ON A PROBLEM OF TRANSIENT TEMPERATURE DISTRIBUTION IN A CHANNEL

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General expressions are presented for the transient temperature distribution in a fully developed flow through a channel of arbitrary cross-section. Conduction along the length of the channel is neglected but internal heat generation including viscous dissipation is taken into account. An example of a non-symmetrical heating problem in a circular pipe is given. The results of a recent investigation are determined as a special case of the general solution.

INTRODUCTION

In a recent investigation Prakash (1966) studied a problem of axisymmetric transient temperature distribution in a fully developed laminar flow inside a circular pipe under the assumption that the temperature field is constant in the direction of flow. With this assumption the problem, in effect, loses its main feature of being a forced convection problem and becomes one of pure conduction in two dimensions, with a prescribed internal heat generation due to viscous dissipation. In this case, a more general problem can be studied. The purpose of the present study is, therefore, to draw attention to the general solution of the relevant heat conduction problem and to show that the problem discussed by Prakash (1966) is a very special case of the general results given here for a channel of arbitrary cross-section. To this end we formulate the problem in general terms as follows.

FORMULATION OF THE PROBLEM

Consider a right cylindrical channel with a constant cross-section which is a two-dimensional region R, bounded by the contour C which has an arbitrary shape. Assuming that the temperature of the fluid flowing in this channel does not vary in the direction of flow, we may express the energy equation for a fully-developed flow as

$$\nabla^2 T(\mathbf{r}, t)+Q(\mathbf{r}) = \frac{1}{\kappa}\frac{\partial T(\mathbf{r}, t)}{\partial t} \quad (\mathbf{r} \text{ in } R,\ t > 0), \qquad \text{(1)}$$

where T is the unsteady temperature distribution of the fluid, $\mathbf{r}$ is the position vector in R, t is the time, κ is the thermal diffusivity of the fluid, Q is a heat generation term representing viscous dissipation and depends on the flow field which, in turn, depends also on the shape of C, and ∇^2 is the Laplacian in $\mathbf{r}$-space. The boundary condition for (1) may be specified in the form

$$\frac{\partial T}{\partial n}+h(\mathbf{r})T = f(\mathbf{r}) \quad (\mathbf{r} \text{ on } C,\ t > 0), \qquad \text{(2)}$$

where n is the outward normal on C, $h \geq 0$ is a function defined on C and may represent a quantity proportional to the relevant heat transfer coefficient between the fluid and the external medium. The quantity f is another function prescribed on C. By appropriate choice of f and h, eqn. (2) can represent any combination of prescribed temperature, prescribed heat flux and Newtonian type of convective cooling. To complete the formulation of the problem we specify the initial condition as

$$T = F(\mathbf{r}) \quad (\mathbf{r} \text{ in } R \text{ and on } C,\ t = 0), \qquad \text{(3)}$$

where $F(\mathbf{r})$ represents the initial temperature field.

General Solution

The solution to the problem defined by the system of eqns. (1), (2) and (3) follows readily from the solution of a more general problem discussed by Ölçer (1964), and is given in two equivalent forms as

$$T(\mathbf{r}, t) = T_0(\mathbf{r})+\sum_{m=1}^{\infty} A_m\phi_m(\mathbf{r})e^{-\lambda_m^2\kappa t}\left\{\iint_R \phi_m(\mathbf{r})F(\mathbf{r})\,dR - \frac{1}{\lambda_m^2}\left[\iint_R \phi_m(\mathbf{r})Q(\mathbf{r})\,dR+\int_C \phi_m(\mathbf{r})f(\mathbf{r})\,dC\right]\right\}, \qquad \text{(4a)}$$

$$T(\mathbf{r}, t) = T_0(\mathbf{r})+\sum_{m=1}^{\infty} A_m\phi_m(\mathbf{r})e^{-\lambda_m^2\kappa t}\iint_R \phi_m(\mathbf{r})[F(\mathbf{r})-T_0(\mathbf{r})]\,dR, \qquad \text{(4b)}$$

where dR denotes surface elements in R, dC denotes line elements on C, and T_0 is a steady temperature defined by

$$\nabla^2 T_0(\mathbf{r})+Q(\mathbf{r}) = 0 \quad (\mathbf{r} \text{ in } R), \qquad \text{(5a)}$$

and

$$\frac{\partial T_0}{\partial n}+h(\mathbf{r})T = f(\mathbf{r}) \quad (\mathbf{r} \text{ on } C). \qquad \text{(5b)}$$

