A NOTE ON THE EQUATIONS OF MOTION OF A PARTICLE IN GENERAL RELATIVITY

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A variation principle of the form

$$\delta \int m \, ds = 0$$

is considered for a curved space-time corresponding to an unspecified gravitational situation. Equation (2.7) and consequences following from it as considered here are believed to be new and not considered so far in this form elsewhere. m is treated as a scalar function of x^i, v^i and the equations of motion can be considered as providing the effect of a non-gravitational field on the background of any specified gravitational field.

1. Introduction

In Newtonian physics the motion of a particle of constant mass m, according to the First Law of Motion, is given by equations of the form $m\ddot{x} = 0$, $m\ddot{y} = 0$, $m\ddot{z} = 0$. In special relativity, the space-time is supposed to be flat and so the gravitational field is excluded from consideration. But in general relativity a curved space-time describes a gravitational situation which has a non-gravitational analogue in flat space-time. A treatment has been given by Brans and Dicke (1961) taking the mass of a test particle as a function of position, that is $m = m(x^i)$. The equations of motion, following the stationary principle

$$\delta \int m \, ds = 0,$$

are given by

$$\frac{d}{ds} \cdot (mg_{ik}v^i) - \tfrac{1}{2} mg_{ij,\,k} v^i v^j - m_{,\,k} = 0 \qquad \text{(1.1)}$$

satisfying the constraint condition

$$g_{ij} v^i v^j = 1. \qquad \text{(1.2)}$$

Here and henceforth a comma preceding a lower suffix implies a simple partial differentiation and a semicolon preceding a suffix means a covariant differentiation, as usual. In eqn. (1.1), the last term on the left-hand side is new implying a deviation from the geodesic path.

We are concerned in this note with departures from the geodesic path, the particles being subject to forces in addition to the usual gravitational forces, such as cosmological and electromagnetic forces. Although we take a finite m, we assume that the interaction between m and the field may be ignored as a first approximation. We assume the mass of the test particle to be a function of its position as well as velocity. *To the best of our knowledge, the problem of motion of a particle in this perspective seems to have received little attention. The assumption of m being a function of x^i and v^i is in agreement with Mach's ideas. There is no reason why a preferred coordinate system be used in which $v^i = 0$. Further, it is well known that a particle may be subject to non-gravitational fields such as an electromagnetic field, creation field and other cosmological and cosmical fields, and its inertial measure may, therefore, be affected by interaction. Generalizing the procedure of Brans and Dicke, we suppose m to be an invariant function of coordinates x^i and velocities v^i, so as to have

$$\left.\begin{aligned} m &= f(\phi, \psi, \chi) \\ \text{where}\quad \phi &= \phi(x^i) \\ \psi &= a_i v^i \\ \chi &= b_{ij} v^i v^j \end{aligned}\right\} \qquad (1.3)$$

a_i and b_{ij} being undefined vector and tensor fields.

2. The Equations of Motion

We consider

$$\delta \int m(x^i, v^i)\, ds = 0. \qquad (2.1)$$

Following the standard procedure for a variational principle we obtain the following equations of motion:

$$mv_{;k}v^k + v^l \cdot \frac{dm}{ds} = \left\{ m_{,k} - \frac{d}{ds} \cdot \left(\frac{\partial m}{\partial v^k}\right) \right\} g^{kl}. \qquad (2.2)$$

The term

$$m_{,k} - \frac{d}{ds}\left(\frac{\partial m}{\partial v^k}\right)$$

on the right-hand side of eqn. (2.2) does not apparently look like a vector but it has to be a vector as otherwise (2.2) cannot be a tensor equation. To understand the interplay of the various terms involved one may see, in particular, what happens when $m = m_1\phi + m_2\psi$ where m_1 and m_2 are constants.

$$m_{,k} = m_1\phi_{,k} + m_2 a_{i,k} v^i$$

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* Trautman (1964) has, in particular, shown that by taking for the Langrangian of the system, a generalized scalar function of x^i and v^i, equations of motion in an electromagnetic field can be obtained. Recently Rohrlich and Winicour (1966) have considered as a Langrangian, function of x^i and v^i, m being constant.

is not a vector. But

$$\frac{\partial m}{\partial v^k} = m_2 a_k$$

and, therefore, it is a vector.

