow[\Pi]{} & B & \rightarrow & O \end{array}$$

where i denotes the inclusion map. By (3) there exists $h': P' \rightarrow B$ such that $h'i = g$. Now P' is projective. Therefore, there exists $h'': P' \rightarrow A$ such that $\Pi h'' = h'$. Let $h = h''i$. Then $\Pi h = \Pi(h''i) = h'i = g$.

Theorem 3.2—The following are equivalent:

(1) R is semi-hereditary.

(2) Every quotient module of an f-injective module is f-injective.

(3) Every quotient module of an injective module is f-injective.

PROOF: (1) $\Rightarrow$ (2). Let A be f-injective and Π be a homomorphism of A on to B. Let $f: I \to B$, where I is a finitely generated right ideal of R. So that we have the following diagram

$$\begin{array}{ccccc} & & i & & \\ O \to & I & \to & R & \\ & \downarrow g & & & \\ A \to & B & \to O & & \\ \Pi & & & & \end{array}$$

where i denotes the inclusion map. I being finitely generated, is projective. Therefore there exists a map $h: I \to A$ such that $\Pi h = g$. Now A being f-injective there exists a map $h': R \to A$ such that $h'i = h$. Now let $h'' = \Pi h'$, then $h''i = \Pi\, h'i = \Pi h = g$. (2) $\Rightarrow$ (3) is trivial. (3) $\Rightarrow$ (1). We prove that every finitely generated right ideal I of R is projective. We appeal to proposition (5.1) of Cartan and Eilenberg (1956). Consider a diagram

$$\begin{array}{ccccc} & & i & & \\ O \to & I & \to & R & \\ & \downarrow g & & & \\ A \to & B & \to O & & \\ \Pi & & & & \end{array}$$

where A is injective and rows are exact, i being the natural inclusion map of I into R. We shall show that $\exists\ h: I \to A$ such that $\Pi h = g$. Since B is f-injective there exists $h': R \to B$ such that $h'i = g$. Now R being projective R-module, there exists h'': $R \to A$ such that $\Pi h'' = h'$. Let $h = h''i$, then $\Pi h = \Pi h''i = h'i = g$.

Corollary—A ring R is semi-hereditary if and only if for every homomorphic image B of an injective R module A, the following are true: (1) $lr(p) = Mp\ \forall\ p \in R$, and (2) $l(I \cap J) = l(I)+l(J)\ \forall$ finitely generated right ideals I and J of R.

Theorem 3.3—A ring R is regular if and only if every R-homomorphion $f: I \to R\text{–}J$, where I is a principal right ideal of R and J any right ideal of R can be realized by left multiplication by an element of $R\text{–}J$.

PROOF: *See* Ikeda and Nakayama (1954).

Theorem 3.4—A right R is semi-hereditary self f-injective if and only if R is regular ring.

PROOF: Let R be a semi-hereditary and R_R f-injective. Then using the characterization (Cartan and Eilenberg 1956, Theorem 3.2) of semi-hereditary rings, it follows that $R\text{–}J$ is a partially injective for each right J. Hence R is regular by Theorem 3.3. Conversely let R be regular. Since every finitely generated right ideal of R is generated, by an idempotent, therefore, R is semi-hereditary. Also any module M over R is f-injective, because, suppose $f: eR \to M$, e idempotent. Let $f(e) = m$. Then $f(ex) = f(eex) = f(e)$. $ex = mex$, $\forall\, x \in R$. Hence M is f-injective. In particular R_R is f-injective.

Corollary—A semi-hereditary self-injective ring is a regular self-injective ring and conversely.

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