The eigenfunctions $\phi_m(\mathbf{r})$ and the eigenvalues λ_m are determined from

$$\nabla^2\phi_m(\mathbf{r})+\lambda_m^2\phi_m(\mathbf{r}) = 0 \quad (\mathbf{r} \text{ in } R), \qquad (6a)$$

$$\frac{\partial\phi_m}{\partial n}+h(\mathbf{r})\phi_m = 0 \quad (\mathbf{r} \text{ on } C), \qquad (6b)$$

and A_m is given by

$$\frac{1}{A_m} = \iint_R \phi_m^2(\mathbf{r})\, dR. \qquad (7)$$

It is worth noting that the steady temperature field T_0 may be represented by the following expansion:

$$T_0(\mathbf{r}) = \sum_{m=1}^{\infty} \frac{A_m}{\lambda_m^2}\phi_m(\mathbf{r}) \left[\iint_R \phi_m(\mathbf{r})Q(\mathbf{r})\, dR + \int_C \phi_m(\mathbf{r})f(\mathbf{r})\, dC\right]. \qquad (8)$$

The solutions (4*a*, *b*) are valid when h does not vanish over the entire length of C. In the event that $h \equiv 0$, expressions (6*a*, *b*) require a fundamental modification discussed by Ölçer (1965).

Application

We now apply the foregoing expressions to the case of a circular channel. Expressing all physical quantities in non-dimensional form, one obtains, from eqns. (1)–(3),

$$\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\phi^2}-\frac{\partial}{\partial t}\right)T(r, \phi, t)+Q(r, \phi) = 0$$

$$(0 \le r < 1,\ 0 \le \phi \le 2\pi,\ t > 0), \qquad (9)$$

$$\left(\frac{\partial}{\partial r}+h\right)T(r, \phi, t) = f(\phi) \quad (r = 1,\ 0 \le \phi \le 2\pi,\ t > 0), \qquad (10)$$

and

$$T(r, \phi, t) = F(r, \phi) \quad (0 \le r \le 1,\ 0 \le \phi \le 2\pi,\ t = 0), \qquad (11)$$

where h is assumed to be constant.

The eigenvalue problem corresponding to (6) is defined by

$$\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\phi^2}+\lambda_{mn}^2\right)\phi_{mn}(r, \phi) = 0 \quad (0 \le r < 1,\ 0 \le \phi \le 2\pi) \qquad (12a)$$

and

$$\left(\frac{\partial}{\partial r}+h\right)\phi_{mn}(r, \phi) = 0 \quad (r = 1,\ 0 \le \phi \le 2\pi). \qquad (12b)$$

The solution of (12*a*) and (12*b*) which is well behaved at $r = 0$ is expressed by

$$\phi_{mn}(r, \phi) = \psi_{mn}(r)\begin{Bmatrix}\cos m\,\phi \\ \sin m\,\phi\end{Bmatrix} \quad (m = 0, 1, 2, \ldots), \qquad (13)$$

where

$$\psi_{mn}(r) = \frac{J_m(\lambda_{mn}r)}{J_m(\lambda_{mn})}, \qquad (14)$$

J_m is the Bessel function of the first kind of order m, and λ_{mn} is the nth root of

$$(m+h)J_m(\lambda_{mn})-\lambda_{mn}J_{m+1}(\lambda_{mn}) = 0. \qquad (15)$$

When $h > 0$, all the roots of eqn. (15) are positive. From eqns. (7) and (13) it follows that

$$\frac{1}{A_{mn}} = \int_0^1 \int_0^{2\pi} \psi_{mn}^2(r) \begin{Bmatrix} \cos^2 m\phi \\ \sin^2 m\phi \end{Bmatrix} r\,dr\,d\phi = (1+\delta_{0m})\frac{\pi}{2}\left(\frac{\lambda_{mn}^2+h^2-m^2}{\lambda_{mn}^2}\right), \qquad (16)$$

where δ_{0m} is the Kronecker delta.

