THEORY OF ANALYTIC SPINORS—II

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In a complex two-dimensional 'analytic' space characterized by two general types of transformation, defined in Part I (Ghosh 1965), two special types of transformation called unitary analytic spinor and co-spinor transformations have been considered. It is shown that the former induces a three-dimensional proper rotation while the latter is associated with the three-dimensional improper rotation. A treatment of symmetric spinors with some of their important properties is also included in the present paper.

Introduction

In Part I (Ghosh 1965) besides a general discussion of complex analytic transformation schemes in a complex two-dimensional space S_2 a special (unimodular) transformation characterized by the invariance of an elementary anti-symmetric spinor $\gamma_{\mu\nu}$ with four components $\gamma_{11} = \gamma_{22} = 0$, $\gamma_{12} = -\gamma_{21} = 1$ was considered. A new spinor representation of Maxwell's equations and the derivation of the associated Dirac equations in Van der Waerden's form were the two principal results obtained in course of the treatment followed in Part I (Ghosh 1965).

The object of the present paper is to study two special types of analytic spinor transformation called unitary spinor and co-spinor transformations characterized by the elementary spinor $\eta_{\mu\dot{\nu}}$ with four components $\eta_{1\dot{1}} = \eta_{2\dot{2}} = i$, $\eta_{1\dot{2}} = n_{2\dot{1}} = 0$ transforming into $p.\eta_{\mu\dot{\nu}}$ where $p = \pm 1$. The well-known Pauli matrices play here an important role as characteristic spin-tensors connecting the above transformation with the induced three-dimensional rotations, proper, when $p = 1$, and improper, when $p = -1$. In the present study we have given a short treatment of symmetric spinors of rank $2j$ under these special transformations noting the reducibility of their direct product by having recourse to an important formula of Cartan (1938).

1. Unitary Analytic Spinor Transformation

In a two-dimensional complex space S_2 under general transformation scheme a unitary spinor transformation is characterized by the invariance of the elementary covariant spinor $\eta_{\mu\dot{\nu}}$ with components

$$\eta_{1\dot{1}} = \eta_{2\dot{2}} = i, \quad \eta_{1\dot{2}} = \eta_{2\dot{1}} = 0. \quad \ldots \quad \ldots \quad \ldots \quad (1.1)$$

In the transformation formula (Ghosh 1965, eqn. 2.2)

$$\eta'_{\mu\dot{\nu}} = \eta_{\alpha\dot{\beta}}\,(\xi'_{\mu}X^{\alpha})\,(\xi'_{\dot{\nu}}X^{\dot{\beta}}) \qquad (1.2)$$

we put $\eta'_{\mu\dot{\nu}} = \eta_{\mu\dot{\nu}}$ and obtain the conditions for a unitary analytic spinor transformation as

$$\left.\begin{aligned} 1 &= \alpha\dot{\alpha}+\beta\dot{\beta} \\ 1 &= \gamma\dot{\gamma}+\delta\dot{\delta} \\ 0 &= \alpha\dot{\gamma}+\beta\dot{\delta} \end{aligned}\right\} \qquad (1.3)$$

where α, β, γ, δ stand for ξ'_1X^1, ξ'_1X^2, ξ'_2X^1, ξ'_2X^2 respectively.

In matrix form the above conditions are expressible as

$$\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \begin{bmatrix} \dot{\alpha} & \dot{\gamma} \\ \dot{\beta} & \dot{\delta} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \qquad (1.4)$$

It may be noted that the contravariant and covariant coefficients of a unitary analytic spinor transformation are connected by the relations

$$\xi'_{\mu}X^{\alpha} = \xi_{\dot{\alpha}}X'^{\dot{\mu}}. \qquad (1.5)$$

We shall denote the associated contravariant spinor by $\eta^{\mu\dot{\nu}}$ having components

$$\eta^{1\dot{1}} = \eta^{2\dot{2}} = -i,\ \eta^{1\dot{2}} = \eta^{2\dot{1}} = 0 \qquad (1.6)$$

so that

$$\eta^{\mu\dot{\nu}}\eta_{\mu\dot{\rho}} = \delta^{\dot{\nu}}_{\dot{\rho}}.$$

Raising and lowering of spinor indices will be performed under the scheme

$$\eta^{\nu\dot{\mu}}\psi_{\dot{\mu}} = \psi^{\nu},\ \psi^{\mu}\eta_{\mu\dot{\nu}} = \psi_{\dot{\nu}}. \qquad (1.7)$$