Now

$$m_{,k} - \frac{d}{ds}\left(\frac{\partial m}{\partial v^k}\right) = m_1 \phi_{,k} + m_2 a_{i,k} v^i - \frac{d}{ds}(m_2 a_k)$$

$$= m_1 \phi_{,k} + m_2 (a_{i;k} - a_{k;i}) v^i. \quad \ldots \quad \ldots \quad (2.3)$$

This is a covariant vector.

In the special theory we have m_0 as the proper mass of a particle connected with the relative mass m and energy E by

$$E = mc^2 = m_0 c^2 \left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}}. \quad \ldots \quad \ldots \quad \ldots \quad (2.4)$$

We now consider a situation in which m_0 is not a constant but a scalar function of x^i, v^i because of local and distant influences affecting inertia. Mach's principle refers to the distant influences. In taking m as a scalar with all the local and distant influences we arrive ultimately at (2.2) where the second term on the left shows how the mass varies along the trajectory under the various influences and the term on the right is the analogue of the Newtonian force. The curvature of the track is thus dependent upon the rate of variation of mass and the force-like term. In eqn. (2.3), $m_1 \phi_{,k}$ brings out the distant influence and $m_2 F_{ik} v^i$

where

$$F_{ik} = a_{i;k} - a_{k;i}$$

may be treated as representing the local influence.

Multiplying eqn. (2.2) throughout by v_l, we obtain an integral from

$$\frac{d}{ds}\left(\frac{\partial m}{\partial v^k} \cdot v^k\right) = 0 \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (2.5)$$

as

$$\frac{\partial m}{\partial v^k} \cdot v^k = \text{constant}. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (2.6)$$

This is the Jacobian integral for the system. By using eqn. (2.5), eqn. (2.2) may be written as

$$m v^l_{;k} v^k = \left\{ m_{,k} - \frac{d}{ds}\left(\frac{\partial m}{\partial v^k}\right) \right\} (g^{kl} - v^k v^l). \quad \ldots \quad \ldots \quad (2.7)$$

On multiplying the above equation by v_l, we note that it becomes an identity, each side being zero, which shows that it is consistent with (2.5) and, of course, with (1.2).

If we take

$$\delta \int m(x^i, \dot{x}^i) \left(\frac{ds}{dt}\right) dt = 0$$

and put

$$\frac{ds}{dt} = (g_{ij}\dot{x}^i\dot{x}^j)^{\frac{1}{2}},\ \dot{x}^i = \frac{dx^i}{dt}$$

etc., in place of eqns. (2.2) and (2.5) we get the equations

$$m(\ddot{x}^l + \Gamma^l_{ij}\dot{x}^i\dot{x}^j) + \dot{x}^l \cdot \frac{dm}{dt} - \left(\frac{ds}{dt}\right)^2 \left\{m_{,k} - \frac{d}{dt}\left(\frac{\partial m}{\partial \dot{x}^k}\right)\right\} g^{kl}$$

$$- \left\{m\dot{x}^l - \left(\frac{ds}{dt}\right)^2 \left(\frac{\partial m}{\partial \dot{x}^k}\right) g^{kl}\right\} \frac{d^2s}{dt^2}\Big/\frac{ds}{dt} = 0 \quad \ldots \quad \ldots \quad \ldots \quad (2.8)$$

$$\frac{d}{dt}\left(\dot{x}^k \cdot \frac{\partial m}{\partial \dot{x}^k}\right) + \dot{x}^k \cdot \frac{\partial m}{\partial \dot{x}^k} \cdot \frac{d^2s}{dt^2}\Big/\frac{ds}{dt} = 0 \quad \ldots \quad \ldots \quad (2.9)$$

which may also be put in the form

$$mv^l_{;k}v^k - \left\{m_{,k} - \frac{d}{ds}\left(\frac{\partial m}{\partial v^k}\right)\right\}(g^{kl} - v^k v^l)$$

$$+ \frac{\partial m}{\partial v^k} \cdot v^k v^l \frac{d^2s}{dt^2}\Big/\left(\frac{ds}{dt}\right)^2 = 0 \quad \ldots \quad \ldots \quad (2.10)$$

$$\frac{d}{ds}\left(v^k \frac{\partial m}{\partial v^k}\right) + v^k \frac{\partial m}{\partial v^k} \cdot \frac{d^2s}{dt^2}\Big/\left(\frac{ds}{dt}\right)^2 = 0. \quad \ldots \quad \ldots \quad (2.11)$$