With $h > 0$, the solution to the system of eqns. (9), (10) and (11) can now be obtained directly from (4):

$$T(r,\phi,t) = T_0(r,\phi)+\sum_{n=1}^{\infty}\sum_{m=0}^{\infty} A_{mn}\psi_{mn}(r)e^{-\lambda_{mn}^2 t}\int_0^{2\pi}\left\{\int_0^1 \psi_{mn}(r)F(r,\phi')r\,dr - \frac{1}{\lambda_{mn}^2}\left[\int_0^1 \psi_{mn}(r)Q(r,\phi')r\,dr+f(\phi')\right]\right\}\cos m(\phi-\phi')\,d\phi', \qquad (17a)$$

$$T(r,\phi,t) = T_0(r,\phi)+\sum_{n=1}^{\infty}\sum_{m=0}^{\infty} A_{mn}\psi_{mn}(r)e^{-\lambda_{mn}^2 t}\int_0^{2\pi}\left\{\int_0^1 \psi_{mn}(r)[F(r,\phi') - T_0(r,\phi')]\,r\,dr\right\}\cos m(\phi-\phi')\,d\phi', \qquad (17b)$$

where the function $T_0(r, \phi)$ is determined from

$$\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\phi^2}\right)T_0(r,\phi)+Q(r,\phi) = 0 \quad (0 \le r < 1,\ 0 \le \phi \le 2\pi), \qquad (18a)$$

$$\left(\frac{\partial}{\partial r}+h\right)T_0(r,\phi) = f(\phi) \quad (r = 1,\ 0 \le \phi \le 2\pi) \qquad (18b)$$

as

$$T_0(r,\phi) = \sum_{m=0}^{\infty}\frac{1}{\pi(1+\delta_{0m})}\int_0^{2\pi}\left\{\int_0^r Q(r',\phi')G_m(r',r)r'dr'+\int_r^1 Q(r',\phi')G_m(r,r')r'dr' + \left(\frac{r^m}{m+h}\right)f(\phi')\right\}\cos m(\phi-\phi')\,d\phi' \quad (h > 0). \qquad (19)$$

In this expression G_m is the Green's function given by

$$\left.\begin{aligned} G_m(r,r') &= \frac{r^m}{2(m+h)}\left[\left(\frac{1}{r'}\right)^m+(r')^m+\frac{h}{m}\left\{\left(\frac{1}{r'}\right)^m-(r')^m\right\}\right] \\ &= \frac{1}{h}+\ln\left(\frac{1}{r'}\right), \quad (m = 0) \end{aligned}\right\} (h > 0). \qquad (20)$$

We note that an alternative expression for $T_0(r, \phi)$ can be obtained directly from (8) as

$$T_0(r, \phi) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \frac{A_{mn}}{\lambda_{mn}^2} \psi_{mn}(r) \int_0^{2\pi} \left\{ \int_0^1 \psi_{mn}(r) Q(r, \phi') r\, dr + f(\phi') \right\} \cos m(\phi - \phi')\, d\phi'. \quad \text{.. (21)}$$

A comparison of eqns. (19) and (21) yields the following summation formulae:

$$\int_0^r Q(r', \phi) G_m(r', r) r' dr' + \int_r^1 Q(r', \phi) G_m(r, r') r' dr'$$
$$= 2 \sum_{n=1}^{\infty} \frac{\psi_{mn}(r)}{(\lambda_{mn}^2 + h^2 - m^2)} \int_0^1 \psi_{mn}(r) Q(r, \phi) r\, dr$$
$$(0 \le r \le 1, h > 0;\ m = 0, 1, 2, \ldots), \quad \text{.. (22}a\text{)}$$

and

$$\frac{r^m}{m+h} = 2 \sum_{n=1}^{\infty} \frac{\psi_{mn}(r)}{(\lambda_{mn}^2 + h^2 - m^2)} \quad (0 \le r < 1, h > 0;\ m = 0, 1, 2, \ldots). \quad (22b)$$

Special Cases

In the event of axial symmetry, i.e. when f and Q are independent of ϕ, expressions (17a, b) yield

$$T(r, t) = T_0(r) + \sum_{n=1}^{\infty} A_n \psi_n(r) e^{-\lambda_n^2 t} \left[\int_0^1 \psi_n(r) F(r) r\, dr \right.$$
$$\left. - \frac{1}{\lambda_n^2} \left\{ \int_0^1 \psi_n(r) Q(r) r\, dr + f \right\} \right], \quad \text{.. .. .. .. (23}a\text{)}$$

$$T(r, t) = T_0(r) + \sum_{n=1}^{\infty} A_n \psi_n(r) e^{-\lambda_n^2 t} \int_0^1 \psi_n(r) [F(r) - T_0(r)] r\, dr, \quad \text{.. (23}b\text{)}$$

where λ_n is the nth root of

$$hJ_0(\lambda_n) - \lambda_n J_1(\lambda_n) = 0. \quad \text{.. .. .. (24)}$$

$$\psi_n(r) = \frac{J_0(\lambda_n r)}{J_0(\lambda_n)}, \quad \text{.. .. .. .. (25)}$$

$$A_n = \frac{2\lambda_n^2}{\lambda_n^2 + h^2}, \quad \text{.. .. .. .. (26)}$$

and $T_0(r)$ is obtained from (19) as

$$T_0(r) = \left[\frac{1}{h} + \ln\left(\frac{1}{r}\right) \right] \int_0^r Q(r') r' dr' + \int_r^1 Q(r') \left[\frac{1}{h} + \ln\left(\frac{1}{r'}\right) \right] r' dr' + \frac{f}{h}. \quad \text{.. (27)}$$