This gives

$$\psi^1 = -i\psi_{\dot{1}},\ \psi^2 = -i\psi_{\dot{2}}$$

consider now a mixed spinor K^{μ}_{ν} satisfying the structural equation

$$K^{\mu}_{\nu}\eta^{\nu\dot{\rho}}\eta_{\mu\dot{\sigma}} = K^{\dot{\rho}}_{\dot{\sigma}} \qquad (1.8)$$

so that

$$K^1_1 = K^{\dot{1}}_{\dot{1}},\ K^2_2 = K^{\dot{2}}_{\dot{2}},\ K^2_1 = K^{\dot{1}}_{\dot{2}}.$$

Assuming further $K^{\mu}_{\mu} = 0$, we can express the components of the mixed spinor K^{μ}_{ν} in terms of three real quantities k_i in the following way:

$$\left.\begin{aligned} &K^1_1 = -k_3,\ K^2_2 = k_3 \\ &K^1_2 = ik_1+k_2,\ K^2_1 = -ik_1+k_2 \end{aligned}\right\}. \qquad (1.9)$$

Introducing the three Pauli spin-tensors

$$P^{1\mu}_{\nu} = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix},\quad P^{2\mu}_{\nu} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\quad P^{3\mu}_{\nu} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \qquad (1.10)$$

let us express (1.9) as

$$K^{\mu}_{\nu} = P^{i\mu}_{\nu} k_i \quad \ldots \quad (1.11)$$

which being inverted gives

$$2k_i = P^{\mu}_{i\nu} K^{\nu}_{\mu} \quad \ldots \quad (1.12)$$

where

$$P^{\mu}_{i\nu} = P^{i\mu}_{\nu}.$$

Observing now that the invariant

$$K^{\mu}_{\nu} K^{\nu}_{\mu} = 2(k_1^2 + k_2^2 + k_3^2)$$

one can remark that K^{μ}_{ν} while undergoing a unitary analytic spinor transformation induces a three-dimensional rotation to the vector k_i which we represent by

$$k'_j = r^i_j k_i. \quad \ldots \quad (1.13)$$

Now by (1.12)

$$K'_j = \tfrac{1}{2} P^{\nu}_{j\mu} K'^{\mu}_{\nu} \quad \ldots \quad (1.14)$$

where

$$K'^{\mu}_{\nu} = K^{\alpha}_{\beta} (\xi_{\alpha} X'^{\mu}) (\xi'_{\nu} X^{\beta}).$$

using (1.11) we obtain the formula

$$r^i_j = \tfrac{1}{2} P^{\nu}_{j\mu} P^{i\alpha}_{\beta} (\xi_{\alpha} X'^{\mu}) (\xi'_{\nu} X^{\beta}) \quad \ldots \quad (1.15)$$

where the spinor coefficients of transformation satisfy conditions (1.3) and (1.5).

Evaluating (1.15) we express it in the following matrix form:

$$\begin{bmatrix} r^1_1 & r^1_2 & r^1_3 \\ r^2_1 & r^2_2 & r^2_3 \\ r^3_1 & r^3_2 & r^3_3 \end{bmatrix} = \tfrac{1}{2} \begin{bmatrix} \dot\alpha & \dot\beta & \dot\gamma & \dot\delta \\ -i\dot\alpha & i\dot\beta & -i\dot\gamma & i\dot\delta \\ -i\dot\beta & -i\dot\alpha & -i\dot\delta & -i\dot\gamma \end{bmatrix} \begin{bmatrix} \delta & i\delta & -i\beta \\ -\gamma & -i\gamma & i\alpha \\ -\beta & i\beta & i\delta \\ \alpha & -i\alpha & -i\gamma \end{bmatrix}. \quad \ldots \quad (1.16)$$