These equations reduce to (2.2) and (2.5) in the limit as $t \to s$.

Let us put

$$m_{,k} - \frac{d}{ds}\left(\frac{\partial m}{\partial v^k}\right) = B_k. \quad \ldots \quad \ldots \quad \ldots \quad (2.12)$$

Equation (2.2) may be written as

$$mv^l_{;k}\, v^k + v^l \frac{dm}{ds} = B_k g^{kl}. \quad \ldots \quad \ldots \quad \ldots \quad (2.13)$$

From this it is clear that

$$\frac{dm}{ds} = B_k v^k. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (2.14)$$

If B_k is perpendicular to v^k giving the direction of motion, m is constant on the world-line. If B_k is not perpendicular to the direction of motion, we have only the invariant $v^k \cdot \frac{\partial m}{\partial v^k}$ constant on the world-line which is true in all cases, that is even when $B_k v^k = 0$. For m being constant on the world-line eqn. (2.13) reduces to

$$mv^l_{;k}v^k = B^l. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (2.15)$$

$B^l \cdot m^{-1}$ is the curvature vector. When m is constant $v^l_{;k}\, v^k = 0$ gives the gravitational effect. For m not being constant

$$mv^l_{;k}\, v^k + \frac{dm}{ds} v^l$$

gives the inertial-gravitational reaction to the cosmological and other forces of the type $B_k g^{kl}$, m being a function of x^i and v^i.

We observe the similarity between eqn. (2.7) and the equation of motion for a perfect fluid (Synge 1964) as given by

$$(\rho+p)v^l_{;k}v^k = p_{,k}(g^{kl}-v^k v^l) \qquad \ldots \quad (2.16)$$

which expresses the absolute acceleration of a streamline in terms of the pressure-gradient. Equation (2.16) is obtained from the divergence equations

$$T^{ij}_{;j} = 0$$

which in turn is deduced (Landau and Lifshitz 1962) from a stationary principle of the form

$$\delta \int \left(\frac{1}{c}\right)(-g)^{\frac{1}{2}} \Lambda d\Omega = 0$$

where Λ is the Langrangian of the system, the coordinate functions x^i themselves being varied. Equation (2.5) may be compared to

$$\frac{dp}{ds} = [(\rho+p)v^l]_{,l} \qquad \ldots \quad (2.17)$$

or

$$\frac{d\rho}{ds} = -(\rho+p)v^l_{;l} \qquad \ldots \quad (2.18)$$

with eqn. (2.14). Equations (2.17) and (2.18) follow from eqn. (2.16). This similarity was expected since the equations of geodesics can be formally deduced from the four divergence equations.

3. Discussion of the Equations of Motion

In classical mechanics the Langrangian of the system gives the equations of motion under the action of different forces. Similarly, we consider a generalized function m to give the equations of motion of a particle under the action of other fields along with gravitation in curved space-time. In the absence of a unified field theory this is only a method of approximations, inasmuch as additional and extraneous field influences are being considered without their producing any change in the metric tensor. The sole novelty of the method is that interactions are considered in a given curved space-time and not in flat space-time. We take m to be of the form as given in eqn. (1.3), and obtain the equations of motion (2.7) as

$$mv^l_{;k}v^k = \left[\frac{\partial m}{\partial \phi}\cdot\phi_{,k}+\frac{\partial m}{\partial \psi}\cdot a_{i,k}v^i+\frac{\partial m}{\partial \chi}\cdot b_{ij,k}v^i v^j-\frac{d}{ds}\left(\frac{\partial m}{\partial \psi}\cdot a_k+2\cdot\frac{\partial m}{\partial \chi}\cdot b_{ik}v^i\right)\right](g^{kl}-v^k v^l) \qquad \ldots \quad (3.1)$$

the analogue of (2.5) being

$$\frac{d}{ds}\left(\frac{\partial m}{\partial \psi}\cdot\psi+2\frac{\partial m}{\partial \chi}\cdot\chi\right) = 0. \qquad \ldots \quad (3.2)$$