We further specialize eqn. (23) for the case where $f = 0$, $F = 1$ and $h \to \infty$. Equation (24) then gives

$$J_0(\lambda_n) = 0, \quad \text{.. .. .. .. (28)}$$

and

$$A_n\psi_n^2(r) = 2\frac{J_0^2(\lambda_n r)}{J_1^2(\lambda_n)}, \quad \ldots \quad (29)$$

so that eqn. (27) becomes

$$T_0(r) = \ln\left(\frac{1}{r}\right)\int_0^r Q(r')r'dr' + \int_r^1 Q(r')\ln\left(\frac{1}{r'}\right)r'dr'. \quad \ldots \quad (30)$$

Expressions (23a, b) now simplify to

$$T(r, t) = T_0(r) + 2\sum_{n=1}^{\infty}\frac{J_0(\lambda_n r)}{J_1^2(\lambda_n)}e^{-\lambda_n^2 t}\left[\frac{J_1(\lambda_n)}{\lambda_n} - \frac{1}{\lambda_n^2}\int_0^1 J_0(\lambda_n r)Q(r)r\,dr\right] \quad \ldots \quad (31a)$$

and

$$T(r, t) = T_0(r) + 2\sum_{n=1}^{\infty}\frac{J_0(\lambda_n r)}{J_1^2(\lambda_n)}e^{-\lambda_n^2 t}\int_0^1 J_0(\lambda_n r)[1 - T_0(r)]r\,dr. \quad \ldots \quad (31b)$$

In the event of laminar flow in a circular pipe the dissipation function is proportional to r^2 and we let

$$Q(r) = cr^2, \quad \ldots \quad (32)$$

where c is a dimensionless positive constant. Introducing (32) into (30) and (31a) we have

$$T_0(r) = \frac{c}{16}(1 - r^4) \quad \ldots \quad (33)$$

and

$$T(r, t) = \frac{c}{16}(1 - r^4) + 2\sum_{n=1}^{\infty}\frac{J_0(\lambda_n r)}{\lambda_n J_1(\lambda_n)}e^{-\lambda_n^2 t}\left[1 - \frac{c}{\lambda_n^2}\left\{1 - \frac{2J_2(\lambda_n)}{\lambda_n J_1(\lambda_n)}\right\}\right]. \quad \ldots \quad (34)$$

It should be noted that the introduction of (33) into (31b) would also result in (34), although this would be at the expense of having to deal with more laborious algebra in the course of the evaluation of the integral

$$\int_0^1 J_0(\lambda_n r)r^5\,dr.$$

The use of (31a) is, therefore, easier and more direct. This also follows from the summation formula (22a) which now reads

$$\left.\begin{aligned}\frac{1}{16}(1 - r^4) &= 2\sum_{n=1}^{\infty}\frac{J_0(\lambda_n r)}{\lambda_n^2 J_1^2(\lambda_n)}\int_0^1 J_0(\lambda_n r)r^3\,dr \\ &= 2\sum_{n=1}^{\infty}\frac{J_0(\lambda_n r)}{\lambda_n^4 J_1^2(\lambda_n)}[\lambda_n J_1(\lambda_n) - 2J_2(\lambda_n)]\end{aligned}\right\} \quad (0 \leq r \leq 1). \quad \ldots \quad (35)$$

The value of c appearing in (32) can be easily shown to be

$$c = 16\left(\frac{\mu}{k}\frac{U^2}{\bar{T}'}\right), \quad \ldots \quad (36)$$

where μ and k are, respectively, the dynamic viscosity and the thermal conductivity of the incompressible fluid, U is the average velocity of flow over the cross-section, and $\overline{T}'$ is the reference temperature used to non-dimensionalize the actual temperature T'.

Expression (34) represents, in effect, the result deduced by a combination of eqns. (10) and (21) obtained by Prakash (1966).

From the expressions (23a, b) it is clear that other problems with $f \neq 0$ and with finite values of h can be similarly solved for the circular pipe and, in the more general case, expressions (4a, b) enable us to resolve readily many problems in channels of a variety of cross-sections.

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