By elementary operations one can reduce the determinant of the matrix involved in (1.16) to the form

$$\begin{vmatrix} \dot\delta\alpha & -i\dot\beta\gamma & i(\dot\beta\alpha - \dot\delta\gamma) \\ i\dot\gamma\beta & \dot\alpha\delta & -\dot\alpha\beta + \dot\gamma\delta \\ i\dot\delta\beta & -i\dot\alpha\nu & \dot\alpha\alpha - \dot\gamma\gamma \end{vmatrix}. \quad \ldots \quad (1.17)$$

Expanding the above and simplifying by means of the conditions (1.3) one obtains the expression

$$\alpha\dot\alpha\delta\dot\delta + \beta\dot\beta\gamma\dot\gamma - 2\dot\alpha\beta\dot\delta\gamma = (\alpha\dot\alpha + \beta\dot\beta)(\gamma\dot\gamma + \delta\dot\delta) = 1.$$

Thus the representation (1.16) belongs to a proper rotation.

2. Unitary Analytic Co-spinor Transformation

Referring to Ghosh [1965, eqns. (1.2) and (1.5)] we notice that the transformation formula of the elementary spinor $\eta_{\mu\dot{\nu}}$, defined in (1.1), undergoing a general analytic co-spinor transformation is given by

$$\eta'_{\mu\dot{\nu}} = \eta_{\dot{\alpha}\beta}(\xi'_{\mu}X^{\dot{\alpha}})(\xi'_{\dot{\nu}}X^{\beta}) . \qquad \text{(2.1)}$$

To define unitary analytic co-spinor transformation we postulate $\eta_{\mu\dot{\nu}}$ transforming into $-\eta_{\mu\dot{\nu}}$ under this special transformation. Putting $\eta'_{\mu\dot{\nu}} = -\eta_{\mu\dot{\nu}}$ in (2.1) we obtain the conditions

$$\left.\begin{aligned} \lambda\dot{\lambda}+\mu\dot{\mu} &= 1 \\ \nu\dot{\nu}+\sigma\dot{\sigma} &= 1 \\ \lambda\dot{\nu}+\mu\dot{\sigma} &= 0 \end{aligned}\right\} \qquad \text{(2.2)}$$

where λ, μ, ν, σ denote respectively $\xi'_1X^{\dot{1}}$, $\xi'_1X^{\dot{2}}$, $\xi'_2X^{\dot{1}}$, $\xi'_2X^{\dot{2}}$.

Expressing (2.1) in the form

$$\eta'_{\mu\dot{\nu}}(\xi_{\rho}X'^{\dot{\nu}}) = \eta_{\dot{\alpha}\rho}(\xi'_{\mu}X^{\dot{\alpha}}) \qquad \text{(2.3)}$$

by means of the identical relation

$$(\xi'_{\dot{\nu}}X^{\beta})(\xi_{\rho}X'^{\dot{\nu}}) = \delta^{\beta}_{\rho}$$

one can show that the contravariant and the covariant transformation coefficients are connected by the relations

$$\xi_{\lambda}X'^{\dot{\mu}} = \xi'_{\mu}X^{\dot{\lambda}} (\lambda,\, \mu = 1,\, 2). \qquad \text{(2.4)}$$

Consider now a mixed spinor K^{μ}_{ν} having structure (1.8) connected with k_i by means of the relations (1.11) and (1.12). Let K^{μ}_{ν} undergo a unitary analytic co-spinor transformation given by

$$K'^{\mu}_{\nu} = K^{\dot{\alpha}}_{\dot{\beta}}(\xi_{\dot{\alpha}}X'^{\mu})\left(\xi'_{\nu}X^{\dot{\beta}}\right). \qquad \text{(2.5)}$$