Particular Cases

(i) $m = m_1\phi$, that is a function of position only, m_1 being a constant. From eqn. (3.1), we have

$$m_1\phi v^l_{;k}v^k = m_1\phi_{,k}(g^{kl}-v^kv^l). \qquad (3.3)$$

Equation (3.2) does not give anything in this case. Equation (3.3) represents the equations of motion of a test particle subject to a field that arises out of the dependence of m on the position of the particle. This is the case considered by Brans and Dicke. For m being constant (3.3) reduces to the equation of a geodesic.

(ii) $m = m_1\phi+m_2\psi$, where m_1 and m_2 are constants.
We have from (3.1) and (3.2),

$$(m_1\phi+m_2\psi)v^l_{;k}v^k = m_1\phi_{,k}(g^{kl}-v^kv^l)+m_2F_{ik}v^ig^{kl} \qquad (3.4)$$

$$\frac{d}{ds}(m_2a_iv^i) = 0 \qquad (3.5)$$

giving

$$m_2a_iv^i = \text{constant} \qquad (3.6)$$

where

$$F_{ik} = a_{i,k}-a_{k,i}.$$

In special theory, the action function for a charge in an electromagnetic field has the form (Landau and Lifshitz 1962)

$$S = \int_{t_1}^{t_2}\left[-mc^2\left(1-\frac{v^2}{c^2}\right)^{\frac{1}{2}}+\left(\frac{e}{c}\right)\hat{A}\,.\,\hat{v}-e\phi'\right]dt. \qquad (3.7)$$

The integrand is just the Langrangian for a charge in an electromagnetic field, viz.

$$L = -mc^2\left(1-\frac{v^2}{c^2}\right)^{\frac{1}{2}}+\left(\frac{e}{c}\right)\hat{A}\,.\,\hat{v}-e\phi' \qquad (3.8)$$

where e is the charge on the particle and A_i is the four-vector potential and its three space components form a three-dimensional vector $\hat{A}$ called the vector potential of the field. The time component of the vector A_i is $A_4 = i\phi'$, the real quantity ϕ' being the scalar potential of the field. This function differs from the Langrangian of a free particle by the term $\left(\frac{e}{c}\right)\hat{A}\,.\,\hat{v}-e\phi'$, that is $\left(\frac{e}{c}\right)A_iv^i$, which describes the interaction of the charge with the field. From the Langrangian (3.8), we can find out the Hamiltonian function H:

$$H = mc^2\left(1-\frac{v^2}{c^2}\right)^{-\frac{1}{2}}+e\phi'. \qquad (3.9)$$

Now if we assume a_i to be the electromagnetic four-vector potential and thence F_{ik} to be the electromagnetic field tensor, by putting $m_2 = e$ in eqn.

(3.4) we obtain

$$(m_1\phi+ea_iv^i)v^l_{;k}v^k = m_1\phi_{,k}(g^{kl}-v^kv^l)+eF_{ik}v^ig^{kl}. \quad .. \quad (3.10)$$

In this case our Langrangian takes the form

$$L = (m_1\phi+ea_iv^i)(g_{ij}v^iv^j)^{\frac{1}{2}} \quad .. \quad .. \quad .. \quad (3.11)$$

so as to give the Hamiltonian function as

$$H = v^i\frac{\partial L}{\partial v^i}-L = ea_iv^i(g_{ij}v^iv^j)^{\frac{1}{2}}. \quad .. \quad .. \quad .. \quad (3.12)$$