Then since $K'^{\mu}_{\nu}K'^{\nu}_{\mu} = K^{\dot{\alpha}}_{\dot{\beta}}K^{\dot{\beta}}_{\dot{\alpha}}$ and $K^{\mu}_{\mu} = 0$, it follows that the vector k_i undergoes a rotation $k_i \to \bar{k}'_j$ leaving $k_1^2+k_2^2+k_3^2$ invariant. Writing $\bar{k}'_j = \bar{r}^i_j \ k_i$ and proceeding as before the formula analogous to (1.15) is now

$$\bar{r}^i_j = \tfrac{1}{2}P^{\nu}_{j\mu}P^{i\dot{\gamma}}_{\dot{\beta}}(\xi_{\dot{\alpha}}X'^{\mu})\left(\xi'_{\nu}X^{\dot{\beta}}\right) \qquad \text{(2.6)}$$

where the coefficients of the co-spinor transformation satisfy the conditions (2.2) and (2.4).

Comparing (2.6) with (1.15) one can see that the determinant of the rotation coefficients has the value -1, showing that the representation (2.6) belongs to an improper rotation.

3. Infinitesimal Unitary Analytic Spinor Transformation and the Corresponding Infinitesimal Rotation

Starting with the infinitesimal spinor transformation equations [Ghosh 1965, eqn. (4.1)]

$$X'^{\mu} = X^{\mu}+\epsilon f^{\mu}(X^1, X^2) \qquad \ldots \quad \ldots \quad \ldots \quad (3.1)$$

and applying the conditions (1.3) in its infinitesimal form we obtain the conditions of unitarity as

$$0 = \frac{\partial f^{\lambda}}{\partial X^{\mu}}+\frac{\partial f_{\mu}}{\partial X_{\lambda}}. \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (3.2)$$

Referring to (1.12) we notice that the infinitesimal unitary analytic spinor transformation of the mixed spinor K^{ν}_{μ} and the corresponding infinitesimal rotation of the vector k_i are connected by means of the equation

$$\delta k_i = \tfrac{1}{2}P^{\mu}_{i\nu}\delta K^{\nu}_{\mu} \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (3.3)$$

where

$$\delta K^{\nu}_{\mu} = \epsilon\left[\frac{\partial f^{\nu}}{\partial X^{\lambda}}K^{\lambda}_{\mu} - \frac{\partial f^{\lambda}}{\partial X^{\mu}}K^{\nu}_{\lambda}\right]$$

by applying eqn. (4.2) given by Ghosh (1965).
Writing $\delta k_i = \rho^j_i k_j$ and making use of (1.11) we get the formula

$$\rho^j_i = \frac{\epsilon}{2}P^{\mu}_{i\nu}\left[P^{j\lambda}_{\mu}\frac{\partial f^{\nu}}{\partial X^{\lambda}} - P^{j\nu}_{\lambda}\frac{\partial f^{\lambda}}{\partial X^{\mu}}\right] \qquad \ldots \quad \ldots \quad \ldots \quad (3.4)$$

connecting the two infinitesimal transformation coefficients.

The above equation shows that $\rho^j_i+\rho^i_j = 0$.
Evaluating (3.4), we get

$$\rho^1_2 = \epsilon\left[i\frac{\partial f^1}{\partial X^1} - i\frac{\partial f^2}{\partial X^2}\right], \quad \rho^1_3 = \epsilon\left[i\frac{\partial f^1}{\partial X^2}+i\frac{\partial f^2}{\partial X^1}\right],$$

$$\rho^2_3 = \epsilon\left[\frac{\partial f^2}{\partial X^1} - \frac{\partial f^1}{\partial X^2}\right] \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (3.5)$$

where the conditions (3.2) are implied.

4. Symmetric Spinors

Let us consider a covariant spinor $\psi_{\mu_1\mu_2\ldots\mu_r}$ of rank r, symmetrical in the indices, each of which can take two values 1 and 2. Out of the 2^r components the number of mutually independent components is $r+1$. The components with S indices equal to 1 and $r-S$ indices equal to 2 are not different because of the symmetry. They are all represented by $\psi_{s,\,r-s}$, their number being $\binom{r}{s}$. In terms of the $(r+1)$ mutually independent components the transformation formula of a symmetric covariant spinor of rank r is given by Wigner (1931)

$$\psi'_{s,\,r-s} = \sum_{p=0}^{r} \psi_{p,\,r-p} \sum_{l=0} \sum_{k=0} \binom{s}{k} \binom{r-s}{l} \alpha^{s-k}\beta^{k}\gamma^{r-s-l}\delta^{l} \qquad .. \quad (4.1)$$

where the summation in k, l extends over all integers such that $k+l = r-p$. In the same way the transformation formula of a symmetric contravariant spinor of rank r is given by

$$\psi'^{s,\,r-s} = \sum_{p=0}^{r} \psi^{p,\,r-p} \sum_{l=0} \sum_{k=0} \binom{s}{k} \binom{r-s}{l} \hat{\alpha}^{s-k}\hat{\gamma}^{k}\hat{\beta}^{r-s-l}\hat{\delta}^{l} \qquad .. \quad (4.2)$$

where $\hat{\alpha}$, $\hat{\beta}$, $\hat{\gamma}$, $\hat{\delta}$ denote respectively $\xi_1 X'^1$, $\xi_1 X'^2$, $\xi_2 X'^1$, $\xi_2 X'^2$, the summation in k, l being restricted as before.

Regarding the invariant

$$\sum_{s=0}^{r} \binom{r}{s} \psi'_{s,\,r-s} \cdot \psi'^{s,\,r-s} = \sum_{p=0}^{r} \binom{r}{p} \psi_{p,\,r-p}\psi^{p,\,r-p} \qquad .. \qquad .. \quad (4.3)$$

of the above spinor transformation we notice that under unitary transformation (1.3) it reduces to

$$\sum_{s=0}^{r} \binom{r}{s} \psi'_{s,\,r-s}\psi'^{*}_{s,\,r-s} = \sum_{p=0}^{r} \binom{r}{p} \psi_{p,\,r-p}\psi^{*}_{p,\,r-p} \text{ (* denoting conj)} \quad .. \quad (4.4)$$

which is real and positive. Thus the unitary analytic spinor transformation induces, in general, a $2(r+1)$-dimensional orthogonal transformation involving real elements suitably associated with the components of the symmetric spinor.

Under general co-spinor transformation the transformation formulae of the above symmetric spinors are respectively

$$\psi'_{s,\,r-s} = \sum_{p=0}^{r} \psi^{*}_{p,\,r-p} \sum_{l=0} \sum_{k=0} \binom{s}{k} \binom{r-s}{l} \lambda^{s-k}\mu^{k}\nu^{r-s-l}\sigma^{l}$$

$$\psi'^{s,\,r-s} = \sum_{p=0}^{r} \psi^{*p,\,r-p} \sum_{k=0} \sum_{l=0} \binom{s}{k} \binom{r-s}{l} \hat{\lambda}^{s-k}\hat{\nu}^{k}\hat{\mu}^{r-s-l}\hat{\sigma}^{l} \qquad .. \quad (4.5)$$

where $\hat{\lambda}$, $\hat{\nu}$, $\hat{\mu}$, $\hat{\sigma}$ denote respectively $\xi_{\dot{1}} X'^1$, $\xi_{\dot{2}} X'^1$, $\xi_{\dot{1}} X'^2$, $\xi_{\dot{2}} X'^2$ which under unitary co-spinor transformation (2.4) are equivalent to $\dot{\lambda}$, $\dot{\mu}$, $\dot{\nu}$, $\dot{\sigma}$ respectively. Here also we can deduce a relation of the type (4.4). Referring to (2.6) we may remark that the associated 2 $(r+1)$-dimensional orthogonal transformation will be improper.