Equation (3.10) gives us the equations of motion of a particle under the action of electromagnetic and gravitational fields. The first term on the right-hand side may be interpreted as bringing out the gravitational influence and the other may be interpreted as bringing out the electromagnetic influence. The latter influence arises because of the presence of the term ea_iv^i, which is supposed to describe the interaction of the particle (charge) with the field, in the Langrangian (3.11). The inertia of the particle may be assumed to have two components, one $m_1\phi$, due to the gravitational-inertial reaction and the second ea_iv^i (in this sum the significant contribution is from ea_4v^4), due to the electromagnetic-inertial reaction. Thus in general the inertia (energy) of the particle increases by the presence of other fields in addition to gravitation. As a first approximation we can neglect the contribution due to the interaction of the particle with the field to the inertia of the particle so as to have (3.10) as follows

$$m_1\phi v^l_{;k}v^k = m_1\phi_{,k}(g^{kl}-v^kv^l)+eF_{ik}v^ig^{kl} \quad .. \quad .. \quad (3.13)$$

which is the familiar form of the equation of motion of a particle subject to electromagnetic and gravitational fields.

(iii) $m = m_3\chi$, m_3 being a constant. It follows from eqns. (3.1) and (3.2) that

$$\chi v^l_{;k}v^k = [b_{ij;k}-2b_{ik;j})v^iv^j-2b_{ik}v^i_{;j}v^j](g^{kl}-v^kv^l) \quad .. \quad (3.14)$$

$$\frac{d}{ds}(b_{ij}v^iv^j) = 0. \quad .. \quad .. \quad .. \quad .. \quad (3.15)$$

If we assume b_{ij} to be a symmetric tensor,

$$b_{ij} = \tfrac{1}{2}(b_{i;j}+b_{j;i}) \quad .. \quad .. \quad .. \quad .. \quad (3.16)$$

which gives

$$\chi = b_{ij}v^iv^j = v^i.\frac{D(b_i)}{ds} \quad .. \quad .. \quad .. \quad (3.17)$$

where $\frac{D}{ds}$ denotes absolute differentiation. From (3.15), we note that the projection of the absolute derivative of b_i along the direction of motion is constant on the world-line of the particle.

Now (3.14) may be put in the two equivalent forms as

$$v^i\left(\frac{D(b_i)}{ds}\right)v^l_{;\,k}v^k+2b_{ik}v^i_{;\,j}v^j(g^{kl}-v^kv^l)$$
$$=\tfrac{1}{2}\,.\,v^iv^j\big[b_\rho R^\rho_{ijk}g^{kl}+\big(b_{j;\,ik}-b_{i;\,jk}-2b_{k;\,ij}\big)(g^{kl}-v^kv^l)\big] \quad .. \quad (3.18)$$

$$v^i\left(\frac{D(b_i)}{ds}\right)v^l_{;\,k}v^k+2(b_{ik}v^i)_{;\,j}v^j(g^{kl}-v^kv^l)$$
$$=\tfrac{1}{2}\,.\,v^iv^j\big[b_\rho R^\rho_{ijk}g^{kl}+\big(b_{j;\,ik}+b_{i;\,kj}\big)(g^{kl}-v^kv^l)\big]. \quad .. \quad .. \quad (3.19)$$

4. Conclusion

By taking m to be a scalar function of x^i and v^i we arrive at the equations of motion (2.2) or (2.7), and treated as a Langrangian it provides the energy integral (2.6). These equations may be said to describe the motion of a particle in an unspecified gravitational field under extraneous forces. Equation (2.5) implies (2.14) and, if m is constant on the track, the motion of the particle is given by (2.15). We have included the treatment of Brans and Dicke in our equations; in particular, (3.3) corresponds to (1.1). Neglecting the interaction of the particle with the field, we derive the familiar equation of motion (3.13) of a particle subject to the forces of gravitation and electromagnetism, as an approximation. Lastly, we should like to say that since a_i, b_{ij} and g_{ij} are not fully specified this method offers a treatment of those cases of motion where the geodesic traced by a particle of small mass is perturbed by forces arising out of other sources. The method has its own limitations. So long as the effects on g_{ij} produced by m are negligible the method should work.

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