Next we consider the infinitesimal transformation of a symmetric spinor. Let us express the components of a symmetric spinor of rank $2j$, j taking integral or half integral values, in the symmetrical form

$$\frac{\psi_{j+m,\, j-m}}{\sqrt{(j+m)!}\ \sqrt{(j-m)!}} \quad \ldots \quad (4.6)$$

m ranging over $-j$ to j. Under infinitesimal analytic spinor transformation (Ghosh 1965, §4), the above spinor component which we denote by V_m undergoes an infinitesimal change δV_m expressed as

$$\partial V_m = -\epsilon\Big[(j+m)V_m\frac{\partial f^1}{\partial x^1}+(j-m)V_m\frac{\partial f^2}{\partial x^2}+[(j+m)(j-m+1)]^{\frac{1}{2}}V_{m-1}\frac{\partial f^2}{\partial x^1}$$
$$+[(j-m)(j+m+1)]^{\frac{1}{2}}V_{m+1}\frac{\partial f^1}{\partial x^2}\Big] \quad \ldots \quad (4.7)$$

ϵ being the infinitesimal parameter attached to the four infinitesimal transformation coefficients.

Introducing operators O_+, O_-, O_0 defined by

$$\left.\begin{aligned} O_+V_m &= \{(j+m+1)(j-m)\}^{\frac{1}{2}}V_{m+1} \\ O_-V_m &= \{(j-m+1)(j+m)\}^{\frac{1}{2}}V_{m-1} \\ O_0V_m &= mV_m \end{aligned}\right\} \quad \ldots \quad (4.8)$$

one can verify that they satisfy the following commutation relations:

$$\left.\begin{aligned} O_+O_--O_-O_+ &= 2O_0 \\ O_0O_+-O_+O_0 &= O_+ \\ O_-O_0-O_0O_- &= O_- \end{aligned}\right\}. \quad \ldots \quad (4.9)$$

Under infinitesimal unimodular transformation (4.7) takes the form

$$\partial V_m = -\left[\epsilon\frac{\partial f^1}{\partial x^1}2O_0+\epsilon\frac{\partial f^2}{\partial x^1}O_- +\epsilon\frac{\partial f^1}{\partial x^2}O_+\right]V_m. \quad \ldots \quad (4.10)$$

In obtaining the irreducible parts to which the elements of the direct product of two symmetric spinors undergoing a unimodular analytic spinor transformation will break up we shall follow Cartan (1938).

Let the elements of the two spinors of ranks $2j$, $2k$ respectively be represented by

$$\left.\begin{aligned} &\psi_{j+m,\, j-m}(m = -j,\ -j+1, \ldots j) \\ &\phi_{k+n,\, k-n}(n = -k,\ -k+1, \ldots k)\, j \geqslant k \end{aligned}\right\}. \quad \ldots \quad (4.11)$$

In the product there will be $(2j+1)$ $(2k+1)$ elements distributed into $(2k+1)$ groups containing respectively $2j+2k+1$, $2j+2k-1 \ldots 2j-2k+1$ elements, all distinct, making up a total of $(2j+1)$ $(2k+1)$.

The groups are represented under the following scheme (Cartan 1938):

$$\left.\begin{aligned} G_0 &= (a\psi_1+b\psi_2)^{2j}(a\phi_1+b\phi_2)^{2k} \\ G_1 &= (\psi_1\phi_2-\phi_1\psi_2)(a\psi_1+b\psi_2)^{2j-1}(\alpha\phi_1+b\phi_2)^{2k-1} \\ G_r &= (\psi_1\phi_2-\phi_1\psi_2)^r(a\psi_1+b\psi_2)^{2j-r}(a\phi_1+b\phi_2)^{2k-r} \\ G_{2k} &= (\psi_1\phi_2-\phi_1\psi_2)^{2k}(a\psi_1+b\psi_2)^{2j-2k} \end{aligned}\right\}. \quad \ldots \quad (4.12)$$

From G_0 we find that the $2j+2k+1$ elements of the product are associated with the terms $a^{j+k+s}b^{j+k-s}$ ($s = -j-k, \; -j-k+1, \ldots . j+k$), expressed in the form

$$\sum_m \sum_n \binom{2j}{j+m} \binom{2k}{k+n} \psi_{j+m,\, j-m} \phi_{k+n,\, k-n} \qquad \ldots \qquad \ldots \quad (4.13)$$

where the summation extends over all values of m, n such that $m+n = s$, m and n ranging over the values $-j$ to j and $-k$ to k respectively.

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